18/July/2014

Calasscal Mechanics

Newton's Laws of Rectilinear Motion :-

First Law :- {Relation of force and motion} :-

A law governing the relationship between forces and motion was completly described by Sir Issac Newton on the basis of inertia law of Galileo which asserted that if there are no external actions on a body it neither moves with a constant velocity or is at rest. Newton formulated this law as follows-

"A body will remain in its state of rest or uniform rectilinear motion (having zero acceleration) unless it is compelled to change its state by application of external force."

OR

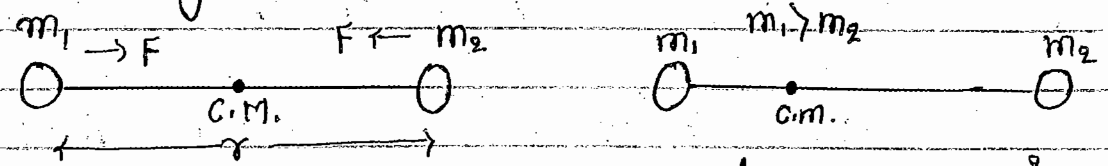
If the (vector) sum of all the forces acting on a given particle is zero then and only then the particle remains unaccelerated (i.e., remains at rest or moves with constant velocity }.

If the sum of all the forces on a given particle is $\vec{F}$ and its acceleration is $\vec{a}$, the above statement may also be written as -

" $\vec{a}=0$ if and only if $\vec{F}=0$ "

Thus, if the sum of the forces acting on a particle is known to be zero, we can be sure that the particle is unaccelerated, or if we know that a particle is unaccelerated, we can be sure that the sum of forces acting on the particle is zero.

* ~~Second Law~~ :- If no external force acts on a system, the state of system does not change.



{ For smooth surface }

{ $F_{external} = 0$ }
then C.m. is at rest

$$\boxed{F = \frac{G m_1 m_2}{r^2}}$$

Here,
State $\longrightarrow$ Centre of mass

If system consists of more than one particle then newton's law is collectively applied to centre of mass.

* Second Law of Motion :-
"The acceleration of a particle as measured from an inertial frame is given by the (vector) sum of all the forces acting on the particle divided by its mass"

In symbols,

$$\vec{a} = \frac{\vec{F}}{m}$$

$$\boxed{\vec{F} = m\vec{a}} \text{ — } ⊛$$

A force $\vec{F}$ acting on a particle of mass m produces an acceleration $\vec{F}/m$ in it with respect to an inertial frame. This is law of nature.

If the forces ceases to act at some instant the acceleration becomes zero at the same instant. In equation ⊛ $\vec{a}$ and $\vec{F}$ are measured at the same instant of time.

OR

"The rate of change of linear momentum of a body is equal to the force acting on it."

Thus,

$$\vec{F}_{ext} = \frac{d(\vec{p})}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt}$$ {taking mass constant} {$\vec{p}$ = linear momentum}

$$\boxed{\vec{F}_{ext} = m\vec{a}} \quad \text{——— (1)}$$

valid only if m = Constant

OR

$$\vec{F}_{ext} = m\frac{d^2\vec{r}}{dt^2} \Rightarrow \text{ or } \frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}}{m} = \vec{a}$$

$$\text{or} \quad \boxed{\vec{F} = m\vec{a}}$$

If mass is variable then law is written as-

$$\boxed{\vec{F} = \frac{d(m\vec{v})}{dt} = \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt}} \text{ ——— (2)}$$

If more than one force acts on the body which consequently acquires acceleration $\vec{a}$, then we write-

$$\boxed{\Sigma\vec{F} = m\vec{a}} \quad \text{——— (3)}$$

Equation (3) is the mathematical form of Newton's Second law and provides for the general equation of rectilinear motion which is the basic equation of the classical mechanics.

Note :-

(i) Newton's first law is simply a special case of second law.

(ii) Both law hold if the frame of reference has no acceleration. {Inertial reference frame}.

(iii) A frame of reference in which Newton's first and second law hold is called an "Inertial" or "Gallilean" or "Newtonian" frame of reference.

* Newton's Third Law of Motion :-

Newton's third law states that " If a body 'A' exerts a force $\vec{F}$ on another body 'B', then 'B' exerts a force $-\vec{F}$ on 'A'.

OR

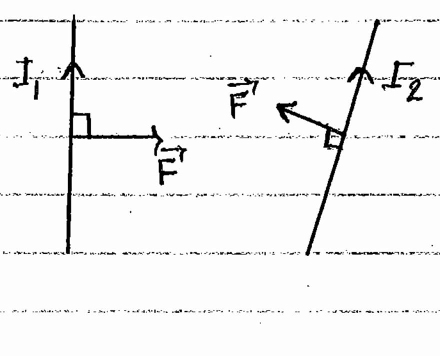
" To every action there is always an equal and opposite reaction."

Note :-

Forces always occur in pairs. Forces on a body A and B is equal & opposite to the force on the body B by A.

$\boxed{\vec{F}_{AB} = -\vec{F}_{21}}$ → always valid in mechani-cal cases.

Violation :-



In this case magnetic force between two parallel wires are not opposite in direction.

Note :-

(i) Newton's third law can be used in non-inertial reference frame, if psudo force are also included.

Resultant known force directly acting on system along X. i.e. $F_x = m a_x$.

(ii) A single isolated force is impossible.

(iii) One of the two forces, involved in mutual interaction between two bodies, is called "action force" and other is termed "reaction force."

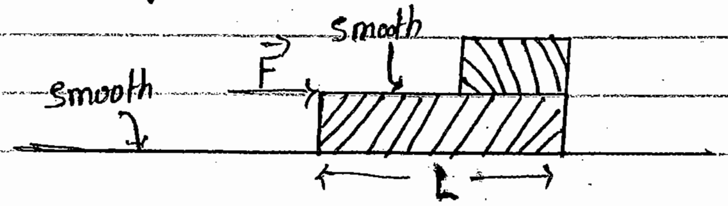
(iv) Third law speaks of mutual and simultaneous interaction.

(v) Action and reaction forces which always occur in pairs, act on different bodies.

Equation of motion $F_x = m\ddot{x}$

* Problem based on Newton's Law's :-

Ques From the given diagram, After what time the two block will separate ?



Sol^n There small block will remain at rest & big block will pass from below

$\therefore$ Acceleration of big block :-

$$\vec{a} = \frac{F_{ext}}{mass} = \frac{\vec{F}}{M}$$

$\because$ $\vec{F}$ and M is constant so $\vec{a}$ is constant.

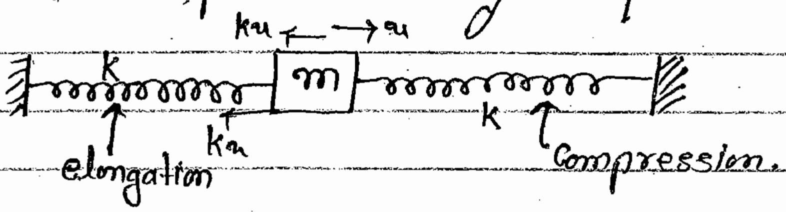
$\therefore$ $s = ut + \frac{1}{2}\vec{a}t^2$

$$L = 0 + \frac{1}{2}\frac{\vec{F}}{M}t^2$$

$$\boxed{t = \sqrt{\frac{2LM}{\vec{F}}}}$$

This is the time after which the two block separate.

A-1 Ques 12 In the fig. shown equation of motion of block which is displaced horizontally is -



Sol^n Direction of motion $= +ve$

Equation of motion,

$$\vec{F} = m\ddot{x}$$

$$-kx - kx = m\ddot{x}$$

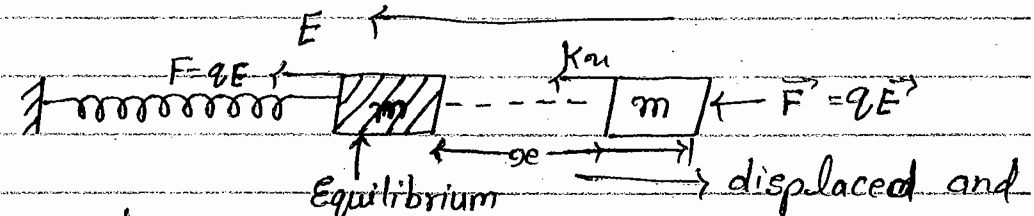
$$-2kx = m\ddot{x}$$

$$-\frac{2k}{m}x = \ddot{x}$$

So $$\boxed{\ddot{x} + \frac{2k}{m}x = 0}$$

Ans

Que Write equation of motion for given figure if the block is displaced to the $+ve$ x-direction & then



Sol^n:- Equation of motion:-

$$-kx - q\vec{E} = m\ddot{x}$$

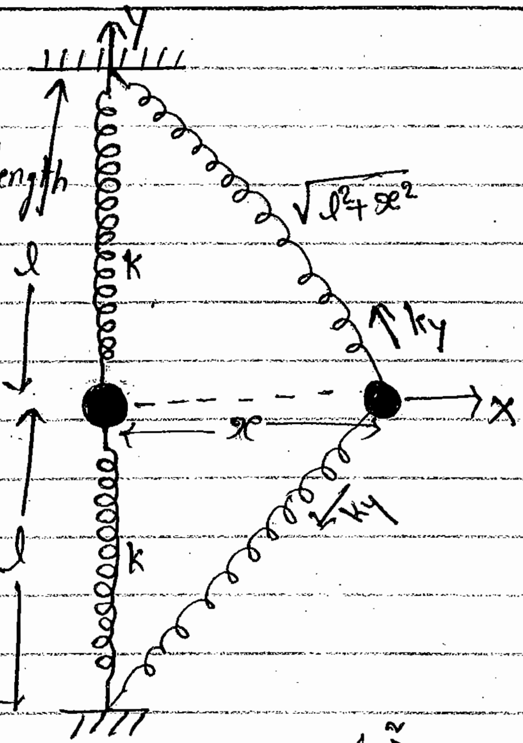
$$-(kx + q\vec{E}) = m\ddot{x}$$

$$\boxed{m\ddot{x} = -(kx + q\vec{E})}$$

Ans

Que Consider a particle of mass m attached to two identical springs each of length 'l' & spring constant K. The equilibrium configuration is the one where the springs are unstretched. There are no other external forces on the system. If the particle is given a small displacement along the x-axis, what will be the equation of motion for small oscillations?

