Gauss'(s) Law

3.1 The Important Stuff

3.1.1 Introduction; Grammar

This chapter concerns an important mathematical result which relates the electric *field* in a certain region of space with the electric *charges* found in that same region. It is useful for finding the value of electric field in situations where the charged objects are highly symmetrical. It is also valuable as an alternate mathematical expression of the inverse–square nature of the electric field from a point charge (Eq. 2.3).

Alas, physics textbooks can't seem to agree on the *name* for this law, discovered by Gauss. Some call it **Gauss' Law**. Others call it **Gauss's Law**. Do we need the extra "s" after the apostrophe or not? *Physicists do not yet know the answer to this question!!!!*

3.1.2 Electric Flux

The concept of **electric flux** involves a *surface* and the (vector) values of the electric field at all points of the surface. To introduce the way that flux is calculated, we start with a simple case. We will consider a *flat* surface of area A and an electric field which is *constant* (that is, has the same vector value) over the surface.

The surface is characterized by the "area vector" $\mathbf{A}$. This is a vector which points perpendicularly (normal) to the surface and has magnitude A. The surface and its area vector along with the uniform electric field are shown in Fig. 3.1.

Actually, there's a little problem here: There are really *two* choices for the vector $\mathbf{A}$. (It could have been chosen to point in the opposite direction; it would still be normal to the surface and have the same magnitude.) However in every problem where we use electric flux, it will be made clear which choice is made for the "normal" direction.

Now, for this simple case, the electric flux Φ is given by

$$\Phi = \mathbf{E} \cdot \mathbf{A} = EA\cos\theta$$

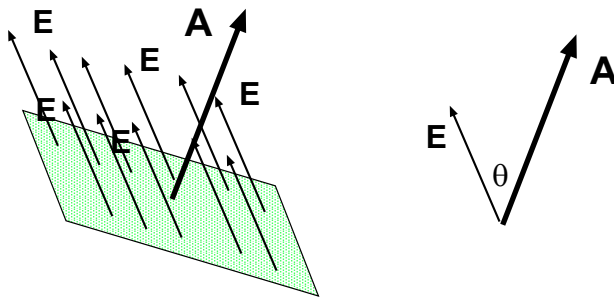where theta is the angle between $\mathbf{E}$ and $\mathbf{A}$.

Figure 3.1: Electric field **E** is uniform over a flat surface whose area vector is **A**.
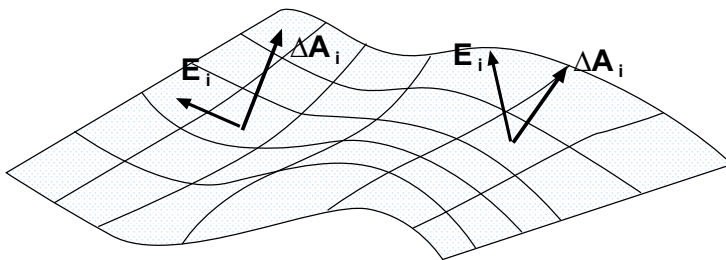

Figure 3.2: How flux is calculated (conceptually) for a general surface. Divide up the big surface into small squares; for each square find the area vector $\Delta\mathbf{A}_i$ and average electric field $\mathbf{E}_i$. Take $\Delta\mathbf{A}_i \cdot \mathbf{E}_i$ and add up the results for all the little squares.

We see that electric flux is a *scalar* and has units of $\frac{\text{N·m}^2}{\text{C}}$.

In general, a surface is *not* flat, and the electric field will not be uniform (either in magnitude or direction) over the surface. In practice one must use the machinery of advanced calculus to find the flux for the general case, but it is not hard to get the basic idea of the process: We divide up the surface into little sections (say, squares) which are all small enough so that it is a good approximation to *treat* them as flat *and* small enough so that the electric field **E** is reasonably constant. Suppose the ith little square has area vector $\Delta\mathbf{A}_i$ and the value of the electric field on that square is close to $\mathbf{E}_i$. Then the electric flux for the little square is found as before,

$$\Delta\Phi_i = \Delta\mathbf{A}_i \cdot \mathbf{E}_i$$

and the electric flux for the whole surface is *roughly* equal to the sum of all the individual contributions:

$$\Phi \approx \sum_i \Delta\mathbf{A}_i \cdot \mathbf{E}_i$$

The procedure is illustrated in Fig. 3.2.

