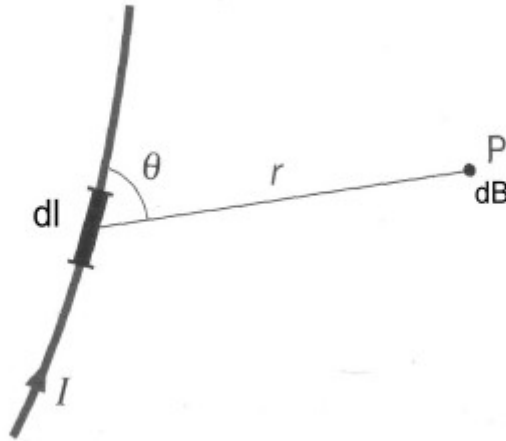


BIOT SAVART LAW AND APPLICATIONS

Biot Savart Law

Let a certain conductor be carrying a current 'I' in a direction on in the figure. Let 'P' be the point where the magnetic field due to the wire is to be studied. Let a small portion be considered which is of length 'dl'. Let the line joining 'dl' and point 'P' from an angle θ with the tangent to 'dl'.



Since the portion considered is very small, the magnetic field given by it at point P will also be small. By experimental observations and empirically also, dB is found to depend on several factors.

- i) Here dB is the measurement of magnetic energy which arises from the electrical energy represented by I which act respectively as output and input. Therefore they should have direct dependency. i.e.

$$dB \propto I$$

- ii) In a certain length of a conductor, certain amount of charge is present at a moment and the magnetic effect it can produce depend on the total number of charges, which in turn depend on the length consider, i.e.

$$dB \propto dl$$

- iii) Any force or phenomena which spread out spherically have inverse proportionality to the square of the distance between the source and point of observation.

$$dB \propto \frac{1}{r^2}$$

- iv) Similarly the magnetic field is found to be least when the angle between r and dl is the smallest (0°) and it is largest when the angle is 90° . So,

$$dB \propto \sin \theta$$

Therefore overall,

$$dB \propto \frac{Idl \sin \theta}{r^2}$$

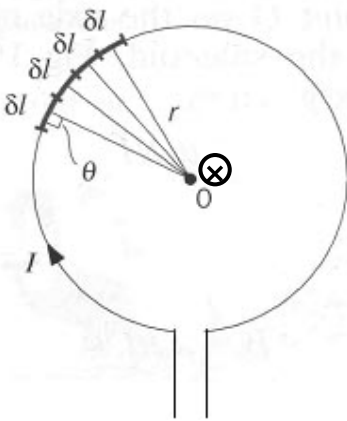
$$\text{Or, } dB = k \frac{Idl \sin \theta}{r^2}$$

In SI units, the value of $k = \frac{\mu_0}{4\pi} = 10^{-7} \text{ Hm}^{-1}$

$$\text{Or, } dB = \frac{\mu_0}{4\pi} \frac{Idl \sin \theta}{r^2}$$

This expression is called as Biot Savart Law or Laplace Law. It is the basic formula to find the magnetic field due to any structures for which all the dB's along the length of the structure have to be added to find out the total B.

MAGNETIC FIELD DUE TO CURRENT CARRYING CIRCULAR COIL AT ITS CENTRE



Let a coil be considered which is bent in the form of almost complete circle. Let a current I be supplied in clockwise direction which will give the overall magnetic field away from the observer at the centre 'O' (according to Fleming's Right-Hand Thumb rule). Let a small portion 'dl' be considered somewhere and a radius be drawn from 'dl' to 'O'. Then according to Biot Savart Law, a small magnetic field 'dB' given by 'dl' can be expressed as:

$$dB = k \frac{Idl \sin \theta}{r^2}, \text{ where } \theta \text{ is the angle between } dl \text{ and } r.$$

Here wherever 'dl' is considered, the angle between it and 'r' is always equal to 90° . Therefore,

$$dB = k \frac{Idl \sin 90^\circ}{r^2} = \frac{kIdl}{r^2}$$

The overall magnetic field will be equal to the sum of all these small magnetic fields.