Sol^n Let y is elongation



Elongation = Final length − initial length

$y = \sqrt{x^2+l^2} - l$

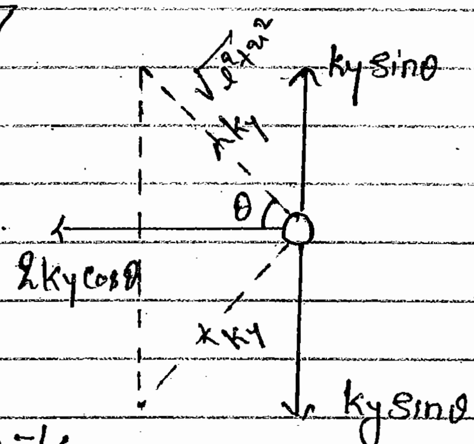
Equation of motion −

$F = m\ddot{x}$

$-2ky\cos\theta = m\ddot{x}$

$-2k\left[\sqrt{x^2+l^2} - l\right]\frac{x}{\sqrt{l^2+x^2}} = m\ddot{x}$

$\Rightarrow -2kx\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right] = m\ddot{x}$



for small displacement −

$x << l$

use binomial expansion −

$\Rightarrow \frac{l}{\sqrt{x^2+l^2}} = \frac{\cancel{l}}{\cancel{l}\sqrt{1+\frac{x^2}{l^2}}} = \left(1+\frac{x^2}{l^2}\right)^{-1/2}$

$= 1 - \frac{x^2}{2l^2} + \cdots\cdots$

$\therefore -2kx\left[\cancel{1}-\cancel{1}+\frac{x^2}{2l^2}\right] = m\ddot{x}$

$-\frac{kx^3}{l^2} = m\ddot{x}$

$\Rightarrow \boxed{m\ddot{x} + \frac{kx^3}{l^2} = 0}$

IInd Method:−

There is another method −

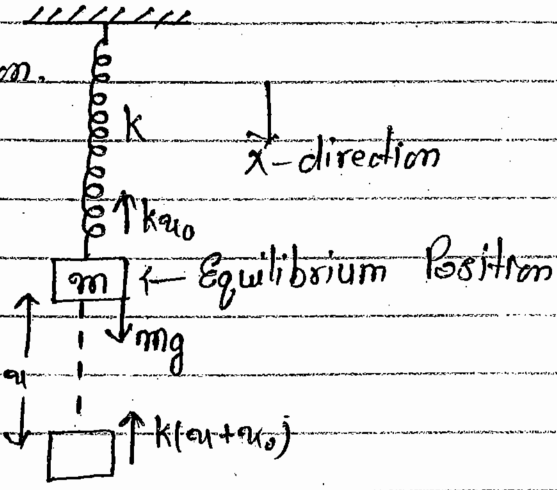
Since there is no friction. So energy is constant.

$$\text{Energy} = K.E. + P.E.$$
$$= \frac{1}{2} m (\dot{x})^2 + 2 \times \frac{1}{2} k \,[\text{elongation}]^2$$
$$= \frac{1}{2} m (\dot{x})^2 + \frac{2.k}{2} \left[\sqrt{x^2 + l^2} - l\right]^2$$

Differentiate w.r. to time and use $\frac{dE}{dt} = 0$

A-1

Ques 4 In the fig. shown if the block is displaced from equilibrium position by a distance 'x' its equation of motion will be-



Sol^n Let x_0 is initial elongation.
for Equilibrium -

$$kx_0 = mg \quad \text{—— (1)}$$

Equation of motion -

$$F_x = m\ddot{x}$$

$$mg - k(x + x_0) = m\ddot{x}$$ displacement = +ve

$$\cancel{mg} - kx - \cancel{kx_0} = m\ddot{x}$$ {∵ at equilibrium $kx_0 = mg$}

$$-kx = m\ddot{x}$$

$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

for Spring mass System:-
If equation of motion is written in terms of displacement from equilibrium position then 'mg' does not cause in final equation.

* If in above question the elongation is y then equation of motion:-

$$\vec{F_y} = m\ddot{y} \Rightarrow mg - ky = m\ddot{y} \Rightarrow \ddot{y} = g - \frac{k}{m}y$$

$$\Rightarrow \boxed{\ddot{y} + \frac{ky}{m} - g = 0}$$ Ans

* Kinematics :-

$$v_x = \frac{dx}{dt}$$

$$a_x = \frac{dv_x}{dt}$$

or $V = \frac{dx}{dt}$, $a_x = \frac{dV_x}{dx} \cdot \frac{dx}{dt} \rightarrow V_x$

$$a_x = v_x \frac{dv_x}{dx}$$

or $a = v\frac{dV}{dx}$

Definations.

Valid only for $\vec{a} \neq$ Const.

If $a =$ Constant

Formulas :-

$$S = ut + \frac{1}{2}\vec{a}t^2$$

$$V = u + \vec{a}t$$

$$V^2 = u^2 + 2\vec{a}\vec{s}$$

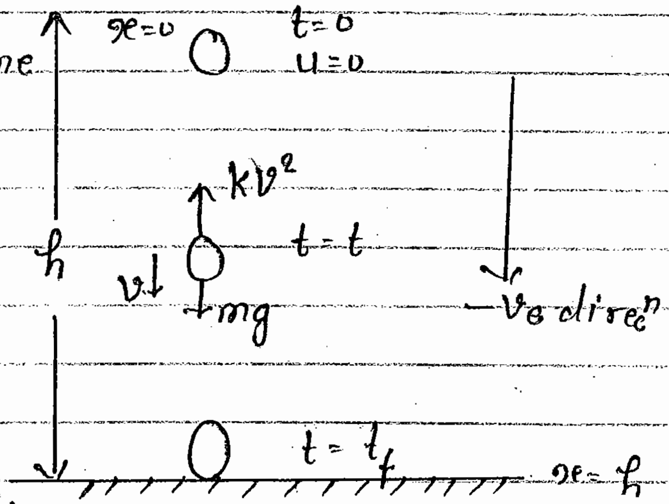
Valid only for $\vec{a} =$ Constant.

A-1

Ques 8 A body of mass 'm' falls from rest at a hight 'h' under gravity (acceleration due to gravity g) through a dense medium which provides a resistive force $F = -kv^2$, where 'k' is a constant & 'v' is the speed. It will hit the ground with what K.E. ?

Solⁿ

Let v is speed at time t.



Equation of motion -

$$F_x = ma_x$$

$$mg - kv^2 = ma$$

$$a = g - \frac{k}{m} v^2$$

∵ Acceleration is variable

$$\therefore \quad v\frac{dv}{dx} = g - \frac{k}{m}v^2$$

$$\Rightarrow \int_0^{v_f} \frac{v\,dv}{\cancel{dx}\left(g - \frac{k}{m}v^2\right)} = \int_0^h dx$$

Let $g - \frac{k}{m}v^2 = y$

$$-\frac{k}{m} 2v\,dv = dy$$

$$v\,dv = -\frac{m}{2k}dy$$

$$\Rightarrow \int_0^{v_f} \frac{-\frac{m}{2k}dy}{y} = h$$

$$-\frac{m}{2k}\left[\log y\right] = h$$

$$-\frac{m}{2k}\left[\log\left(g - \frac{kv^2}{m}\right)\right]_0^{v_f} = h$$

$$h = -\frac{m}{2k}\log\left\{\frac{g - \frac{kv_f^2}{m}}{g}\right\}$$

$$\log\left(1-\frac{kv_f^2}{mg}\right) = -\frac{2k}{m}h$$

$$1-\frac{kv_f^2}{mg} = e^{-\frac{2k}{m}h}$$

$$v_f^2 = \frac{mg}{k}\left(1-e^{-2\frac{kh}{m}}\right)$$

$$\boxed{K.E. = \frac{1}{2}mv_f^2 = \frac{m^2g}{2k}\left(1-e^{-\frac{2kh}{m}}\right)}$$

Ans

Trick to test options :-

If here k tends to zero, i.e. $k \to 0$
then this should give $K.E. = mgh$.
Take limit $k \to 0$

$$K.E. = \frac{m^2g}{2k}\left(1-e^{-\frac{2kh}{m}}\right)$$

Let $x = \frac{2kh}{m}$

$$e^{-x} = 1 - x + \frac{x^2}{\lfloor 2} - \cdots$$

$$\approx 1 - x \quad \{\because x \to 0\}$$

$$\therefore \quad K.E. = \frac{m^2g}{2k}\left(\cancel{1}-\cancel{1}+\frac{2kh}{m}\right)$$

$$\boxed{K.E. = mgh}$$

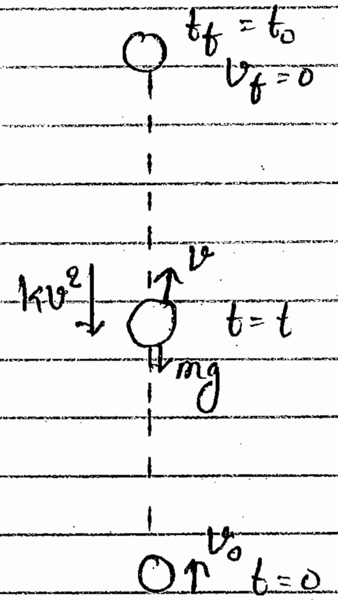
Ans

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

A-1

Qus. 9 A particle of mass m is thrown upward with velocity v & there is retarding air resistance proportional to the square of the velocity with proportionality constant k. If the particle attains the maximum hight after time t, and g is the gravitetional acceleration what is the velocity v?

Sol^n: Equation of motion :-



$$-mg - kv^2 = ma$$

$$a = -\left(g + \frac{k}{m}v^2\right)$$

$$\frac{dv}{dt} = -\left(g + \frac{k}{m}v^2\right)$$

$$\int_{v_0}^{0} \frac{dv}{(\sqrt{g})^2 + \left(\sqrt{\frac{k}{m}}\,v\right)^2} = -\int_0^{t_0} dt$$

$$\frac{\sqrt{m}}{\sqrt{k}}\,\frac{1}{\sqrt{g}}\left[\tan^{-1}\frac{\sqrt{\frac{k}{m}}\,v}{\sqrt{g}}\right]_{v_0}^{0} = -t_0$$

$$\sqrt{\frac{m}{kg}} \times \left[\tan^{-1}0 - \tan^{-1}\sqrt{\frac{k}{mg}}\,v_0\right] = -t_0$$

$$\sqrt{\frac{m}{kg}}\,\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) = t_0$$

$$\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) = \sqrt{\frac{kg}{m}}\,t_0$$

$$\sqrt{\frac{k}{mg}}\,v_0 = \tan\left(\sqrt{\frac{kg}{m}}\,t_0\right)$$

$$v_0 = \sqrt{\frac{mg}{k}} \tan\left(\sqrt{\frac{kg}{m}}\, t_0\right)$$

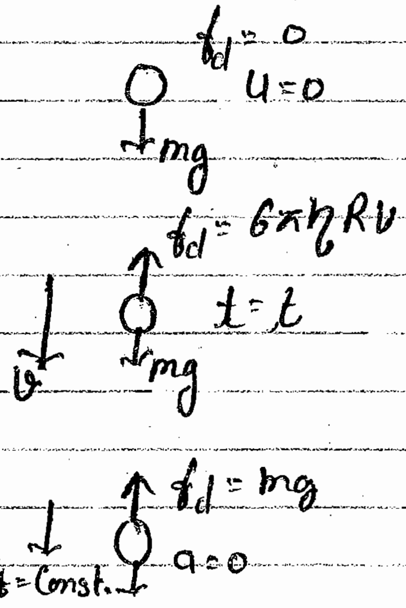
Put $m = 1$

$$\boxed{v_0 = \sqrt{\frac{g}{k}} \tan\left(\sqrt{kg}\, t_0\right)}$$

Net 2013 Dec.