The procedure outlined above gets closer and closer to the real value of the electric flux Φ when we make the little squares more numerous and smaller. A similar procedure in beginning calculus gives an integral (for one variable). Here, we arrive at a **surface integral** and the *proper* way to write out definition of the electric flux over the surface S is

$$\Phi = \int_S \mathbf{E} \cdot d\mathbf{A} \tag{3.1}$$

3.1.3 Gaussian Surfaces

For now at least, we are only interested in finding the flux for a special class of surfaces, ones which we call **Gaussian surfaces**. Such a surface is a *closed* surface... that is, it encloses a particular volume of space and doesn't have any holes in it. In principle it have any shape at all, but in our problem–solving we will have the most use for surfaces which have a high degree of symmetry, for example spheres and cylinders.

When we find the electric flux on a *closed* surface, it is agreed that the unit normal for all the little surface elements $d\mathbf{A}$ points *outward*.

There is a special notation for a surface integral done over a *closed* surface; the integral sign will usually have a circle superimposed on it. Thus for a *Gaussian* surface S, the electric flux is written

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} \tag{3.2}$$

We will be considering Gaussian surfaces constructed around different configurations of charges, configurations for which we are interested in finding the electric field. We get an interesting result for the electric flux for a Gaussian surface when it *encloses some electric charge*... and also when it doesn't!

3.1.4 Gauss'(s) Law

Suppose we choose a *closed* surface S in some environment where there are charges and electric fields. We can (in principle, at least) compute the electric flux Φ on S. We can also find the total electric charge enclosed by the surface S, which we will call q_{enc}. **Gauss'(s) Law** tells us:

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0} \tag{3.3}$$

3.1.5 Applying Gauss'(s) Law

Gauss'(s) Law is used to find the electric field for charge distributions which have a symmetry which we can exploit in calculating both sides of the equation: $\oint \mathbf{E} \cdot d\mathbf{A}$ and $q_{\text{enc}}/\epsilon_0$.

● **Point Charge**

Of course, we already know how to get the magnitude and direction of the electric field due to a point charge q. Here we show how this result follows from Gauss'(s) Law. (The purpose here is to give a patient discussion of how we get a *known* result so that we can use Gauss'(s) Law to obtain *new* results.

We imagine a spherical surface of radius r centered on q, as shown in Fig. 3.3. The spherical shape takes advantage of the fact that a single point gives no preferred direction in space. When we are done with the calculation, we will know the electric field for any point a distance r away from the charge.

Having drawn the proper surface, we have to use a little "common sense" for determining the direction of the electric field. From symmetry we can see the the the E field *must* point radially. Imagine looking at the point charge from any direction. It doesn't look any different!
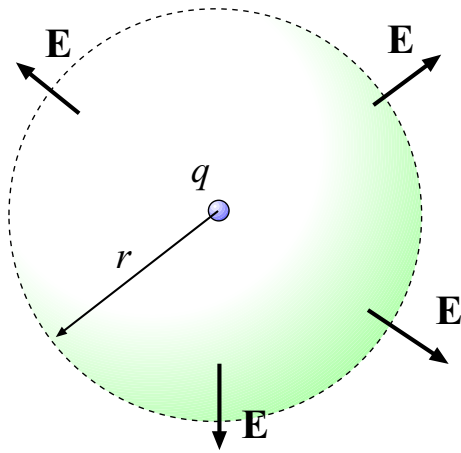
Figure 3.3: Gaussian surface of radius r centered on a point charge q. Symmetry dictates that the E field must point in the radial direction so that for points on the surface it is (locally) perpendicular to the surface.