$$\text{i.e. } B = \int dB \quad \text{Or, } B = \int \frac{kIdl}{r^2}$$

$$\text{Or, } B = \frac{kI}{r^2} \int dl$$

Here the variable is 'l'. if all the small "dl's" are added one by one, "l" will extend from $l = 0$ to $l = \text{circumference} (= 2\pi r)$.

$$\therefore B = \frac{kI}{r^2} \int_0^{2\pi} dl \quad \text{Or,} \quad B = \frac{kI}{r^2} [l]_0^{2\pi} \quad \text{Or,} \quad B = \frac{kI}{r^2} 2\pi r$$

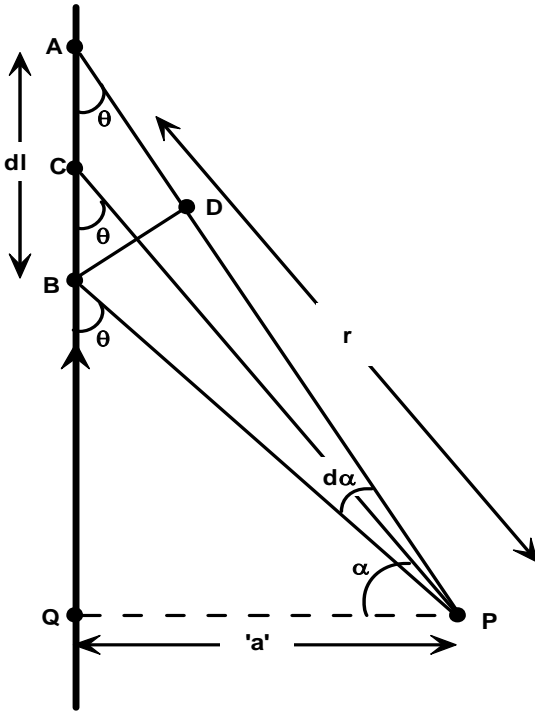
$$\text{Or,} \quad B = \frac{kI}{r} 2\pi \quad \text{Or,} \quad B = \frac{2\pi I}{r} \quad \text{Or,} \quad B = \frac{\mu_0}{4\pi} \frac{2\pi I}{r}$$

$$\text{Or,} \quad B = \frac{\mu_0 I}{2r}, \text{ (r is the radius of the coil)}$$

If the number of coils is more than one, for example 'n', the magnetic field will be

$$B = \frac{\mu_0 nI}{2r}$$

Magnetic field due to a straight conductor



Let a straight wire be considered whose magnetic field is to be determined at a certain point P which is nearby the conductor at a distance 'a'. Let a small portion of length dl be considered whose distance from P is 'r'.

Therefore the magnetic field at point P due to this small length is given by,

$$dB = \frac{kI dl \sin \theta}{r^2}$$

This magnetic field due to the whole wire is found by adding all the magnetic fields due to all these dB's of the whole wire for which the expression has to be changed to integrable form. For this, the point 'P' is joined to A, B & C. Similarly a perpendicular BD is drawn to AP at D.

Let $\angle CPQ = \alpha$. Then $\angle APB$ is the small variation in $d\alpha$ due to the consideration of the angles at the two ends of small length dl.

Since dl is very small, points A and C lie very close to each other.

Therefore $\angle BAD = \angle BCP = \theta$.

So in triangle ABD,

$$\sin \angle BAD = \frac{BD}{AB}$$

$$\text{Or, } \sin \theta = \frac{BD}{AB} \quad \text{Or, } AB \sin \theta = BD$$

$$\text{Or, } dl \sin \theta = BD \dots \dots \dots (i)$$

$$\text{Similarly in triangle BDP, } \sin \angle BPD = \frac{BD}{BP}$$

$$\text{Or, } \sin d\alpha = \frac{BD}{BP}$$

Since dl is very small, B & C also lie close together. So $BP = CP = r$. Similarly the angle $d\alpha$ is also very small. So, $\sin d\alpha = d\alpha$.