Ques6 A-1 Spherical particle of a given material of density ρ are released from rest inside a liquid medium of lower density. The viscous drag force may be approximated by the stoke's law i.e. $F_d = 6\pi\eta R v$, where η is the viscosity of the medium, R the radius of a particle & v its instantaneous velocity. If $\tau(m)$ is the time taken by a particle of mass m to reach half its terminal velocity, then the ratio $\tau(8m)/\tau(m)$ is - ?

Sol^n When terminal velocity is reached -



$$F_d = mg$$

density $=$ $\frac{\text{mass}}{\text{Volume}}$

$$\rho = \frac{m}{\frac{4}{3}\pi R^3}$$

$$\cancel{R^3 = 6\pi\eta R v = 6\pi\eta \left[\frac{3m}{4\pi\rho}\right]^{1/3} v}$$

$$R^3 = \frac{3m}{4\pi\rho} \Rightarrow R = \left[\frac{3m}{4\pi\rho}\right]^{1/3}$$

$$F_d = 6\pi\eta R v = 6\pi\eta\left[\frac{3m}{4\pi\rho}\right]^{1/3} v$$

$$\boxed{F_d = k m^{1/3} v} \quad \text{where } k = 6\pi\eta\left[\frac{3}{4\pi\rho}\right]^{1/3}$$

Now

$$F_d = mg \Rightarrow k m^{1/3} v_t = mg \Rightarrow \boxed{v_t = \frac{m^{2/3}}{k} g} \quad (1)$$

here $v_t \Rightarrow$ terminal velocity

Now,

Equation of motion (to find acc^n)

$$mg - F_d = ma$$

$$mg - k m^{1/3} v = ma$$

$$a = g - k m^{-2/3} v$$

$$\frac{dv}{dt} = g - k m^{-2/3} v$$

$$\int_0^{v_t/2} \frac{dv}{g - k m^{-2/3} v} = \int_0^{\tau} dt$$

$$\left[\frac{-1}{k m^{-2/3}} \log\left(g - k m^{-2/3} v\right)\right]_0^{\frac{m^{2/3} g}{2k}} = \tau$$

$$\Rightarrow \frac{-m^{2/3}}{k}\left[\log\left(g - \frac{g}{2}\right) - \log g\right] = \tau$$

$$\Rightarrow \frac{-m^{2/3}}{k} \log \frac{g/2}{g} = \tau$$

$$\Rightarrow \frac{-m^{2/3}}{k} \log\left(\frac{1}{2}\right) = \tau$$

$$\Rightarrow \boxed{\frac{m^{2/3}}{k} \log 2 = \tau}$$

Now

$$\tau(m) = \frac{m^{2/3}}{k} \log 2$$

$$\tau(8m) = \frac{(8m)^{2/3}}{k} \log 2$$

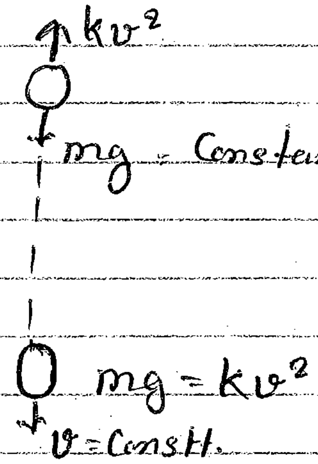
$$\boxed{\frac{\tau(8m)}{\tau(m)} = 8^{2/3} = 4}$$

Ans

A-1
Ques7 A particle is observed after it has been moving for a long time under the influence of a constant force in a medium that applies a drag force proportional to the square of its velocity. Distance versus time graph made by the observer will look like?

Solⁿ for constant velocity, distance v/s time graph will be straight line.

{∵ Particle is moving for a long time. So after reaching to the terminal velocity, the velocity becomes constant.}



∴ Terminal velocity, $v = \frac{dx}{dt}$

$$dx = v\,dt$$

∵ v is constant.

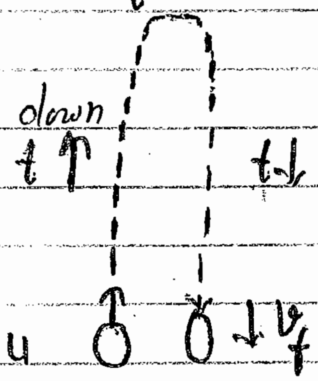
$$\therefore \int dx = v \int dt$$

$$\boxed{x = vt + c}$$ ← It is eqⁿ of straight line.

So it is straight line.

* Vertical motion in absence of air resistance:

time for going up = time for going down

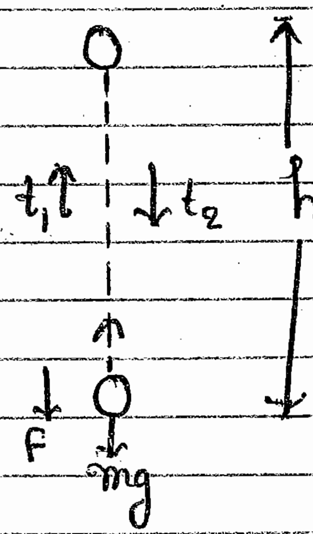


$$\boxed{\begin{aligned} t\uparrow &= t\downarrow \\ u\uparrow &= v_f\downarrow \end{aligned}}$$

A-2

Ques 3 An object of mass m is thrown vertically up. It is acted upon by a constant resistive force F. If t_1 & t_2 be time of ascent & time of descent then value of t_1/t_2 is ?

Sol^n:- Let h be the maximum hight reached.



Equation of motion -

$$-mg - F = ma$$

$$\boxed{a = -\left(g + \frac{F}{m}\right)} = \text{Constant}$$

$$v = u + at_1$$

$$0 = u - \left(g + \frac{F}{m}\right)t_1$$

$$\boxed{t_1 = \frac{u}{\left(g + \frac{F}{m}\right)}} \longrightarrow (1)$$

Express u in terms of h :-

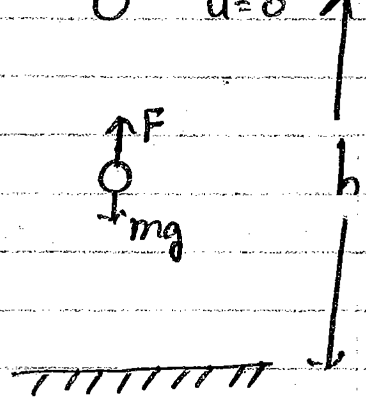
$$v^2 = u^2 + 2as$$

$$\Rightarrow \quad 0 = u^2 - 2\left(g + \frac{F}{m}\right) \cdot h$$

$$u = \sqrt{2\left(g + \frac{F}{m}\right)h}$$

$$\therefore \quad \boxed{t_1 = \frac{\sqrt{2h}}{\sqrt{g + \frac{F}{m}}}} \quad \text{——— (ii)}$$

For downward motion :-



Equation of motion :-

$$mg - F = ma$$

$$\boxed{a = g - \frac{F}{m}} = \text{Constant}$$

$$s = ut + \frac{1}{2}at^2$$

$$h = 0.t + \frac{1}{2}\left(g - \frac{F}{m}\right)t_2^2$$

$$h = 0 + \frac{1}{2}\left(g - \frac{F}{m}\right)t_2^2$$

$$t_2 = \sqrt{\frac{2h}{(g - F/m)}}$$

$$\boxed{\frac{t_1}{t_2} = \sqrt{\frac{g - F/m}{g + F/m}}} \quad (t_1 < t_2)$$

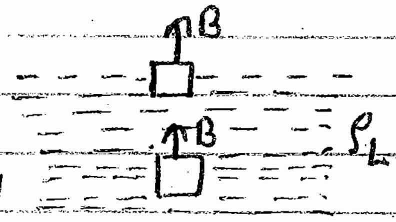
Ans

mass = Volume x density

* Buoyancy Force (B) :-

B = weight of displaced

B = mass of liquid displaced x g.



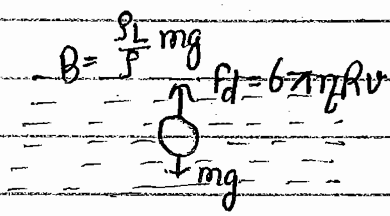
B = Volume of liquid displaced . ρ_L . g.

$$\boxed{B = \text{Volume of object inside liquid} \cdot \rho_L \cdot g}$$

ρ_L = density of liquid.

A-1

Q. 6 {Icluding Bouyancy force}



$f_d = 6\pi\eta R v = k m^{1/3} v$

$B = \text{Vol.} \, \rho_L . g$

When v_t is reached

$B + f_d = mg$

$\text{Vol.} \, \rho_L \cdot g + k m^{1/3} v = mg$

$\text{Vol.} \, \rho . \frac{\rho_L}{\rho} g + k m^{1/3} v_t = mg$

$m \frac{\rho_L}{\rho} g + k m^{1/3} v_t = mg$

$k m^{1/3} v_t = mg \left(1 - \frac{\rho_L}{\rho}\right)$

Equation of motion -

$$mg - \frac{\rho_L}{\rho} mg - k v m^{1/3} = ma$$

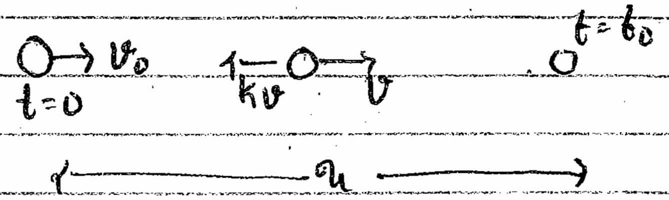
$$\boxed{g\left(1 - \frac{\rho_L}{\rho}\right) - K m^{-2/3} v = a}$$

A-2

1.4 A particle of mass m is thrown with initial speed v_0. A resistance force $= kv$ acts on the particle. Distance moved by particle in time t_0?

Sol^n

Equation of motion -



$$-kv = ma$$

$$-kv = m\frac{dv}{dt}$$

$$-\frac{k}{m}\int_0^t dt = \int_{v_0}^{v}\frac{dv}{v} \Rightarrow \frac{-k}{m}t = \log\frac{v}{v_0}$$

$$\Rightarrow v = v_0 e^{-k/m\, t}$$

$$\Rightarrow \frac{dx}{dt} = v_0 e^{-\frac{k}{m}t}$$

$$\Rightarrow \int_0^x dx = v_0\int_0^{t_0} e^{-\frac{k}{m}t}\,dt$$

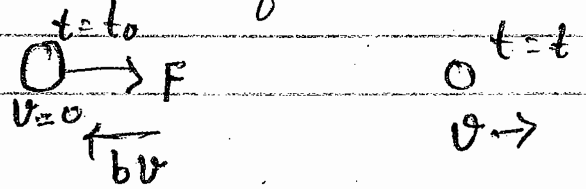
$$x = \frac{v_0}{-k/m}\left[e^{-\frac{k}{m}t}\right]_0^{t_0}$$

$$\boxed{x = \frac{m v_0}{k}\left(1 - e^{-\frac{k}{m}t_0}\right)} \qquad \{e^{-0} = 1\}$$

A-2

Ques 2 A constant force F is applied on a particle of mass 'm' which is initially at rest. As the particle starts moving a resistive force $-bv$ begins to act on it. Speed of the particle at any instant of time t is?

