Physics 181

Hamilton's Principle and Lagrange's Equation

Overview

The contributions of Galileo to the development of classical mechanics are underplayed in many textbooks. In fact they were crucial. Not only did he formulate the law of inertia (which became Newton's 1st law), he also recognized that in general the effect of the external world on an object is to provide the object with *acceleration*. This was given precise formulation in Newton's 2nd law, but it has wider consequences. For example, if it is acceleration that really counts, then changing from one reference frame to another moving at *constant velocity* relative to the first will not change anything important, because it does not alter the acceleration. This is called Galilean relativity. (The equations one uses to make such a change of reference frame had to be revised by Einstein's relativity, but the basic point remains valid.)

So if the external world only affects accelerations, then what we need to know initially about a particle is only its position and velocity. These specify the **state** of the particle. Our task in mechanics is to describe changes in that state. That is what we do when we invoke the 2nd law and find the acceleration.

But there are many situations in which use of the 2nd law is clumsy at best. Consider a particle sliding without friction on a vertically curved track, subject to gravity. At any point in its motion there are two forces on it, gravity and the normal force exerted by the track. Of course gravity has a simple formula (at least near the earth's surface), but the normal force is complicated. Its direction changes because the track is curved, and its magnitude depends on the particle's speed. For these reasons the actual acceleration of the particle is a quite complicated function, continually changing both magnitude and direction. We know the path followed by the particle (assuming it doesn't leave the track at any point) but we would be hard pressed to say at what time it reaches a particular location.

This is an example of a problem with a **constraint** force. It was partly to find a better method of approach for such problems, partly to find a more mathematically elegant way of stating the rules, that the successors of Newton studied new ways to state the theory. Foremost in these developments were Euler, Lagrange, and Hamilton, in that chronological order. In this unit we study their work to some extent. The discussion is less detailed than in T&M, especially concerning the calculus of variations in general.

Hamilton's principle

Since the state of the particle is specified by its location and velocity at a particular time, we look for some function of those variables to work with. Then we look for a general principle involving this function that tells us how the external world influences the particle's state.

It was recognized early on that cartesian coordinate axes are not the only way to specify location. For the curved track referred to above it would be helpful to have a coordinate that just told us how far the particle has moved along the track. Such specifications are called **generalized coordinates**, denoted by q_i. There are as many of these as there are independent ways for the particle to move; these ways are called *degrees of freedom*. Each coordinate has its corresponding velocity $\dot{q}_i = dq_i / dt$. Then the function we seek will be called $L(q_i, \dot{q}_i, t)$. (For brevity, we will often omit the subscripts *i*.)

The general principle we need was given in its final form by Hamilton, and is often called the *principle of least action*. The term "action" refers to the integral of L over time:

$$J = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \,.$$

Here the limits are two times at which the particle has two different states. We imagine that these times and the corresponding states are fixed, but that we can vary both q and $\dot{q}$ during the time in between, making the particle follow different paths, so that J is varied. Calling these variations δq, $\delta \dot{q}$, and δJ, we have

$$\delta J = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t)\, dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \,.$$

Hamilton's principle says that for the actual motion of the particle, $\delta J = 0$ to first order in the variations δq and $\delta \dot{q}$. That is, the actual motion of the particle is such that small variations do not change the action.

Now by Taylor's theorem we can write to 1st order

$$L(q + \delta q, \dot{q} + \delta \dot{q}, t) \approx L(q, \dot{q}, t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \,,$$

Where the partial derivatives are evaluated for $\delta q = \delta \dot{q} = 0$. Thus we find

$$\delta J = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right] dt \,.$$

Since $\dot{q} = dq / dt$ we have $\delta \dot{q} = d(\delta q) / dt$, so the 2nd term in the integral is

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d(\delta q)}{dt} dt ,$$

and we integrate by parts to convert this to

$$\frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \cdot \delta q \, dt .$$

Because the states at the initial and final times are fixed, δq vanishes at both times, so the first term above is zero. We have then

$$\delta J = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] \delta q \, dt = 0 .$$

Since δq is arbitrary, the quantity in [ ] must vanish, so we have finally

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 . \qquad (1)$$

The becomes a differential equation (2^{nd} order in time) to be solved. It is the equation of motion for the particle, and is called Lagrange's equation. The function L is called the Lagrangian of the system.

Here we need to remember that our symbol q actually represents a set of different coordinates. Because there are as many q's as degrees of freedom, there are that many equations represented by Eq (1).

Properties of the Lagrangian

So Hamilton's principle has given us Eq (1) for the Lagrangian. What do we know about L beyond the variables it depends on? We assume we are in an inertial reference frame. Then all coordinate axes are equivalent, so L must be a scalar. And our choice of when to start the clocks is arbitrary, so L cannot depend explicitly on t.