But if the electric field's direction were anything but radial (straight out from the charge) we *could* distinguish the direction from which we were observing the charge.

Furthermore at a given distance r from the charge, the magnitude of the E field must be the same, although for all we know right now it could depend upon r. So we conclude that at all points of our (spherical) surface the E field is radial everywhere and its magnitude is the same. This fact is indicated in Fig. 3.3.

With this very reasonable assumption about $\mathbf{E}$ we can evaluate $\oint \mathbf{E} \cdot d\mathbf{A}$ *without explicitly doing any integration.* We note that every where on the surface the vector $\mathbf{E}$ is *parallel* to the area vector $d\mathbf{A}$, so that $\mathbf{E} \cdot d\mathbf{A} = E\,dA$. Since the magnitude of E is constant over the surface it can be taken outside the integral sign:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint E dA = E \oint dA \ .$$

But the expression $\oint dA$ just tells us to add up all the area elements of the surface, giving us the total area of the spherical surface, which is $4\pi r^2$. So we find:

$$\oint \mathbf{E} \cdot d\mathbf{A} = E(4\pi r^2) \ .$$

Now the charge enclosed by the Gaussian surface is simply q, that is:

$$q_{\text{enc}} = q \ .$$

Putting these facts into Gauss'(s) Law (Eq. 3.3) we have:

$$E(4\pi r^2) = \frac{q}{\epsilon_0} \qquad \Longrightarrow \qquad E = \frac{q}{4\pi \epsilon_0 r^2} \ ,$$

which we know is the correct answer for the electric field due to a point charge q.
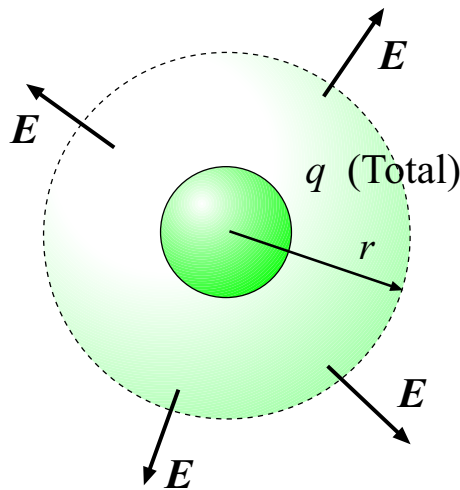
• **Spherically–Symmetric Distributions of Charge**

Figure 3.4: Gaussian surface of radius r centered on *spherically symmetric* charge distribution with total charge q. E field points radially outward on the surface.

Using Gauss'(s) Law and a spherical Gaussian surface, we can find the electric field outside of *any* spherically symmetric distribution of charge. Suppose we have a ball with total charge q, where the charge density only depends on the distance from the center of the ball. (That is to say, is has spherical symmetry.) We can draw a Gaussian surface of radius r (r being large than the radius of the ball) and use the same arguments as for the point charge to find the electric field.

We again argue that since the system looks the same regardless of the direction from which we view it, the E field on the spherical surface must point in the radial direction. (See Fig. 3.4.) So for the surface integral in Gauss'(s) Law, we get *exactly* the same thing we had before:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint E dA = E(4\pi r^2) \ .$$

As for the charge enclosed, since the *total* charge is given as q, Gauss'(s) Law gives us:

$$E(4\pi r^2) = \frac{q}{\epsilon_0}$$

so as before we find that the magnitude of the electric field at a distance r from the center of the ball of charge is

$$E = \frac{q}{4\pi \epsilon_0 r^2}$$

From a mathematical point of view, this result is quite interesting. It is the same as the field due to a point charge (as long as r is bigger than the ball's radius). The exact nature of the distribution of charge does not matter, just so long as it is spherically symmetric and its total is q. If you were to try to calculate the electric field *explicitly* by doing a integral over the volume elements of the sphere it would be a lot of work! Using Gauss'(s) Law the calculation is very easy.