Sol^n



Equation of motion -

$$-bv + F = ma$$

$$\boxed{a = \frac{-bv+F}{m}} \qquad \left\{ \begin{array}{l} \text{accelaration} \\ \text{is variable} \end{array} \right\}$$

$$\frac{dv}{dt} = \frac{-bv+F}{m}$$

$$\Rightarrow m\int_0^v \frac{dv}{(-bv+F)} = \int_0^t dt$$

$$\Rightarrow m \times \frac{1}{-b}\left[\log(-bv+F)\right]_0^v = t$$

$$\Rightarrow \log(-bv+F) - \log F = -\frac{bt}{m}$$

$$\Rightarrow \log\frac{(-bv+F)}{F} = \frac{-bt}{m}$$

$$\Rightarrow \frac{-bv+F}{F} = e^{-bt/m}$$

$$\Rightarrow \frac{\cancel{+}bv}{F} = \cancel{+}(1-e^{-bt/m})$$

$$\Rightarrow \boxed{v = \frac{F}{b}(1-e^{-bt/m})}$$

Ans

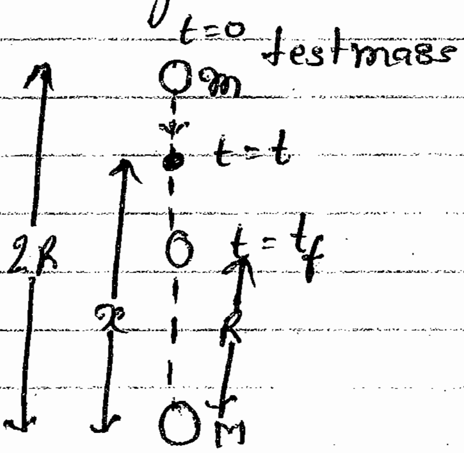
A-1

Q.10 Find the free fall time of a test mass on an object of mass M from a hight 2R to R.

Sol^n Let at time t distance of test mass from the given object is x

Then eq^n of motion -

$$\frac{G\cancel{m}M}{x^2} = \cancel{m}a$$



$2\sin^2\theta = 1 - \cos 2\theta$

$$\frac{GM}{x^2} = -v\frac{dv}{dx}$$ $\left\{ \because x \text{ is decreasing so } -ve \text{ sign is taken} \right.$

$$\Rightarrow \int_{2R}^{R} \frac{GM}{x^2}\,dx = -\int_{0}^{v} v\,dv$$

$$\Rightarrow GM\left[\frac{1}{x} - \frac{1}{2R}\right] = \frac{v^2}{2}$$

$$\Rightarrow GM\left[\frac{2}{x} - \frac{1}{R}\right] = v^2$$

$$\sqrt{GM}\sqrt{\frac{2R-x}{Rx}} = v$$

Note :-

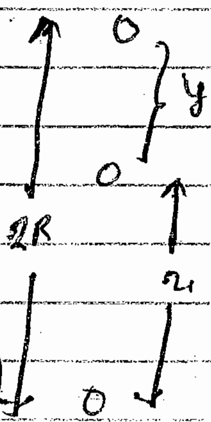
$v = \frac{dy}{dt}$

$v = \frac{d}{dt}(2R - x)$

$v = -\frac{dx}{dt}$

$a = \frac{dv}{dt} = \frac{dv}{dx}\left(\frac{dx}{dt}\right)$

$\boxed{a = -v\frac{dv}{dx}}$



$$\sqrt{\frac{GM}{R}}\sqrt{\frac{2R-x}{x}} = -\frac{dx}{dt}$$

$$\int_{0}^{t_f}\sqrt{\frac{GM}{R}}\,dt = -\int_{2R}^{R}\sqrt{\frac{x}{2R-x}}\,dx$$

$$\Rightarrow \sqrt{\frac{GM}{R}}\,t_f = -\int_{2R}^{R}\sqrt{\frac{2R\sin^2\theta}{2R\cos^2\theta}}\,.\,4R\sin\theta\cos\theta\,d\theta$$

$$= -2R\int_{2R}^{R} 2\sin^2\theta\,d\theta$$

$$= -2R\int_{2R}^{R}[1 - \cos 2\theta]\,d\theta$$

$$= -2R\left[\theta - \frac{\sin 2\theta}{2}\right]_{2R}^{R}$$

$$= -2R\left[\sin^{-1}\sqrt{\frac{x}{2R}} - \sqrt{\frac{x}{2R}}\sqrt{1-\frac{x}{2R}}\right]_{2R}^{R}$$

Let $x = 2R\sin^2\theta$

$dx = 4R\sin\theta\cos\theta\,d\theta$

$\sin^2\theta = \frac{x}{2R}$

$\sin\theta = \sqrt{\frac{x}{2R}}$

$\theta = \sin^{-1}\sqrt{\frac{x}{2R}}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$\cos\theta = \sqrt{1-\frac{x}{2R}}$

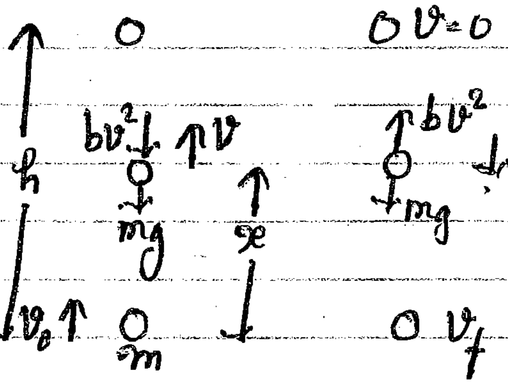
$$= \left[\frac{\pi}{4} - \frac{\pi}{2} - \frac{1}{2} + 0\right]$$

$$\sqrt{\frac{GM}{R}}\, t_f = R\left[\frac{\pi}{2} + 1\right]$$

$$\boxed{t_f = \left[\frac{\pi}{2} + 1\right]\sqrt{\frac{R^3}{GM}}}$$

A-2
Q.11 A particle of unit mass is thrown vertically upward with initial speed v_0. The particle is acted upon by a drag force bv^2 in addition to gravity. Here 'b' is a constant and v is instantaneous speed. Speed of the particle when it returns to the point from where it had been thrown is -

Sol^n For Upward motion:-



$$-mg - bv^2 = ma$$

$$-\left(g + \frac{bv^2}{m}\right) = v\frac{dv}{dx}$$

$$-\int_0^h dx = \int_{v_0}^{0} \frac{v\,dv}{\left(g + \frac{bv^2}{m}\right)}$$

$$h = -\frac{m}{2b}\left[\log\left[g + \frac{bv^2}{m}\right]\right]_{v_0}^{0}$$

$$\boxed{h = \frac{-m}{2b}\left[\log\left(\frac{g}{g + \frac{bv^2}{m}}\right)\right]} \quad \text{—} (1)$$

For downward motion :-

$$mg - bv^2 = ma \Rightarrow v\frac{dv}{dx} = g - \frac{bv^2}{m}$$

$$\Rightarrow \int_0^{v_f} \frac{v\,dv}{\left(g - \frac{bv^2}{m}\right)} = \int_0^{h} dx \Rightarrow \frac{-m}{2b}\left[\log \frac{\left(g - \frac{bv_f^2}{m}\right)}{g}\right] = h$$

$$\boxed{h = \frac{-m}{2b}\left[\log \frac{\left(g - \frac{bv_f^2}{m}\right)}{g}\right]} \quad \text{— (ii)}$$

from eq^n (i) and (ii), we get:-

$$\cancel{\frac{-m}{2b}}\left[\log \frac{g}{\left(g + \frac{bv_0^2}{m}\right)}\right] = \cancel{\frac{-m}{2b}}\left[\log \frac{\left(g - \frac{bv_f^2}{m}\right)}{g}\right]$$

$$\Rightarrow g^2 = g^2 - \frac{b^2 v_0^2 v_f^2}{m^2} - \frac{gbv_f^2}{m} + g\frac{bv_0^2}{m}$$

$$\Rightarrow v_f^2 \frac{b}{m}\left(\frac{v_0^2}{m} + g\right) = g\frac{bv_0^2}{m}$$

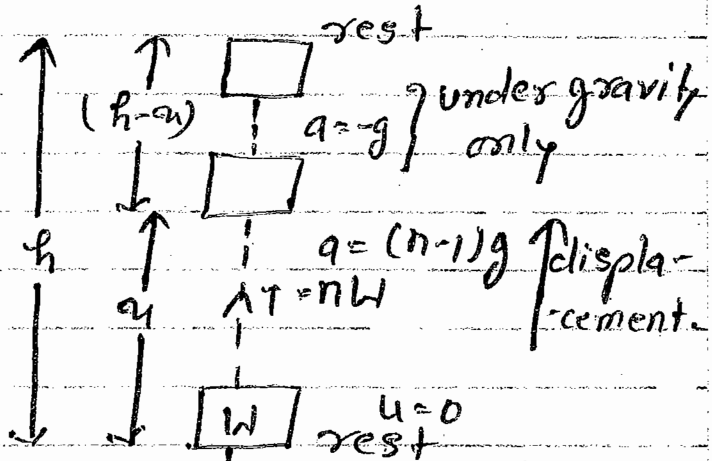
$$\Rightarrow v_f^2 = \frac{\cancel{g}\, v_0^2}{\cancel{g}\left(1 + \frac{v_0^2}{mg}\right)}$$

$$\Rightarrow \boxed{v_f = \frac{v_0}{\sqrt{\left(1 + \frac{v_0^2}{mg}\right)}}}$$

Ans

Que. A load of weight W is to be raised by a rope from rest to rest through a height h. The greatest tension which the rope can safely bear is nW. The least time in which ascent can be made is -?