$$\text{So, } d\alpha = \frac{BD}{r}$$

Therefore,

$$r d\alpha = BD \dots \dots \dots (ii)$$

$$\text{Equations (i) and (ii) give } dl \sin \theta = r d\alpha$$

$$\text{The expression for dB becomes, } dB = \frac{kI dl \sin \theta}{r^2} \quad \text{or, } dB = \frac{kI r d\alpha}{r^2}$$

$$\text{Or, } dB = \frac{kI d\alpha}{r}$$

$$\text{In triangle CPQ, } \cos \angle CPQ = \frac{PQ}{CP} \quad \text{or, } \cos \alpha = \frac{PQ}{CP}$$

$$\text{or, } \cos \alpha = \frac{a}{r} \quad \text{or, } \frac{\cos \alpha}{a} = \frac{1}{r}$$

$$\text{which means } dB = \frac{kI \cos \alpha}{a} d\alpha$$

The total magnetic field is given by summing up all these small dB's throughout the whole length of the conductor.

$$\text{Total magnetic field (B)} = \int dB d\alpha = \int \frac{kI \cos \alpha}{a} d\alpha$$

Here the variable α varies within certain given values α_1 and α_2 , where α_1 is the angle formed by the lower tip of the conductor at P and α_2 is by the upper tip. But when the angle goes below PQ, its value becomes negative, since $\alpha_1 < 0^\circ$.

$$B = \frac{kI}{a} \int_{\alpha_1}^{\alpha_2} \cos \alpha d\alpha = \frac{kI}{a} [\sin \alpha]_{\alpha_1}^{\alpha_2}$$

$$\text{Or, } B = \frac{kI}{a} [\sin \alpha_2 - \sin \alpha_1]$$

This is the expression for the magnetic field at a certain point due to a straight conductor of finite length at a distance 'a' such that the angles formed at the two ends α_1 and α_2 .

Special case:

In most cases the wires are very long compared to the distance of the point of observation from the wire in such cases, the angles will be 90° at both sides.

$$B = \frac{kI}{a} \left[\sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2} \right) \right] = \frac{kI}{a} \left[1 + \sin \left(\frac{\pi}{2} \right) \right] = \frac{kI}{a} [1 + 1]$$

$$\text{Or, } B = \frac{2kI}{a}$$

Using the value of k as $\frac{\mu_0}{4\pi}$ gives $B = \frac{2\mu_0 I}{4\pi a}$

$$\therefore B = \frac{\mu_0 I}{2\pi a}$$

MAGNETIC FIELD AT THE AXIS OF A CIRCULAR COIL.

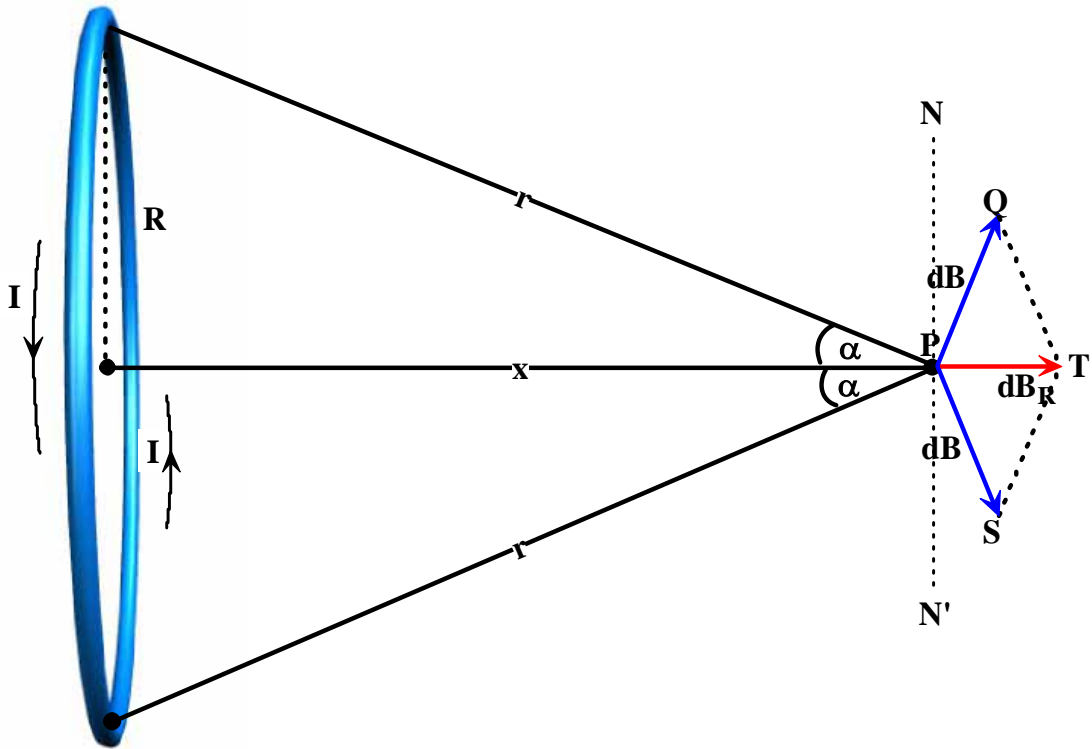
Let a circular coil be considered of radius R in which current is flowing as shown in the figure. Let P be a point in the axis of the coil where the total magnetic field due to the coil is to be determined. For this let two small position of the coil each of length dl be considered at the top & bottom, exactly opposite of to each other.