As a further example involving spherical symmetry, we consider a *hollow* spherically-symmetric charge distribution. We can find the value of the electric field *inside* all of charge.
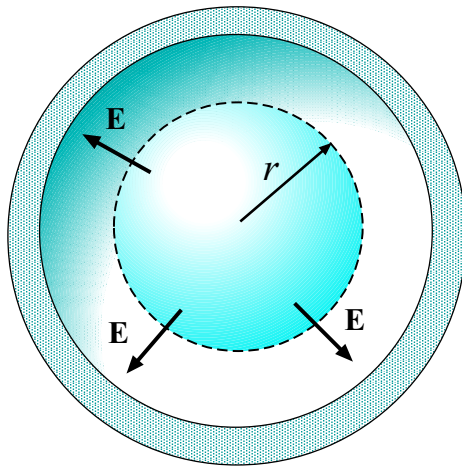
Figure 3.5: Gaussian surface of radius r centered in the interior of a *spherically symmetric* charge distribution with total charge q. **E** must point in the radial direction everywhere on the surface, but in fact **E** is zero.

To do this we once again draw a spherical Gaussian surface, this time of radius r, where r is smaller than the inner radius of the hollow ball.

What can we say about the electric field on *this* Gaussian surface? Symmetry tells us exactly the same thing as before: The electric field (if there is one!) must point in the radial direction because of the symmetry of the problem, and it must have the same magnitude everywhere on the surface. This is shown in Fig. 3.5. So again we have

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint E dA = E(4\pi r^2) \ .$$

But this time the Gaussian surface *encloses no charge at all*. So Gauss'(s) Law gives

$$E(4\pi r^2) = \frac{q_{\text{enc}}}{\epsilon_0} = 0$$

so that $E = 0$ anytime we are inside the hollow sphere of charge. This result comes about very simply using Gauss'(s) Law but it is rather challenging to show it by doing all the integrals by hand.

● **Other Geometries**

We can use Gauss'(s) Law to find the electric field around other charge distributions which have some type of symmetry, but we need to chose Gaussian surfaces of different shapes in order to take advantage of the symmetry.

If a charge distribution has symmetry about an axis (that is, *cylindrical* symmetry, like a long line of charge) then it is most useful to choose a *cylindrical* Gaussian surface, as shown in Fig. 3.6.

Using a cylindrical Gaussian surface, one can show that for a line of charge with a (positive) linear charge density λ, the electric field **E** at a distance r from the points radially outward and has magnitude

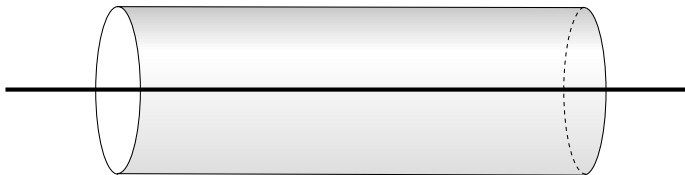$$E = \frac{\lambda}{2\pi \epsilon_0 r} \tag{3.4}$$

Figure 3.6: Gaussian surface for a line charge or more generally a distribution with *cylindrical* symmetry.
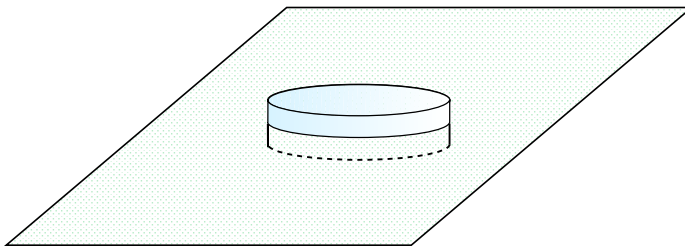

Figure 3.7: Gaussian surface for a sheet of charge (or, more generally, a charge distribution with planar symmetry).

If the charge density λ is negative, the E field points radially inward with a magnitude given by 3.4 with λ being the magnitude of the charge density.

If a charge distribution has planar symmetry that is, it stretches out uniformly and forever in the x and y directions) then it turns out to be quite useful to choose a Gaussian surface shaped like a "pillbox", that is, a cylindrical shape of very small thickness. Such a construction is shown in Fig. 3.7.