Sol^n Equation of motion -



$$nW - W = ma$$

$$W(n-1) = \frac{W}{g}a$$

$$\boxed{a = (n-1)g = \text{Constt.}}$$

Let block is accelerated (pulled for a distance x)

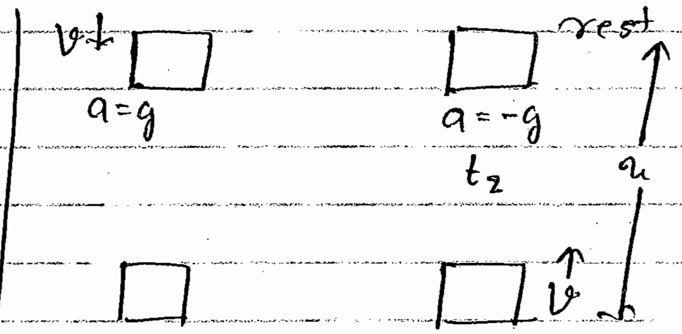
$mg = W$ & $m = \frac{W}{g}$

Calculation of t_1 (time for which it is pulled)

$$s = ut + \frac{1}{2}at^2$$

$$x = 0 + \frac{1}{2}(n-1)g t_1^2$$

$$\boxed{t_1 = \sqrt{\frac{2x}{(n-1)g}}}$$



Calculation of time t_2 :-

$$s = ut + \frac{1}{2}at^2$$

$$(h-x) = 0 + \frac{1}{2}g t_2^2$$

$$\boxed{t_2 = \sqrt{\frac{2(h-x)}{g}}}$$

$\boxed{\text{Time for up} = \text{Time for down}}$

if only g acting

Total time :-

$$t = t_1 + t_2 = \sqrt{\frac{2}{g}}\left[\sqrt{\frac{x}{(n-1)}} + \sqrt{h-x}\right] \quad \text{--- (1)}$$

for least time -

$$\frac{dt}{dx} = 0$$

So from (1)

$$\frac{dt}{dx} = \sqrt{\frac{2}{g}}\left[\frac{1}{\sqrt{n-1}} \times \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{(h-x)}}\right] = 0$$

Squaring.

$$\frac{1}{(n-1)x} = \frac{1}{h-x}$$

$$h - x = nx - x$$

$$\Rightarrow \boxed{x = \frac{h}{n}}$$

$\therefore$ for least time we put $x = \frac{h}{n}$

$$t = \sqrt{\frac{2}{g}}\left(\sqrt{\frac{h/n}{n-1}} + \sqrt{h - \frac{h}{n}}\right]$$

$$= \sqrt{\frac{2}{g}}\left[\sqrt{\frac{h}{n(n-1)}} + \sqrt{\frac{h(n-1)}{n}}\right]$$

$$= \sqrt{\frac{2h}{gn}}\left[\frac{1}{\sqrt{n-1}} + \sqrt{n-1}\right]$$

$$t = \sqrt{\frac{2h}{gn}} \times \frac{n}{\sqrt{n-1}}$$

$$\Rightarrow \quad t = \sqrt{\frac{2hn^2}{gn(n-1)}}$$

$$\Rightarrow \quad \boxed{t = \sqrt{\frac{2nh}{g(n-1)}}}$$

A-1

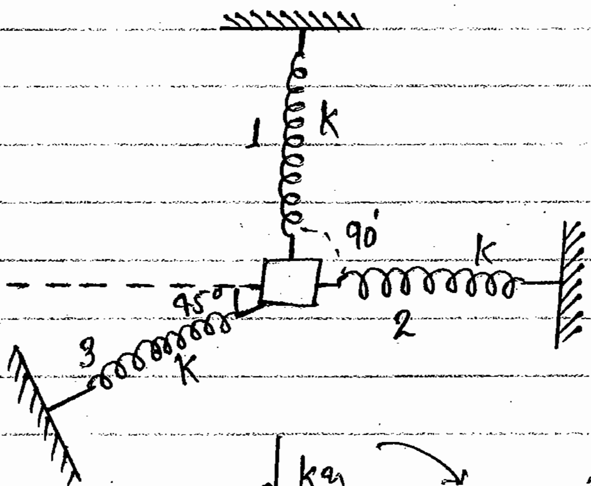
Q.3 In the figure shown the block is given a small displacement 'x' along the spring 3. Equation of the motion of the block is (Assume that system is lying on table).

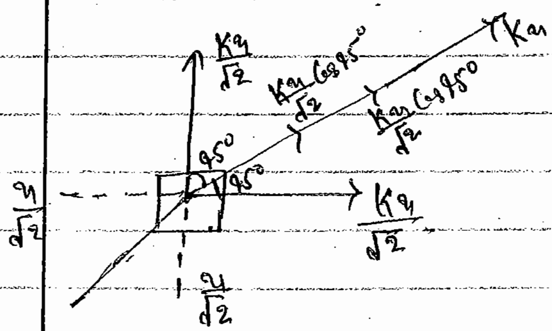
Sol^n Approximate method:-

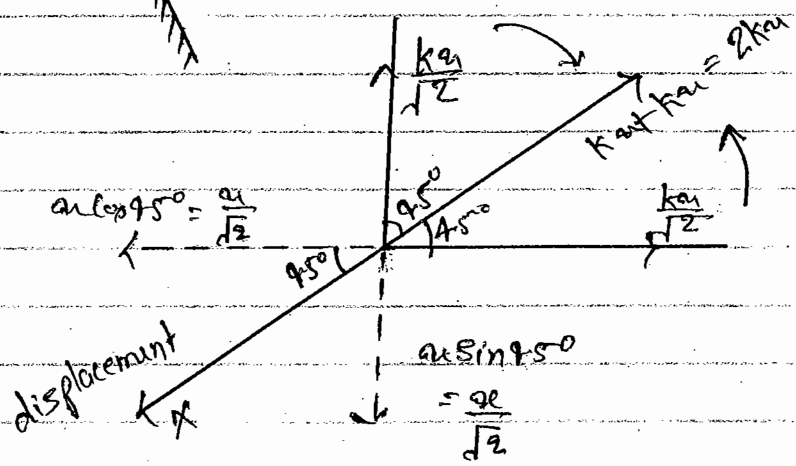
$$f_x = m a_x$$

$$-2kx = m\ddot{x}$$

$$\boxed{\ddot{x} + \frac{2k}{m}x = 0}$$



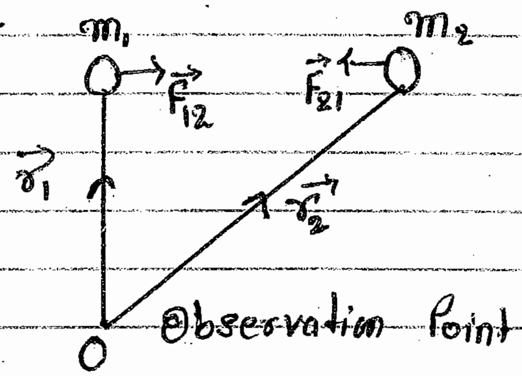




Reduced Mass

When two objects are moving under their mutual interaction force. Then problem can be simplified by assuming that, one object is at rest. when we do so mass of other object had to be replaced by reduced mass.

Equation of motion of m_1 :-



$$\vec{F}_{12} = m_1 \frac{d^2\vec{r}_1}{dt^2}$$

$$\frac{\vec{F}}{m_1} \Leftarrow \frac{\vec{F}_{12}}{m_1} = \frac{d^2\vec{r}_1}{dt^2} \quad \text{——— (1)}$$

Equation of motion of m_2 :-

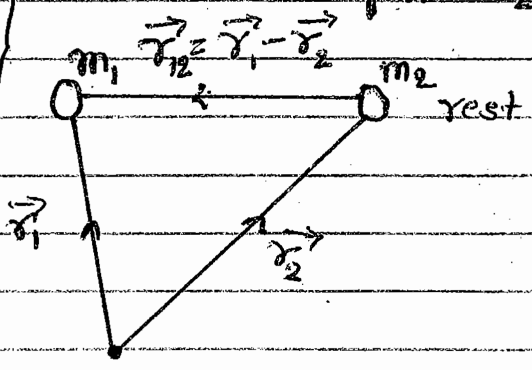
$$\frac{\vec{F}_{21}}{m_2} = \frac{d^2\vec{r}_2}{dt^2}$$

$$\Rightarrow \frac{-\vec{F}}{m_2} = \frac{d^2\vec{r}_2}{dt^2} \quad \text{——— (2)}$$

$\because$ from Newtons Law -

$$\left\{ \vec{F}_{12} = -\vec{F}_{21} = \vec{F} \ (\text{let}) \right\}$$

observation from m_2



Subtracting (2) in (1)

$$\frac{\vec{F}}{m_1} + \frac{\vec{F}}{m_2} = \frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{F}\ \frac{m_1 + m_2}{m_1 \cdot m_2} = \frac{d^2\vec{r}_{12}}{dt^2}$$

$$\Rightarrow \vec{F} = \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{d^2 \vec{r}_{12}}{dt^2}$$

$$\Rightarrow \vec{F} = \mu \vec{a}_{12}$$

$$\Rightarrow \boxed{a_{12} = \frac{\vec{F}}{\mu}}$$

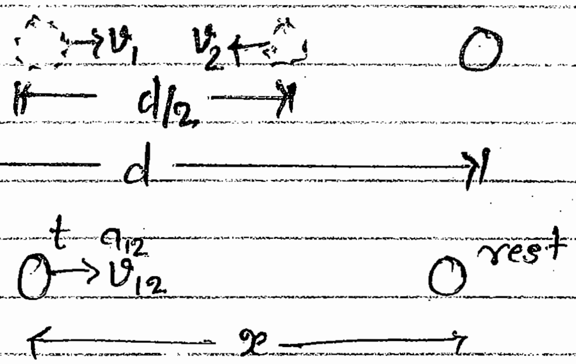
here $\mu = \frac{m_1 m_2}{m_1 + m_2}$

In this case velocity will be relative.

A-2
Q.7 Two particle of masses m_1 & m_2 are 'd' distance apart. Due to gravitational attraction they move towads each other. What is speed of m_1 when their separation reduces to $d/2$.

Sol^n:-

Let m_2 is at rest.



$$a_{12} = \frac{F}{\mu} = \frac{Gm_1 m_2 / x^2}{\frac{m_1 m_2}{m_1 + m_2}}$$

$$a_{12} = \frac{G(m_1 + m_2)}{x^2}$$

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

$$\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$$

$$-v_{12} \frac{dv_{12}}{dx} = \frac{G(m_1 + m_2)}{x^2}$$

{∵ here −ve sign is taken because x is decreasing}

$$-\int_0^{v_{12}} v_{12}\, dv_{12} = \int_d^{d/2} G(m_1 + m_2) \frac{dx}{x^2}$$

$$-\frac{v_{12}^2}{2} = -G(m_1 + m_2) \frac{1}{d}$$

$$v_{12} = \sqrt{\frac{2G(m_1 + m_2)}{d}}$$

$$v_1 + v_2 = \sqrt{\frac{2G(m_1 + m_2)}{d}} \quad \text{——— (1)}$$

Here no external force acts on system (m_1+m_2)

$\therefore \vec{P} = \text{Constant}$

$\Rightarrow P_i = P_f \Rightarrow 0 = m_1v_1 - m_2v_2$

$\Rightarrow m_1v_1 = m_2v_2 \Rightarrow v_2 = \frac{m_1v_1}{m_2}$ — (11)

from (1) and (2)

$$v_1\left(1+\frac{m_1}{m_2}\right) = \sqrt{\frac{2G(m_1+m_2)}{d}}$$

$$v_1 = \frac{m_2}{(m_1+m_2)}\sqrt{\frac{2G(m_1+m_2)}{d}}$$

$$\Rightarrow \boxed{v_1 = m_2\sqrt{\frac{2G}{(m_1+m_2)d}}}$$

Conservative Forces

If work done by a force is independent of path followed, the force is called Conservative

⇒ Work done along closed path = zero.

$$\boxed{\oint \vec{F}.d\vec{l} = 0} \qquad \{F \text{ is conservative}\}$$

Using Stoke's theorem to convert into surface integral.

$$\int (\vec{\nabla} \cdot \vec{F}) . d\vec{s} = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{F} = 0} \quad \text{— (i)}$$

From mathematics

$$\pm \vec{\nabla} \times \vec{\nabla} \,(\text{scalar}) = 0 \quad \text{— (ii)}$$

Calculating (i) & (ii)

$$F = \pm \vec{\nabla} \text{ scalar}$$

Or

F from U → $\boxed{F = -\nabla U}$ — ve sign gives current direction of force.

$$U = P.E.$$

It is defined only for conservative forces.
If we fined U from F

Actual definition → $\boxed{U(r) = \int_{r}^{\text{ref.pt.}} \vec{F}.d\vec{l}}$ Ref.pt. the pt. where P.E. is zero & also force is zero.