Beyond that we can make some reasonable requirements. Suppose we have two systems A and B separated by large distances so they do not interact with each other. Then the Lagrangian for this composite system must consist of separate parts for each, i.e., $L(A+B) = L(A) + L(B)$. Furthermore, multiplying L by some constant would change nothing in the equations so far. Choice of that constant simply involves choosing a system of units.

Another thing that does not change the physical content of the Lagrangian is adding to it the total time derivative of a function of q and t. Suppose we define

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{df(q,t)}{dt} .$$

Then since

$$\int_{t_1}^{t_2} \frac{df}{dt} dt = f(t_2) - f(t_1),$$

for the action we have

$$J' = J + f(t_2) - f(t_1).$$

The terms in f evaluated at the end points do not change when we perform the variation, so $\delta J' = \delta J$. The two Lagrangians give the same variation and are thus equivalent in physical content.

Now we take the simplest system, a particle moving without any interaction with the external world. We know its velocity is constant (the 1st law). Since all points in space are equivalent for such a particle, L cannot depend on its position **x**. It must therefore depend only on the velocity **v**. But it is a scalar, so it can depend only on v^2. We have that $\partial L / \partial x_i = 0$, so by Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = 0,$$

showing that $\partial L / \partial v_i$ is constant. But

$$\frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial v^2} \cdot \frac{\partial v^2}{\partial v_i} = 2\frac{\partial L}{\partial v^2} \cdot v_i.$$

and we know that v_i is constant. This means

$$\frac{\partial L}{\partial v^2} = \text{const.}$$

We conclude that $L = (\text{const}) \cdot v^2$. We choose the constant to be $\frac{1}{2}m$ and have

$$L = \tfrac{1}{2}mv^2,$$

the kinetic energy T of the particle.

Now we introduce interactions of the particle with its environment, In Newtonian mechanics these are described by forces, the connection to the motion being given by the 2nd law. We try to introduce these into to the Lagrangian by adding a term to the one we already have.

Suppose the interaction term in L does not depend explicitly on the particle's velocity. Then we will have $\partial L / \partial v_i = \partial T / \partial v_i = mv_i$, and Lagrange's equation becomes

$$\frac{d}{dt}(mv_i) - \frac{\partial L}{\partial x_i} = 0,$$

or

$$m\ddot{x}_i = \frac{\partial L}{\partial x_i}.$$

For this to give us the 2nd law we need the right side to be the force. We know this to be given by $-\partial U(\mathbf{x}) / \partial x_i$, where U is the potential energy function for the force. We are thus led to the final form for the Lagrangian:

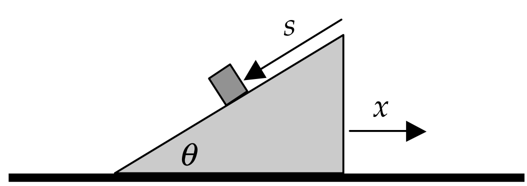
$$L(x_i, v_i, t) = T - U(x_i, t). \tag{2}$$

The possible dependence of U on t might arise if the locations of objects with which our particle interacts are changing with time in a known way. In most of our cases U will not depend on t.

Advantages of the Lagrange formulation

Perhaps the main advantage of the Lagrange approach is its use of generalized coordinates. This allows use of different coordinates for different parts of the system.

Example. A small block of mass m starts from rest at the top of a frictionless wedge of mass M which is on a frictionless horizontal floor. The block slides down the wedge, while the wedge slides across the floor. We wish to find the equations of motion for the block and the wedge.



Choose s to represent the distance the block moves down the wedge, and x to represent the distance the wedge moves across the floor. These are our generalized coordinates.

First we construct the total kinetic energy, T. The velocity of the wedge is $\dot{x}$. We break the block's velocity (in the inertial frame of the floor) into vertical and horizontal components: $v_x = \dot{x} - \dot{s}\cos\theta$, $v_y = -\dot{s}\sin\theta$. So for the combined system we have

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{s}^2 - 2\dot{x}\dot{s}\cos\theta).$$

The potential energy (taking $U = 0$ at the top of the wedge) is

$$U = -mgs\sin\theta.$$

Therefore the Lagrangian is

$$L = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{s}^2 - 2\dot{x}\dot{s}\cos\theta) + mgs\sin\theta.$$

We have $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial s} = mg\sin\theta$, $\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - m\dot{s}\cos\theta$, $\frac{\partial L}{\partial \dot{s}} = m\dot{s} - m\dot{x}\cos\theta$. The Lagrange equations are therefore

$$\begin{aligned}(M+m)\ddot{x} - m\ddot{s}\cos\theta = 0 \\ m\ddot{s} - m\ddot{x}\cos\theta - mg\sin\theta = 0\end{aligned}$$