The position dl at the top will give magnetic lines of force in the direction PQ at P and its value is given by

$$dB = k \frac{Idl \sin \theta}{r^2}$$

Here θ is the angle between 'dl' & 'r' which is equal to 90° in this case $\sin \theta = \sin 90^\circ = 1$, therefore,

$$dB = \frac{kIdl}{r^2} \dots\dots(i)$$



Similarly the magnetic field at 'P' due to 'dl' at the bottom is also given by $dB = \frac{kIdl}{r^2}$ along PS.

Because of these two dB's the resultant magnetic field will be along the direction PT. Let its value be dB_R .

Let the angle between 'r' and OP is ' α ', then angle $\angle NPQ = \angle N'PS = \alpha$ i.e.

$$\angle QPT = \angle SPT = 90^\circ - \alpha$$

$$\angle QPS = 180^\circ - 2\alpha$$

Therefore the resultant dB_R is given by the relation,

$$\begin{aligned} (dB_R)^2 &= (dB)^2 + 2 (dB)(dB) \cos\angle QPS \\ &= 2(dB)^2 + 2(dB)^2 \cos(180^\circ - 2\alpha) \\ &= 2(dB)^2 - 2(dB)^2 \cos 2\alpha \\ &= 2(dB)^2 (1 - \cos 2\alpha) \\ &= 2(dB)^2 2\sin^2 \alpha \end{aligned}$$

$$\begin{aligned} \text{or, } dB_R &= \{4(dB)^2 \sin^2 \alpha\}^{1/2} \\ &= 2dB \sin \alpha \dots \text{(ii)} \end{aligned}$$

The total magnetic field will be equal to the sum of all the dB_R 's due to all the pairs of dl 's considered throughout the circle for which 'l' will extend from '0' to ' πR ' (i.e. half of circumference).

$$\begin{aligned} \therefore B &= \int_0^{\pi R} dB_R & \text{or, } B &= 2 \sin \alpha \int_0^{\pi R} dB \\ \text{or, } B &= 2 \sin \alpha \int_0^{\pi R} \frac{kI dl}{r^2} & \text{or, } B &= \frac{2 \sin \alpha kI}{r^2} \int_0^{\pi R} dl \\ \text{or, } B &= \frac{2 \left(\frac{R}{r} \right) \left(\frac{\mu_0}{4\pi} \right) I \pi R}{r^2} & \text{or, } B &= \frac{\mu_0 IR^2}{2r^3} \end{aligned}$$

Since, $r^2 = R^2 + x^2$ or, $r^3 = (R^2 + x^2)^{3/2}$

$$\therefore B = \frac{\mu_0 IR^2}{2(R^2 + x^2)^{3/2}}$$

If the point P is exactly at the centre, i.e. $x = 0$, in such case, $R = r$ and the value of B would be given by

$$\begin{aligned} \therefore B &= \frac{\mu_0 IR^2}{2(R^2)^{3/2}} = \frac{\mu_0 IR^2}{2R^3} \\ \therefore B &= \frac{\mu_0 I}{2R}, \text{ (R is the radius of the coil).} \end{aligned}$$

This value is exactly same as the value of the magnetic field of a circular coil at its center.