Gravitational Force [$F = mgh$ is used when $h <<< R_e$]

Short Method :-

$$U(x) = -\int F\,dx$$

$$U(r) = -\int F\,dr$$

If force is repulsive it is taken +ve
If force is attractive it is taken -ve.

Example of conservative forces :-

1. Gravitational force b/w two masses :-

$$\boxed{F = \frac{G\,m_1 m_2}{r^2}} \quad \text{(attractive)}$$

{ -ve sign is taken because force is attractive

$$U = -\int F.dr = -\int \frac{-G m_1 m_2}{r^2}\,dr = -\frac{G m_1 m_2}{r}$$

2. **Potential Energy of Earth Mass System :-**

$$U = -\frac{G M_e m}{R_e + h}$$

P.E. = 0 at surface with $h = 0$

$$\Rightarrow \frac{-G M_e m}{R_e\left(1+\frac{h}{R_e}\right)} = -\frac{G M_e m}{R_e}\left(1+\frac{h}{R_e}\right)^{-1}$$

$$P.E. = U_{ref} = -\frac{G M_e m}{R_e}$$

$$\Rightarrow U = -\frac{G M_e m}{R_e}\left(1-\frac{h}{R_e}\right)$$

$$\Rightarrow U = -\frac{G M_e m}{R_e} + \frac{G M_e m}{R_e^2}\,h$$

$\left\{ \because U_{ref} = -\frac{G M_e m}{R_e}, \quad g = \frac{G M_e}{R_e^2} \right.$

$$\Rightarrow U = U_{ref} + mgh$$

(true for small h) $h <<< R_e$

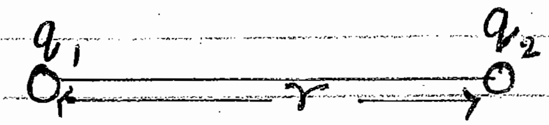
$$\Rightarrow U - U_{ref} = mgh$$

{ If $h = 2R_e$ or $\frac{R_e}{2}$ etc then this relation will not be used }

To convert electrostatic formulae into Gravitational Formulas
Replace $q \longrightarrow m$, & $\frac{1}{4\pi\epsilon_0} \longrightarrow G$

if $U_{ref} = 0$
Then, $\boxed{U = mgh}$

3. Electrostatic force :- {It is a conservative force.}

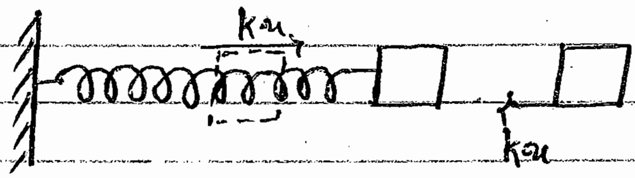


$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$

$q_1 > 0$, $q_2 > 0$

$U = -\int F . dr \Rightarrow \boxed{U = \frac{q_1 q_2}{4\pi\epsilon_0 r}}$

4. Spring force {Conservative} :-



$U(x) = -\int F dx$

$= -\int (-kx) dx$

$\boxed{U(x) = \frac{1}{2} kx^2}$

* Conservation of Mechanical Energy :-

If only conservative forces are acting on a system then total mechanical energy of the system remains conserved.

$\boxed{\text{Total Energy} = \text{Constant}}$

$$K.E. + P.E. = \text{Constant}$$

$$(K.E. + P.E.)_i = (K.E. + P.E.)_f.$$

Mechanical energy is not conserved if non conservative forces act on the system.

Non-Conservative forces → friction and any other dissipative forces.

{If there is other than above three forces } then we use Work Energy Theorem }.

* Work Energy Theorem :- Total work done by all forces is equal to change in K.E.

$$\boxed{\text{Total Work Done} = K.E_f - K.E_i}$$

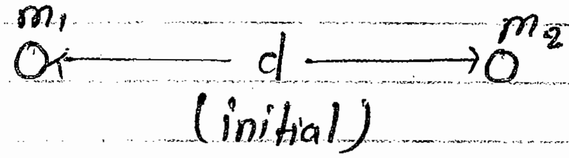
IInd Method.

A-2

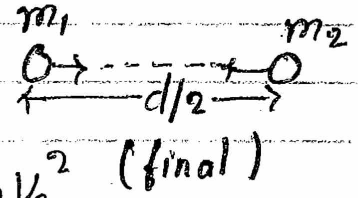
Ques7 Two particles of masses m_1 & m_2 are placed 'd' distance apart. Due to gravitational attraction they move towards each other. What is speed of m_1 when their separation reduces to $d/2$.

Sol?

Use conservation of mechanical energy:-



$$(K.E.+P.E.)_i = (K.E.+P.E.)_f$$



$$0 - \frac{Gm_1m_2}{d} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2V_2^2 - \frac{Gm_1m_2}{d/2}$$

$$\frac{Gm_1m_2}{d} = \frac{1}{2}\left[m_1v_1^2 + m_2V_2^2\right] \quad \text{—— (I)}$$

from Conservation of momentum:-

$$P_{initial} = P_{final}$$

$$0 = m_1V_1 - m_2V_2 \Rightarrow v_2 = \frac{m_1V_1}{m_2} \quad \text{—— (II)}$$

Put v_2 in (I) -

$$\frac{Gm_1m_2}{d} = \frac{1}{2}\left[m_1v_1^2 + \frac{m_1^2v_1^2}{m_2}\right]$$

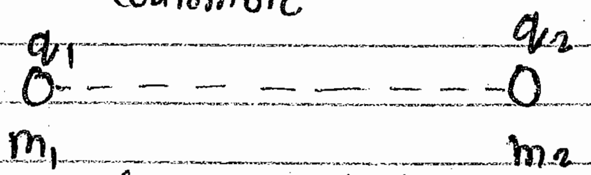
$$= \frac{1}{2}m_1v_1^2\left[1 + \frac{m_1}{m_2}\right]$$

$$m_1\left(\frac{m_1+m_2}{m_2}\right)v_1^2 = \frac{2Gm_1m_2}{d}$$

$$v_1 = \sqrt{\frac{2G\cancel{m_1}m_2^2}{\cancel{m_1}(m_1+m_2)d}} \Rightarrow \boxed{v = \sqrt{\frac{2Gm_2^2}{(m_1+m_2)d}}}$$

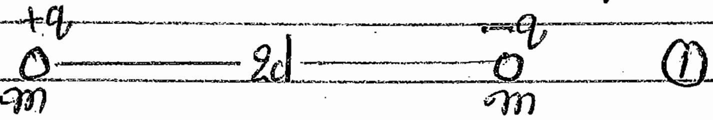
Note:-
When two or many Particle having mass as well as Charge :-

∴ $F_{gravitational} << F_{coulombic}$

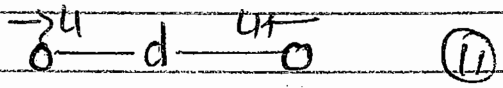


∴ Neglect effect of gravitation in comparision to electrostatic force.

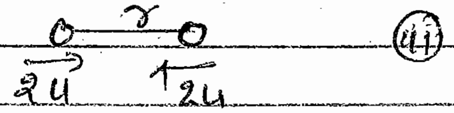
Ques5 Two particles of equal masses & having opposite charges are placed 2d distance apart. Due to electrostatic force they move towards each other. When separation between them reduces to 'd' their speeds becomes 'u'. At what separation speed of each particle is '2u'.



$(K.E.+P.E.)_{(i)} = (K.E.+P.E.)_{(ii)}$



$$0 + \frac{q^2}{4\pi\epsilon_0 (2d)} = \cancel{2}\times\frac{1}{\cancel{2}} mu^2 - \frac{q^2}{4\pi\epsilon_0 d}$$



$$\frac{q^2}{8\pi\epsilon_0 d} = mU^2 \quad \text{--- (1)}$$

Now
$$(K.E.+P.E.)_{(i)} = (K.E.+P.E.)_{(iii)}$$

$$0 + \frac{q^2}{8\pi\epsilon_0 d} = \cancel{2}\times\frac{1}{\cancel{2}} m(2u)^2 - \frac{q^2}{4\pi\epsilon_0 r}$$

$$\frac{-q^2}{8\pi\epsilon_0 d} = 4mu^2 - \frac{q^2}{4\pi\epsilon_0 r}$$

Put value of mu^2 from (1)

$$\Rightarrow \frac{-q^2}{8\pi\epsilon_0 d} = \frac{4 q^2}{8\pi\epsilon_0 d} - \frac{q^2}{4\pi\epsilon_0 r}$$

$$\Rightarrow \frac{-1}{\cancel{8\pi\epsilon_0} d} = \frac{4}{\cancel{8\pi\epsilon_0} d} - \frac{2\times 1}{\cancel{2\times 4\pi\epsilon_0} r}$$

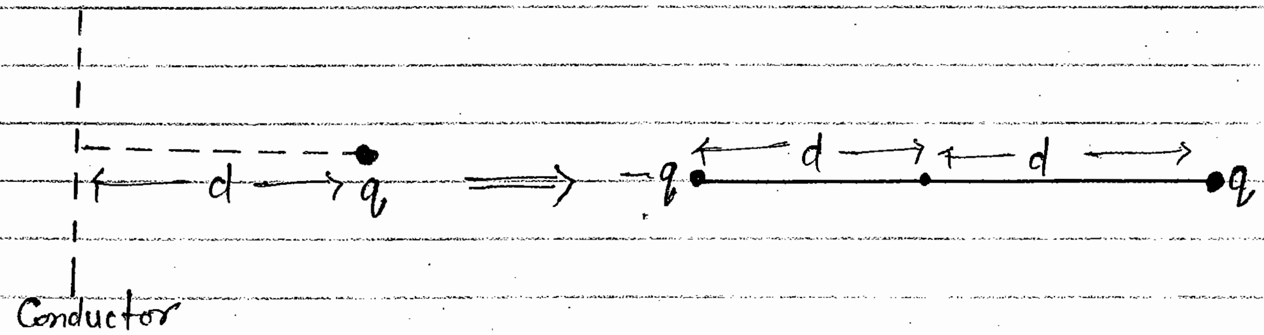
$$\Rightarrow \frac{-1}{d} = \frac{4}{d} - \frac{2}{r}$$

$$\Rightarrow \frac{\cancel{-}5}{d} = \frac{\cancel{-}2}{r}$$

$$\boxed{r = \frac{2d}{5}}$$

Note:-

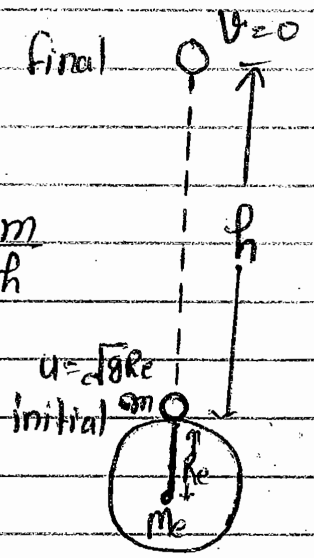
If a conductor is placed at a distance d from a charge q, then we can remove conductor by a charge $-q$, putting at a distance $2d$ from q. i.e.