Solving the first one for $\ddot{x}$ and substituting in the second one, we find after some rearrangement

$$\ddot{s} = \frac{M+m}{M+m\sin^2\theta} \cdot g\sin\theta \,.$$

This is a constant acceleration, so it is easy to find s at any time. Note that as $M \to \infty$ this gives $\ddot{s} = g\sin\theta$, as it should, since the wedge will not move. For the acceleration of the wedge we find, using the first Lagrange equation above,

$$\ddot{x} = \frac{m}{M+m\sin^2\theta} \cdot g\sin\theta\cos\theta \,.$$

This is also constant.

To solve this problem using the 2nd law we would need to bring in the normal force between the wedge and the block, which we do not know. To use the free body diagram for the block on the wedge we would have to take into account that the wedge is accelerating, so this is a non-inertial frame. The Lagrange method is much easier.

We will return to this example when we discuss non-inertial frames in terms of effective gravity.

In this example there is a *constraint* on the motion of the block: it must move along the incline of the wedge. We built that in when we wrote the components of the block's velocity, using $(-\dot{s}\cos\theta, -\dot{s}\sin\theta)$ for the components of the vector $\dot{\mathbf{s}}$. One can approach this problem by using as generalized coordinates the horizontal and vertical coordinates of the block plus the horizontal coordinate of the wedge, and then imposing separately the constraint that it must move along the incline. The best way to do this is through the use of what are called undetermined multipliers. This is discussed in T&M, Sec. 7.5.

Conservation laws

As the importance of energy became clearer in the first half of the 19th century the reformulation of classical mechanics took another turn, especially through the work of Hamilton.

We consider a *closed* system, which means one where there are no interactions with anything outside the system. Different parts of the system can interact in complicated ways, but it is free of outside influences. Presumably it occupies a finite region of space. So to observers inside the system the space outside is infinite in all directions and all directions are equivalent to it. That is, where one puts the origin of a coordinate system, and how one orients the axes, is totally arbitrary. Equally arbitrary is when one decides to start clocks to count time. We say that for such a system space and time are *homogeneous*.

We start with the consequence of homogeneity in time. This means that the Lagrangian L describing the system cannot depend explicitly on t. The time derivative of L is thus

$$\frac{dL}{dt}=\frac{\partial L}{\partial \dot{q}_i}\cdot\ddot{q}_i+\frac{\partial L}{\partial q_i}\cdot\dot{q}_i\,.$$

(Sum over i implied.) We use the Lagrange equations to substitute for $\frac{\partial L}{\partial q_i}$ and find

$$\frac{dL}{dt}=\frac{\partial L}{\partial \dot{q}_i}\cdot\ddot{q}_i+\dot{q}_i\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=\frac{d}{dt}\left[\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right].$$

We write this as

$$\frac{d}{dt}\left[\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L\right]=0\,. \qquad (3)$$

The quantity in [ ] is thus constant in time. Since $L=T-U$, where T is a quadratic function of the $\dot{q}_i$ and U is independent of them, it follows that

$$\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}=2T\,.$$

Exercise. This is true for any generalized coordinates, by a theorem of Euler, but it is easy to show in cartesian coordinates. Show it.

So the quantity in [ ] above is $T+U$, and what we have found is that this quantity is constant in time. This is the mechanical energy of the system, so we have proved that homogeneity in time implies that the total mechanical energy of a closed system is conserved. A remarkable conclusion, showing just how fundamental energy is.

A similar argument follows from the fact that for a closed system we can move every particle the same small distance without changing any behavior of the system. That is, the system's location in space is irrelevant because space is homogeneous.

We will use cartesian coordinates. Let the ath particle's position vector, x_i^a, be changed to $x_i^a+\varepsilon_i$, where ε_i is a small distance. This will change the Lagrangian by a small amount:

$$\delta L=L(x_i^a+\varepsilon_i)-L(x_i^a)=\sum_a\varepsilon_i\frac{\partial L}{\partial x_i^a}=\varepsilon_i\sum_a\frac{\partial L}{\partial x_i^a}\,,$$

to first order in the ε's. But this movement of all the particles by the same amount cannot have any effect, so we require $\delta L=0$. Since the ε's are arbitrary this implies

$$\sum_a\frac{\partial L}{\partial x_i^a}=0\,.$$

Now we use the Lagrange equations: $\frac{\partial L}{\partial x_i^a} = \frac{d}{dt}\left(\frac{\partial L}{\partial v_i^a}\right)$, where $v_i^a = \dot{x}_i^a$ is the corresponding velocity component, to find

$$\sum_a \frac{d}{dt}\left(\frac{\partial L}{\partial v_i^a}\right) = \frac{d}{dt}\sum_a\left(\frac{\partial L}{\partial v_i^a}\right) = 0 .$$