Using such a "pillbox" Gaussian surface, one can show that for a plane of charge with a (positive) charge density (charge per unit area) σ, the electric field **E** points outward from the sheet and has magnitude

$$E = \frac{\sigma}{2\epsilon_0} \tag{3.5}$$

If the charge density is *negative* the electric field points inward toward the sheet and has a magnitude given by 3.5 with σ being the magnitude of the charge density. This is the same result as Eq. 2.7.

Note that the magnitude of the E field in 3.5 does not depend on the distance from the sheet of charge.

3.1.6 Electric Fields and Conductors

For the *electrostatic* conditions that we are considering all through Vol. 4, the electric field is *zero* inside any conductor. Using Gauss'(s) law it follows that if a conductor carries any net charge, the charge will reside on the surface(s) of the conductor.

Also using Gauss'(s) law one can show that the electric field just outside a conducting surface is perpendicular to the surface and is given by

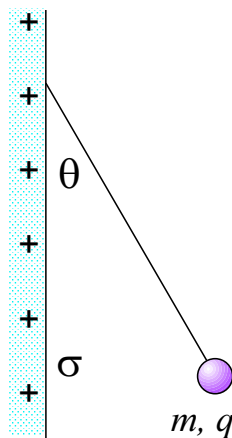$$E = \frac{\sigma}{\epsilon_0} \tag{3.6}$$

Figure 3.8: Example 1.

where σ is the surface charge density at the chosen location on the conductor and where we mean that if σ is positive the E field points outward and if it is negative the E field points toward the surface. Note that Eq. 3.6 differs from Eq. 3.5; the reasons are subtle! Be careful in choosing which one to use!

3.2 Worked Examples

3.2.1 Applying Gauss'(s) Law

1. In Fig. 3.8, a small nonconducting ball of mass $m = 1.0\,\text{mg}$ and charge $q = 2.0 \times 10^{-8}\,\text{C}$ (distributed uniformly through its volume) hangs from an insulating thread that makes an angle $\theta = 30°$ with a vertical, uniformly charged nonconducting sheet (shown in cross section). Considering the gravitational force on the ball and assuming that the sheet extends far vertically and into and out of the page, calculate the surface charge density σ of the sheet. [HRW6 24-29]

Draw a free–body diagram for the sphere! This is done in Fig. 3.9. The forces on the ball are gravity, mg downward, the tension in the string T and the force of electrostatic repulsion (F_{elec}, straight out from the sheet), arising from the sheet of positive charges. We know that the electrostatic force must point straight out from the sheet because the electric field arising from the charge points straight out, so the force exerted on the ball must point straight out as well. (We can assume the ball acts like a point charge with the charge concentrated at its center.)

First, find F_{elec}. The ball is in static equilibrium, so that the vertical and horizontal forces sum to zero. This gives us the equations:

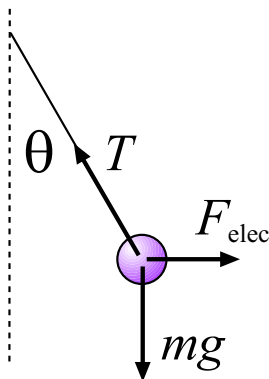$$T \cos \theta = mg \qquad\qquad T \sin \theta = F_{\text{elec}}$$

Figure 3.9: Free–body diagram for the small ball in Example 1.