→ In a question if speed is given in formula form then we consider speed large.

Q. A particle is thrown upward vertically with initial speed $\sqrt{gR_e}$ where g is acceleration due to gravity on earth's surface. What is the maximum height attain by the particle?

Sol^n



$$(K.E.+P.E.)_i = (K.E.+P.E.)_f$$

$$\frac{1}{2}m\left(\sqrt{gR_e}\right)^2 - \frac{GM_em}{R_e} = 0 - \frac{GM_em}{R_e+h}$$

$$\Rightarrow \frac{1}{2}\frac{G\cancel{M_e}}{R_e^{\cancel{2}}}\cancel{R_e} - \frac{G\cancel{M_e}}{R_e} = -\frac{G\cancel{M_e}}{R_e+h}$$

$$\Rightarrow \frac{1}{2R_e} - \frac{1}{R_e} = \frac{-1}{R_e+h}$$

$$\Rightarrow \frac{\cancel{-}1}{2R_e} = \frac{\cancel{-}1}{R_e+h}$$

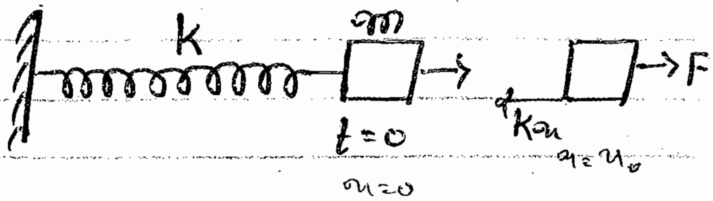
$$\Rightarrow R_e+h = 2R_e$$

$$\Rightarrow \boxed{h = R_e}$$ Ans

Basic Assignment

Ques 5 In the fig. shown the block is pulled with a constt. force F. What is speed of the block at the instant when acc^n is zero?

Sol^n Equation of motion:-



$$F - kx = ma \quad \text{—} (1)$$

Let $a = 0$ when $x = x_0$

$$F - kx_0 = 0$$

$$\Rightarrow \boxed{x_0 = \frac{F}{k}}$$

Now from (1)

$$F - kx = mv\frac{dv}{dx}$$

$$\int_0^{F/k} (F - kx)\,dx = m\int_0^{v} v\,dv$$

$$\left[Fx - \frac{kx^2}{2}\right]_0^{F/k} = \frac{mv^2}{2}$$

$$\frac{F^2}{k} - \frac{F^2}{2k} = \frac{mv^2}{2}$$

$$\Rightarrow \frac{F^2}{\cancel{2}k} = \frac{mv^2}{\cancel{2}}$$

$$\boxed{v = \frac{F}{\sqrt{mk}}}$$

Second Method:-

Since there is another force so we can use work energy theorem.

Work Energy Theorem,

$$T.W.D. = K.E_f - K.E_i$$

$$W_S + W_F = \frac{1}{2}mv^2 - 0$$

$$\int_0^{F/k} \vec{F}_{spring} \cdot d\vec{l} + \int_0^{F/k} \vec{F} \cdot d\vec{l} = \frac{1}{2}mv^2$$

$$\int_0^{F/k} kx\,dx \cos 180^\circ + \int_0^{F/k} F\,dx \cos 0^\circ = \frac{1}{2}mv^2$$

$$\left[-\frac{1}{2}kx^2\right]_0^{F/k} + F \cdot \frac{F}{K} = \frac{1}{2}mv^2$$

$$\Rightarrow -\frac{1}{2}\cancel{K}\frac{F^2}{K^{\cancel{2}}} + \frac{F^2}{K} = \frac{1}{2}mv^2$$

$$\Rightarrow \frac{1}{2}\frac{F^2}{K} = \frac{1}{2}mv^2$$

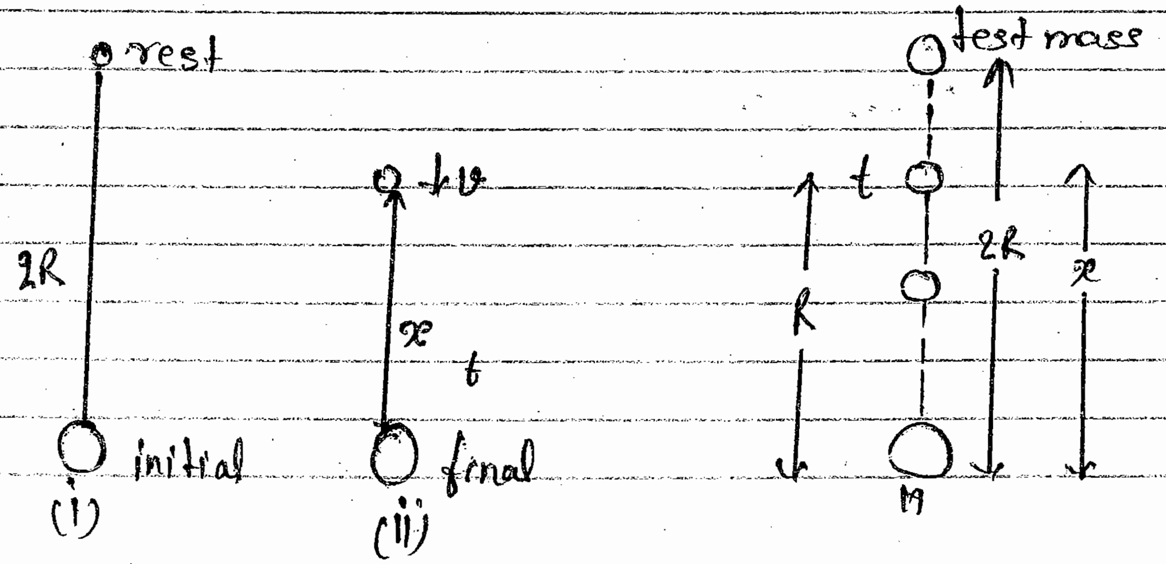
$$\Rightarrow v^2 = \frac{F^2}{mK}$$

$$\Rightarrow \boxed{v = \frac{F}{\sqrt{mK}}}$$

Ans

A-1 (Method-II)

Q.10 The free fall time of a test mass on an object of mass M from a hight 2R to R is -



Apply Conservation of energy.

$$(K.E. + P.E.)_{I} = (K.E. + P.E.)_{II}$$

$$0 + \frac{GM\cancel{m}}{2R} = \frac{1}{2}\cancel{m}v^2 - \frac{GM\cancel{m}}{x}$$

$$GM\left(-\frac{1}{2R} + \frac{1}{x}\right) = \frac{1}{2}v^2$$

$$v = \sqrt{GM}\sqrt{\frac{2}{x} - \frac{1}{R}}$$

$$\Rightarrow \frac{-dx}{dt} = \sqrt{GM}\sqrt{\frac{2R-x}{xR}}$$

{ $v = \frac{-dx}{dt}$ is used because x is decreasing.

$$\Rightarrow \frac{-dx}{dt} = \sqrt{\frac{GM}{R}}\sqrt{\frac{2R-x}{x}}$$

$$\Rightarrow -\int_{2R}^{R} \frac{\sqrt{x}\,dx}{\sqrt{2R-x}} = \sqrt{\frac{GM}{R}}\int_0^t dt \qquad \text{—— (1)}$$

Put $x = 2R\sin^2\theta \Rightarrow \sin\theta = \sqrt{\frac{x}{2R}} \Rightarrow \theta = \sin^{-1}\sqrt{\frac{x}{2R}}$

$dx = 4R\sin\theta\cos\theta\,d\theta$

Integrating -

$$\int_{2R}^{R} \frac{\sqrt{x}}{\sqrt{2R-x}}\,dx = \int_{2R}^{R} \frac{\sqrt{\cancel{2R}}\sin\theta\,4R\sin\theta\,\cancel{\cos\theta}\,d\theta}{\sqrt{\cancel{2R}}\,\cancel{\cos\theta}}$$

$$\Rightarrow 2R\int 2\sin^2\theta\,d\theta = 2R\int(1-\cos 2\theta)\,d\theta$$

$$\Rightarrow 2R\left[\theta - \frac{\sin 2\theta}{2}\right]_{2R}^{R} = 2R\left[\theta - \sin\theta\cos\theta\right]$$

$$\Rightarrow 2R\left[\sin^{-1}\sqrt{\frac{x}{2R}} - \sqrt{\frac{x}{2R}}\sqrt{1-\frac{x}{2R}}\right]_{2R}^{R} = 2R\left[\frac{-\pi}{4} - \frac{\pi}{2} - \frac{1}{2} + 0\right]$$

$$\Rightarrow 2R\left[-\frac{\pi}{4}-\frac{1}{2}\right] = -R\left[\frac{\pi}{2}+1\right]$$ {Putting this value of integration in (1)

$$\Rightarrow -\left\{-R\left[\frac{\pi}{2}+1\right]\right\} = \sqrt{\frac{GM}{R}}\,t$$

$$\Rightarrow t = R\left[\frac{\pi}{2}+1\right]\sqrt{\frac{R}{GM}}$$

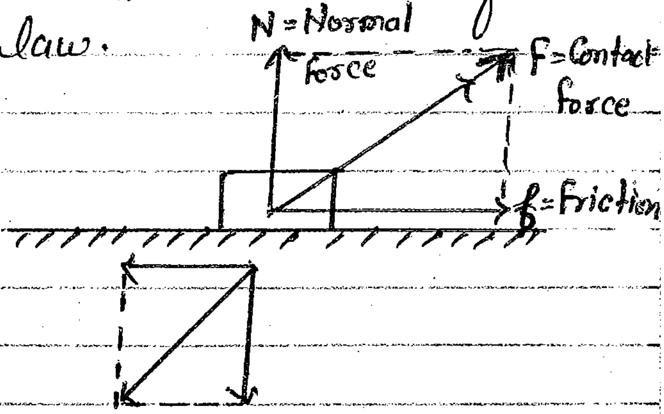
$$\Rightarrow \boxed{t = \left(\frac{\pi}{2}+1\right)\sqrt{\frac{R^3}{GM}}}$$