MAGNETIC FIELD INSIDE A SOLENOID (AT A POINT IN THE AXIS)

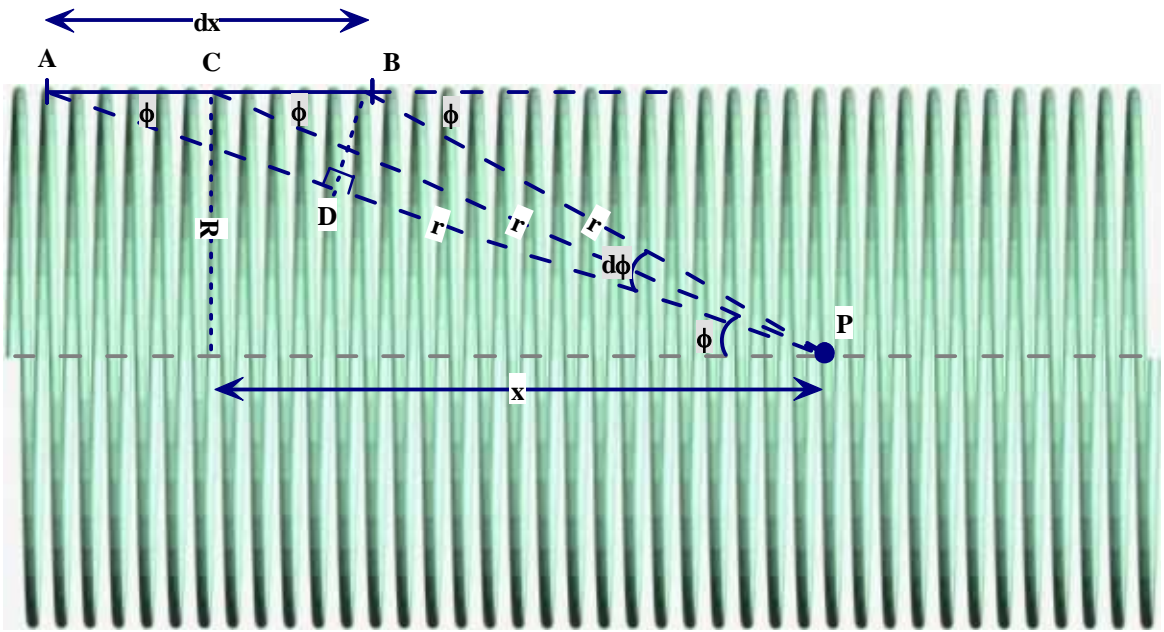
Let the figure represent the linear cross section of a solenoid whose coiling is such that it has 'n' number of coils per unit length. Let 'P' be a point where the magnetic field due to the whole solenoid is to be determined. For this, let a small length 'dx' of the solenoid be considered first and the magnetic field at 'P' due to the coils within this length be determined first.

The magnetic field due to one circular coil at a certain distance 'x' from point 'P' is,

$$B = \frac{2\pi kIR^2}{(R^2 + x^2)^{3/2}}$$

Here the term $R^2 + x^2 = r^2$ denotes the square of the distance of the point 'P' from each point of the circumference of that single coil. The number of coils present in 'dx' length is equal to 'ndx'. Therefore the magnetic field given by coils present in 'dx' length or 'ndx' number of coils is given by

$$B_{ndx} = \frac{2\pi kIR^2}{(R^2 + x^2)^{3/2}} \times (ndx)$$



However compared to the magnetic field exerted by the whole solenoid, this magnetic field is very small. So it is denoted by dB_{ndx}

$$\text{i.e. } dB_{ndx} = \frac{2\pi kIR^2}{(R^2 + x^2)^{3/2}} \times (ndx)$$

Let P be connected to A, B as well as C, where C is the centre of AB. Let $\angle CPQ = \phi$, then $\angle BCP = \phi$.