Divide the second of these by the first to cancel out T and give:

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{F_{\text{elec}}}{mg} \qquad \Longrightarrow \qquad F_{\text{elec}} = mg\tan\theta$$

Plug in the numbers (note: $1.0\,\text{mg} = 1.0 \times 10^{-6}\,\text{kg}$) and get

$$F_{\text{elec}} = (1.0 \times 10^{-6}\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2})\tan 30° = 5.7 \times 10^{-6}\,\text{N}$$

From $F_{\text{elec}} = |q|E$ we can get the magnitude of the electric field:

$$E = F_{\text{elec}}/|q| = (5.7 \times 10^{-6}\,\text{N})/(2.0 \times 10^{-8}\,\text{C}) = 2.8 \times 10^2\,\tfrac{\text{N}}{\text{C}}$$

This is the magnitude of an E field on one side of an infinite sheet of charge so that from Eq. 3.5 we can find the charge density of the sheet:

$$\sigma = 2\epsilon_0 E = 2(8.85 \times 10^{-12}\,\tfrac{\text{C}^2}{\text{N·m}^2})(2.8 \times 10^2\,\tfrac{\text{N}}{\text{C}}) = 5.0 \times 10^{-9}\,\tfrac{\text{C}}{\text{m}^2} = 5.0\,\tfrac{\text{nC}}{\text{m}^2}$$

Since the E field points away from the sheet, this is the correct sign for the charge density; the charge density of the sheet is $+5.0\,\tfrac{\text{nC}}{\text{m}^2}$.

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2. In a 1911 paper, Ernest Rutherford said: "In order to form some idea of the forces required to deflect an α particle through a large angle, consider an atom [as] containing a point positive charge Ze at its center and surrounded by a distribution of negative electricity $-Ze$ uniformly distributed within a sphere of radius R. The electric field E at a distance r from the center for a point *inside* **the atom [is]**

$$E = \frac{Ze}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right)\;."$$

Verify this equation. [HRW6 24-37]

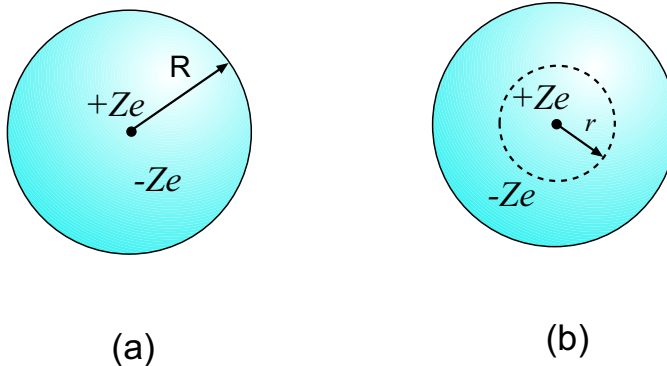Rutherford's model of the atom is shown in Fig. 3.10(a). The charge density of the

Figure 3.10: (a) Rutherford's atomic model. Point charge $+Ze$ is at the center, with a ball of uniform charge density of radius R and total charge $-Ze$ surrounding it. (b) Spherical Gaussian surface of radius r.

distribution of "negative electricity" is found by dividing the total charge $-Ze$ by the volume of the ball:

$$\rho_{-Ze} = \frac{-Ze}{\frac{4}{3}\pi R^3} = -\frac{3Ze}{4\pi R^3}$$

To find the electric field at a distance r from the center (where $r < R$), we will assume from the spherical symmetry of the problem that the **E** field points *radially*, and its magnitude depends on the distance from the center, r. Then a spherical Gaussian surface will be useful, and such a surface is shown in Fig. 3.10(b). The surface has radius r and is centered on the point charge $+Ze$. Since the **E** field is (assumed) radial, the surface integral of **E** will give a simple result, and it won't be too hard to find the charge enclosed by this surface. Then Gauss'(s) Law will give us E.

What is the charge enclosed by the surface in Fig. 3.10(b)? It encloses the point charge $+Ze$ but it also encloses *some* of the continuous charge distribution. How much? The volume of our surface is $\frac{4}{3}\pi r^3$ and multiplying this volume by the charge density found above gives the amount of the charge from the ball of negative charge which is enclosed. Thus the *total* charge enclosed by the surface is

$$
\begin{aligned}
q_{\text{enc}} &= +Ze + \left(\tfrac{4}{3}\pi r^3\right)\left(\frac{-3Ze}{4\pi R^3}\right) = +Ze - Ze\left(\frac{r^3}{R^3}\right) \\
&= Ze\left(1 - \frac{r^3}{R^3}\right)
\end{aligned}
$$

(Notice that when $r = R$ the total charge enclosed is *zero*, as it should be.)