So the sum is constant in time. What physical quantity is it? Since the potential energy does not depend on the velocities, only the kinetic energy is involved, and it can be written as

$$T = \sum_a \tfrac{1}{2} m_a (v_i^a v_i^a) .$$

Thus we have

$$\frac{\partial L}{\partial v_i^a} = m_a v_i^a = p_i^a ,$$

the *i*th component of the momentum of the *a*th particle. So what we have proved is that

$$\frac{d}{dt}\sum_a p_i^a = 0 .$$

The sum is the total linear momentum (*i*th component) of the system. We have shown that the homogeneity of space implies conservation of the total linear momentum of a closed system. Another remarkable result.

The conservation law holds for each component of momentum separately. That is, if the Lagrangian remains unchanged when the system is moved in a particular direction, the total momentum component in that direction is conserved, whether it is true for other directions or not.

A similar argument, given in T&M, Sec 7.9, shows that if rotation of the whole system about some axis does not change the Lagrangian (as it will not for a closed system) then the total angular momentum about that axis is conserved. And again, this holds regardless of whether it is valid about other axes.

The lesson learned is that the conservation laws are direct consequences of *symmetries* of the system. In Newtonian mechanics those symmetries take the form of statements about external force and/or torques being zero. Here they are simple statements about the Lagrangian.

Hamilton's formulation

We saw in our discussion of conservation of linear momentum that the linear momentum of a particle is given by

$$p_i = \frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial \dot{x}_i}.$$

We were using cartesian coordinates, of course. But the same kind of relation holds when we use generalized coordinates. What we get is called the *generalized* momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

It may not have the dimensions of mass times velocity if q_i is not a length, but that doesn't matter. In Hamilton's reformulation the variables describing the state of a particle are taken to be the set (q_i, p_i) rather than the set $(q_i, \dot{q}_i)$.

We return to Eq (3) above and write it as

$$\frac{d}{dt}\left(\dot{q}_i p_i - L\right) = 0,$$

Where L is now regarded as function of the set (q_i, p_i). The function in ( ) above is called the *Hamiltonian* of the particle:

$$H(q_i, p_i) = \dot{q}_i p_i - L. \tag{4}$$

For a system, one sums the first term over all the particles and uses the proper L for the whole system.

We have seen that if L does not depend explicitly on time, the Hamiltonian is the same as the total energy $E = T + U$, and it is conserved. But the function is useful even if it does depend explicitly on t and is not the conserved total energy.

The total differential of H is

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \tag{5}$$

but by Eq (4) it is also

$$dH = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt. \tag{6}$$

In this expression the 2nd and 4th terms on the right are the same, so they cancel. Now from the Lagrange equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt}(p_i) = \dot{p}_i,$$

so Eq (6) becomes

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt\,.$$

Comparing with Eq (5) we find

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \tag{7}$$

which are called Hamiltion's equations of motion.

We also find $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, which is of interest only for systems depending explicitly on time. Extending our earlier analysis to include such a case, we find $\frac{dH}{dt} = \frac{\partial H}{\partial t}$. That is, H is conserved unless it depends explicitly on time.

The content of Hamilton's equations can be made clear by looking at a single partcle subject to a conservative force with potential energy function $U(q_i)$. The kinetic energy is

$$T = \tfrac{1}{2} m v_i v_i = \frac{p_i p_i}{2m},$$

so we have

$$H(q_i, p_i) = \frac{p_i p_i}{2m} + U(q_i)\,.$$

Hamilton's equations then give us

$$\dot{q}_i = \frac{p_i}{m} \text{ and } \dot{p}_i = -\frac{\partial U}{\partial q_i}.$$

The first is the definition of momentum, the second is Newton's 2nd law.

Turning to generalized coordinates, the second Hamilton equation tells us that if some variable q_i is missing from the Hamiltonian, then the corresponding momentum is conserved. Such variables are called *cyclic*. For example, if, in polar coordinates, we find that H does not depend explicitly on the azimuthal angle ϕ, then the angular momentum along the polar axis will be conserved.

Hamilton's formulation is of course logically equivalent to Lagrange's and Newton's. Its two first order (in time) differential equations are mathematically equivalent to the second order Lagrange equations. But the effects of the symmetry of the situation are often much easier to find and make use of in the Hamiltonian version. And, it turns out, the transition from the classical approximation — which is what this is — to the more general theory, quantum mechanics, is easier to make if one starts from the Hamiltonian formulation.