Since AB is very short length, points A & C lie very close to each other. Therefore $\angle BQP = \phi$. Similarly $\angle BPA$ denotes the small variation in ϕ , so, $\angle BPA = d\phi$

Let BD be drawn perpendicular to AP at P. In triangle BAD,

$$\begin{aligned} \sin \phi &= \frac{BD}{AB} & \text{or, } AB \sin \phi &= BD \\ \text{or, } \sin \phi \, dx &= BD \dots \dots (i) \end{aligned}$$

In triangle BDP,

$$\sin d\phi = \frac{BD}{BP} \quad \text{Or, } BP \sin d\phi = BD$$

Here, points B and c are very close to each other due to short length of AB. Therefore, $BP = CP = r$. Similarly $d\phi$ is also a very small angle so, $\sin d\phi \approx d\phi$

$$\text{Therefore, } rd\phi = BD \dots \dots (ii)$$

Comparing (i) and (ii) gives,

$$\sin \phi \, dx = r \, d\phi \qquad \text{Or, } dx = \frac{r \, d\phi}{\sin \phi}$$

$$\text{Or, } dB_{\text{ndx}} = \frac{2\pi k I R^2 n}{(R^2 + x^2)^{3/2}} \frac{r \, d\phi}{\sin \phi}$$

$$\text{Or, } dB_{\text{ndx}} = \frac{2\pi k I R^2 n}{r^3} \frac{r \, d\phi}{\sin \phi} \quad (\text{since, } R^2 + x^2 = r^2)$$

$$\text{Or, } dB_{\text{ndx}} = 2\pi k n I \sin^2 \phi \frac{d\phi}{\sin \phi}$$

$$\text{Or, } dB_{\text{ndx}} = 2\pi k n I \sin \phi \, d\phi$$

Therefore, the total magnetic field is given by, $B = \int dB_{\text{ndx}}$

Here from one end to the other end of the solenoid, ϕ varies from a minimum value ϕ_1 , to a maximum value ϕ_2

$$B = 2\pi k n I \int_{\phi_1}^{\phi_2} \sin \phi \, d\phi = 2\pi k n I [-\cos \phi]_{\phi_1}^{\phi_2}$$

i.e. $B = 2\pi k n I (\cos \phi_1 - \cos \phi_2)$

Generally solenoids are designed in such a way that the radius is very-very small compared to the length. In such case angles ϕ_1 and ϕ_2 will range from the minimum 0° through the maximum 180° . In such case,

$$B = 2\pi k n I [\cos 0^\circ - \cos 180^\circ]$$

$$\text{Or, } B = 2\pi k n I [1 - (-1)]$$

$$\text{Or, } B = \mathbf{4\pi k n I}$$

Using $k = \frac{\mu_0}{4\pi}$ gives

$$\mathbf{B = \mu_0 n I}$$

Helmholtz Coil Arrangement

It is a special arrangement of coils which have a uniform magnetic field within an appreciable region in between them. The simplest of them is made by two circular coils separated by a particular distance. The distance is determined mathematically, as follows:

The magnetic field due to the circular coil at its axis is given by

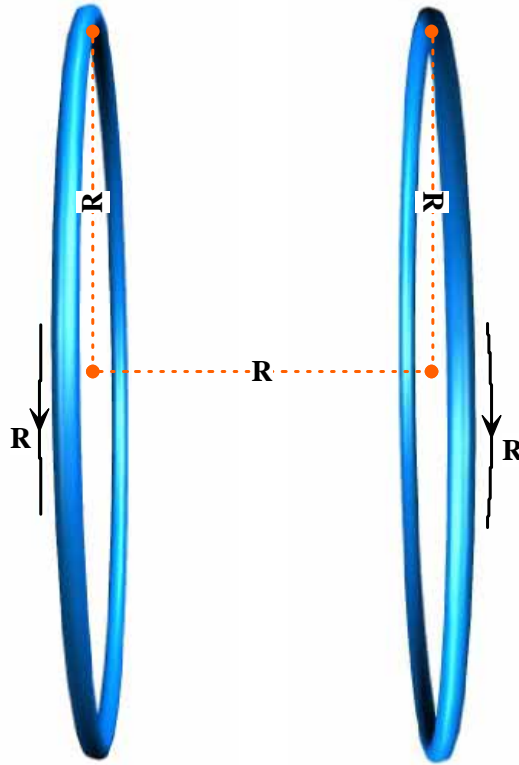
$$B = \frac{2\pi k I R^2}{(R^2 + x^2)^{3/2}} = 2\pi k I R^2 (R^2 + x^2)^{-3/2}$$