Now, the surface integral of **E** is just the (common) magnitude of **E** on the surface multiplied by its area, $4\pi r^2$. Putting *all* of this together, Gauss'(s) Law gives us:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0} \qquad \Longrightarrow \qquad E(4\pi r^2) = \frac{Ze}{\epsilon_0}\left(1 - \frac{r^3}{R^3}\right)$$

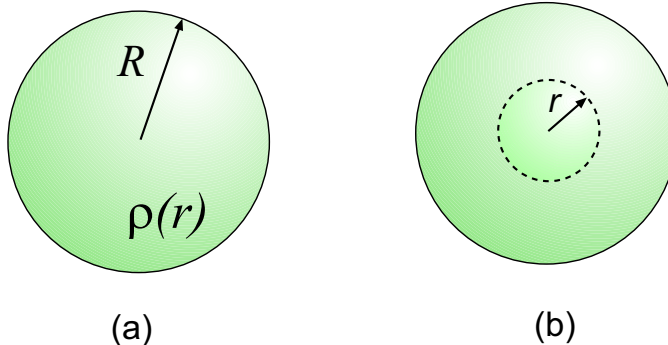Divide through by $4\pi r^2$ to get E, the radial component of the **E** field inside the "atom"

Figure 3.11: (a) Ball of charge, radius R. The charge density depends on the distance r. (b) Spherical Gaussian surface of radius r drawn inside the sphere.

(which is also the magnitude of the **E** field):

$$E = \frac{Ze}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right)$$

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3. A solid nonconducting sphere of radius R has a nonuniform charge distribution of volume charge density $\rho = \rho_s r/R$, where ρ_s is a constant and r is the distance from the center of the sphere. Show (a) that the total charge on the sphere is $Q = \pi\rho_s R^3$ and (b) that

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^4}r^2$$

gives the magnitude of the electric field inside the sphere. [HRW6 24-41]

(a) The ball of charge with nonuniform density $\rho(r)$ is drawn in Fig. 3.11(a). To get the total charge, integrate the charge density $\rho(\mathbf{r})$ over the volume of the sphere. (We must do an integral since the density is not uniform.) When integrating functions like $\rho(r)$ which depend only on the distance r over a spherical volume, we multiply $\rho(r)$ by the volume of the spherical shell element $4\pi r^2 dr$ and sum up from $r = 0$ to $r = R$:

$$Q = \int_{\text{sphere}} \rho(r)d\tau = \int_0^R \rho(r)4\pi r^2 dr$$

Substitute the given expression for $\rho(r)$ and get:

$$
\begin{aligned}
Q &= \int_0^R \frac{\rho_s r}{R}(4\pi)r^2 dr = \frac{4\pi\rho_s}{R}\int_0^R r^3 dr = \frac{4\pi\rho_s}{R}\frac{r^4}{4}\Big|_0^R \\
&= \frac{4\pi\rho_s}{R}\frac{R^4}{4} = \pi\rho_s R^3
\end{aligned}
$$

(b) To find the E field inside the sphere: Assume that the E field points in the radial direction (from the spherical symmetry of the problem). Imagine a spherical Gaussian surface of radius

r centered on the center of the charge distribution, as drawn in Fig. 3.11(b). Then the surface integral of $\mathbf{E}$ will have a simple expression and if we can calculate the charge enclosed by this surface, we can find E using Gauss'(s) Law.

To get the enclosed charge, perform an integral as in part (a) but this time only integrate out to the radius r. This gives us:

$$q_{enc} = \int_0^r \rho(r')(4\pi r'^2)dr' = \int_0^r \frac{\rho_s r'}{R}(4\pi r'^2)dr'$$

$$= \frac{4\pi\rho_s}{R}\int_0^r r'^3 dr' = \frac{4\pi\rho_s}{R}\frac{r^4}{4} = \frac{\pi\rho_s r^4}{R}$$

As usual (yawn) the surface integral of the E field over the spherical Gaussian surface is

$$\oint \mathbf{E} \cdot d\mathbf{A} = E(4\pi r^2)$$

Putting all of this into Gauss'(s) Law, we find:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\epsilon_0} \quad \Longrightarrow \quad E(4\pi r^2) = \frac{\pi\rho_s r^4}{\epsilon_0 R}$$

Divide out the $4\pi r^2$ and get:

$$E = \frac{\rho_s r^2}{4\epsilon_0 R}$$

This answer is correct, but we would like to express E in terms of the *total charge* of the sphere (instead of ρ_s). In part (a) we found that we can write:

$$Q = \pi\rho_s R^3 \quad \Longrightarrow \quad \rho_s = \frac{Q}{\pi R^3}$$

so then our answer for E is

$$E = \frac{r^2}{4\epsilon_0 r}\left(\frac{Q}{\pi R^3}\right) = \frac{Qr^2}{4\pi\epsilon_0 R^4}$$

This is the radial component of the E field as well as its magnitude.

3.2.2 Electric Fields and Conductors

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4. An isolated conductor of arbitrary shape has a net charge of $+10 \times 10^{-6}$ C. Inside the conductor is a cavity within which is a point charge $q = +3.0 \times 10^{-6}$ C. What is the charge (a) on the cavity wall and (b) on the outer surface of the conductor? [HRW6 24-15]

(a) The system is shown in Fig. 3.12(a). Consider any Gaussian surface which lies within the material of the conductor and encloses the cavity, as shown in Fig. 3.12(b). Since $\mathbf{E} = 0$
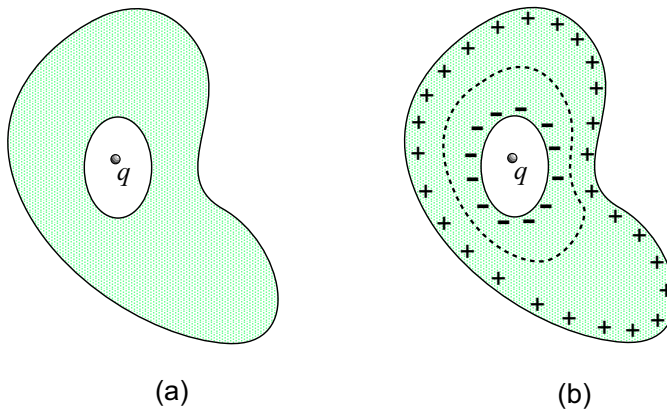
Figure 3.12: (a) Conductor carrying a net charge has a cavity inside of it. Cavity contains a charge $q = 3.0 \times 10^{-6}$ C. (b) Charges in the conductor arrange themselves; negative charges collect on the inner surface; the Gaussian surface shown must enclose a net charge of zero! An even larger number of positive charges collect on the outer surface since the conductor must have a net positive charge.

everywhere inside the conductor the integral $\oint \mathbf{E} \cdot d\mathbf{A}$ on this surface gives zero and then by Gauss'(s) Law the total charge enclosed is zero. The surface encloses the charge $q = +3.0 \times 10^{-6}$ C and also the charge which accumulates on the inner surface of the conductor. This implies that the charge which collects on the inner surface is $q_{\text{inner}} = -3.0 \times 10^{-6}$ C.

(b) The rest of the free charge on the conductor accumulates on the outer surface. But the total must come out to $+10 \times 10^{-6}$ C, as advertised! Thus:

$$q_{\text{outer}} + q_{\text{inner}} = +10 \times 10^{-6} \text{ C} \ .$$

Then:

$$
\begin{aligned}
q_{\text{outer}} &= +10 \times 10^{-6} \text{ C} - q_{\text{inner}} \\
&= +10 \times 10^{-6} \text{ C} - (-3.0 \times 10^{-6} \text{ C}) = 13 \times 10^{-6} \text{ C}
\end{aligned}
$$

The charge on the outer surface is $+13 \times 10^{-6}$ C.