$$\begin{aligned} \text{Or, } \frac{dB}{dx} &= 2\pi kIR^2 \frac{d(R^2 + x^2)^{-3/2}}{dx} = 2\pi kIR^2 \left(-\frac{3}{2}\right)(R^2 + x^2)^{-5/2} (2x) \\ &= -6\pi kIR^2 x(R^2 + x^2)^{-5/2} \end{aligned}$$

$$\begin{aligned} \text{Or, } \frac{d^2B}{dx^2} &= -6\pi kIR^2 \frac{d}{dx} [x(R^2 + x^2)^{-5/2}] \\ &= -6\pi kIR^2 \left[x \left\{ \frac{d}{dx} (R^2 + x^2)^{-5/2} \right\} + (R^2 + x^2)^{-5/2} \frac{dx}{dx} \right] \\ &= -6\pi kIR^2 \left[x \left(-\frac{5}{2} \right) (R^2 + x^2)^{-7/2} (2x) + (R^2 + x^2)^{-5/2} \right] \\ &= -6\pi kIR^2 \left[-5x^2 (R^2 + x^2)^{-7/2} + (R^2 + x^2)^{-5/2} \right] \\ &= -6\pi kIR^2 \left[\frac{-5x^2}{(R^2 + x^2)^{7/2}} + \frac{1}{(R^2 + x^2)^{5/2}} \right] \\ &= -6\pi kIR^2 \left[\frac{-5x^2 + (R^2 + x^2)}{(R^2 + x^2)^{7/2}} \right] = -6\pi kIR^2 \left[\frac{(R^2 - 4x^2)}{(R^2 + x^2)^{7/2}} \right] \end{aligned}$$

The quantity $\frac{d^2B}{dx^2} = 0$, when $R^2 - 4x^2 = 0$,

i.e. $\frac{dB}{dx} = \text{constant}$ when $R^2 = 4x^2$, or $x = \pm \frac{R}{2}$



This means in the regions where the distance of a point from the coil is around $\pm \frac{R}{2}$, i.e. it is at a distance of $\frac{R}{2}$ at both the sides, the magnetic field would be decreasing at a constant rate, i.e. it will decrease by the same amount for the same distance from the coil if taken away from the coil and increase by the same amount for the same distance from the coil if brought near.

So if a point is at the specified distance $\left(\frac{R}{2}\right)$ of a coil and another similar coil is placed at a further distance of $\left(\frac{R}{2}\right)$, it will be at the right of one coil and left of the other, i.e. the distance will be $\left(\frac{R}{2}\right)$ from one coil and $\left(-\frac{R}{2}\right)$ from the other. So, first coil is causing reduction in the magnetic field at a constant rate if an object is taken from left to right at that region but at the same time, the coil at the right will be increasing the value of the magnetic field there by the same rate. That means the value of the magnetic field will be constant in that region. So in order to have uniform magnetic field, the overall distance between two identical coils will be $\left(\frac{R}{2} + \frac{R}{2} = R\right)$. This is the Helmholtz Coil Arrangement.

The total magnetic field at those regions ($x = \frac{R}{2}$) is given by

$$B_{\text{Helmholtz}} = 2 \times \frac{2\pi k I R^2}{(R^2 + x^2)^{3/2}}, \text{ where } x = \pm \frac{R}{2}$$
$$\text{i.e. } B_{\text{Helmholtz}} = 2 \times \frac{2\pi k I R^2}{\left[R^2 + \left(\pm \frac{R}{2}\right)^2\right]^{3/2}} = \frac{4\pi k I R^2}{\left[R^2 + \frac{R^2}{4}\right]^{3/2}} = \frac{4\pi k I R^2}{\left[\frac{4R^2 + R^2}{4}\right]^{3/2}} = \frac{4\pi k I R^2}{\left[\frac{5R^2}{4}\right]^{3/2}}$$

Since $k = \frac{\mu_0}{4\pi}$, $4\pi k = \mu_0$. Therefore,

$$B_{\text{Helmholtz}} = \mu_0 I R^2 \times \left(\frac{4}{5R^2}\right)^{3/2} = \mu_0 I R^2 \times \frac{4^{3/2}}{(5R^2)^{3/2}} = \frac{4^{3/2} \mu_0 I R^2}{5^{3/2} R^3} = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 I}{R}$$

Or, $B_{\text{Helmholtz}} = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 I}{R} = \frac{0.7155 \mu_0 I}{R}$