Ans

{Friction Force}

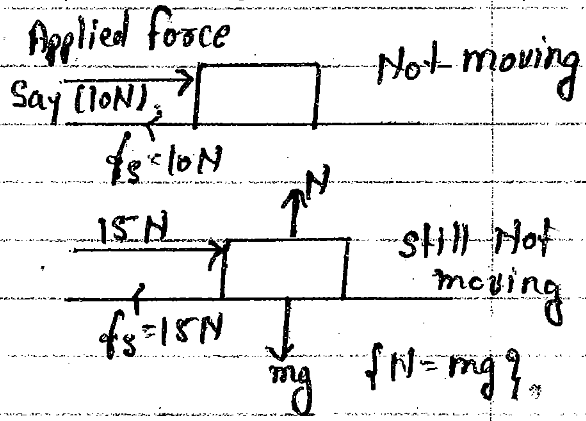
When two bodies are kept in contact, electro-magnetic forces act between the charged particles at the surfaces of the bodies. As a result each body exerts a contact force on the other.

The magnitude of the contact forces on the two bodies are equal but their direction are opposite and hence the contact forces obey Newton's IIIrd law.



Static Friction (f_s) :-

When two bodies do not slip on each other the force of friction is called static friction.



$\therefore$ $\boxed{f_s \leq \mu_s N}$

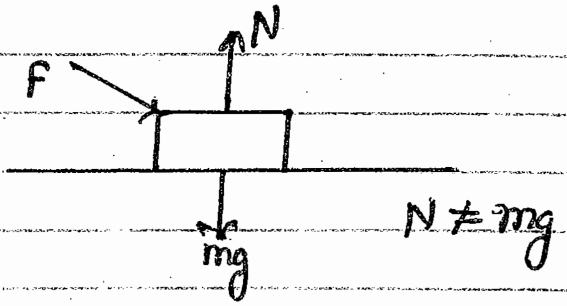
$N \rightarrow$ Normal Reaction

$\mu_s \rightarrow$ Coefficient of static friction.

The limiting value of static friction is -

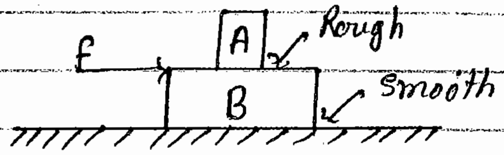
$$\boxed{(f_s)_{max} = \mu_s N}$$

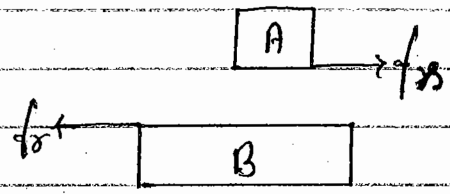
N may or may not be equal to mg.



$\Rightarrow$ friction may not be opposite to direction of motion always.

Ex.- Here A will move with B since B is applying friction on A.



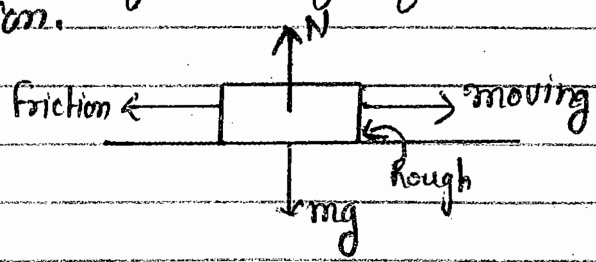


by Newton's Law.

Kinetic Friction :- (f_k) :-

When two solid bodies slip over each other, the force of friction is called kinetic friction.

$$\boxed{f_k = \mu_K N}$$



$\mu_k < \mu_s$ (slightly)

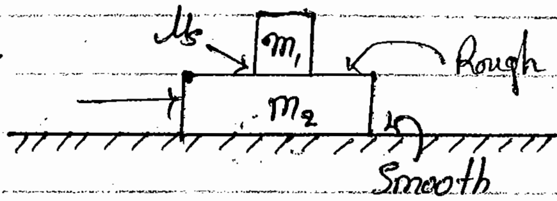
- Maximum force that can be applied without moving an object is $\mu_s N$.
- Minimum force required to move an object is almost equal to $\mu_K N$.

* Two Object Cases :-

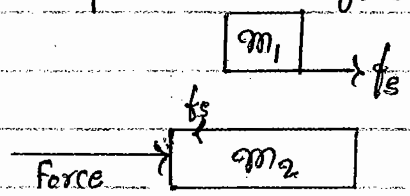
Q. What maximum force can be applied on lower block so that the two blocks move together (no relative motion between them).

Sol^n

The acceleration of upper block has a limit because it is moving due to static friction.



{ Upper block will move due to static friction



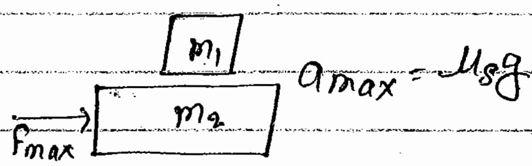
Maximum acceleration of upper block $= \dfrac{(f_s)_{max}}{m_1}$

$$= \frac{\mu_s N}{m_1}$$

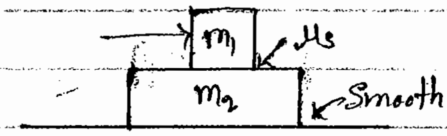
$$= \frac{\mu_s \cancel{m_1} g}{\cancel{m_1}} = \mu_s g.$$

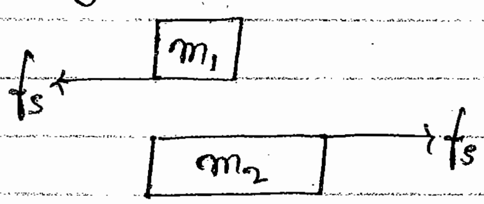
The two blocks will move together without any relative motion if their acceleration is less or equal to $\mu_s g$.

$$\boxed{F_{max} = (m_1 + m_2)\mu_s g}$$



Q. What is maximum acc^n so that the two blocks move together.





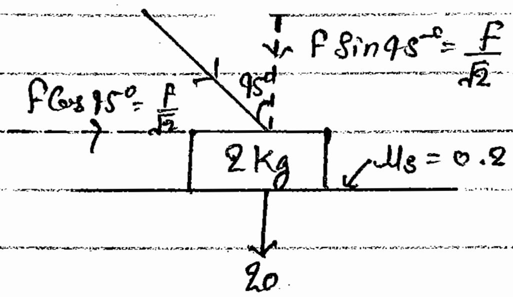
$$\text{Maximum acc}^n \text{ of } m_2 = \frac{(f_s)_{max}}{m_2} = \frac{\mu_s N}{m_2}$$

$$\boxed{\text{Maximum acc}^n \text{ of } m_2 = \frac{\mu_s m_1 g}{m_2}}$$

Basic.

Q.15 A 2 kg block is lying on a rough surface. If coefficient of static friction between the block & ground is 0.2. What max^m force can be applied on the block without moving it.

Sol^n



The block will not move if –

$$\frac{F}{\sqrt{2}} = (f_s)_{max} = \mu_s N$$

$$\frac{F}{\sqrt{2}} = 0.2 N \quad \text{——— (1)}$$

Vertical Equilibrium :-

$$20 + \frac{F}{\sqrt{2}} = N$$

Put N from (1)

$$20 + \frac{F}{\sqrt{2}} = \frac{F}{0.2\sqrt{2}}$$

$$\frac{F}{\sqrt{2}}\left[1 - \frac{1}{0.2}\right] = -20 \Rightarrow -\frac{F}{\sqrt{2}}\left[\frac{0.8}{0.2}\right] = -20$$

$$\Rightarrow F = \frac{20\sqrt{2}}{4} \Rightarrow F = 5\sqrt{2}$$

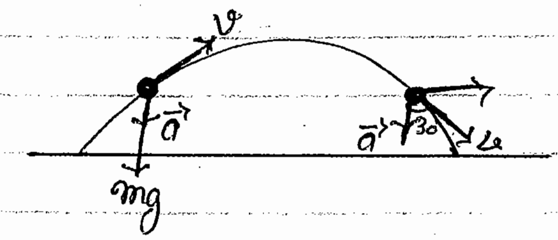
$$\Rightarrow \boxed{F \approx 7N} \text{ Ans}$$

To match the option

Basic

Q.10 A projectile is thrown at some angle. What is tangential accⁿ of projectile at the instant its velocity vector makes 30° with its accⁿ vector.

Solⁿ ∵ Direction of accⁿ will be towards force direction



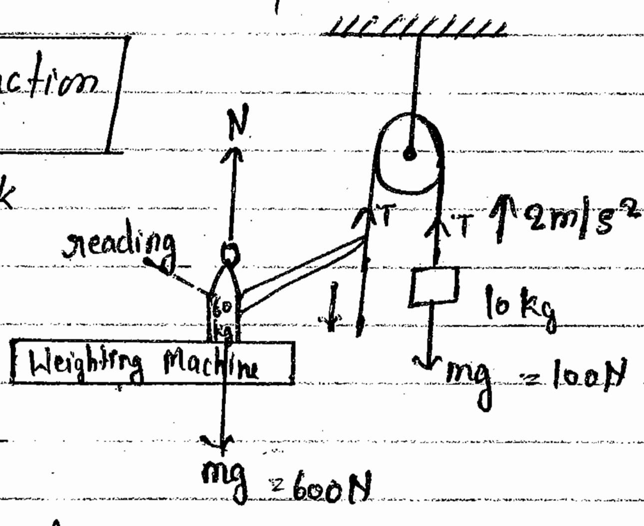
$$\vec{a} = \frac{\vec{F}}{m} = \frac{mg}{m} = g$$

$$a_t = a\cos 30°$$

$$\boxed{a_t = \frac{g\sqrt{3}}{2}}$$ Ans

Q.11 A person standing on a weighting machine as shown in the fig. pulls the string attached to a block in vertically downward direction. What is reading shown by the machine.

Solⁿ



$\boxed{\text{Reading} = \text{Normal Reaction}}$

Eqⁿ of motion of block

$$T - mg = ma$$
$$T - 100 = 10 \times 2$$
$$T = 120$$

Consider equilibrium of person

$$N + T = mg \Rightarrow N = Mg - T$$
$$= 600 - 120$$
$$= 480 \text{ Newton}$$

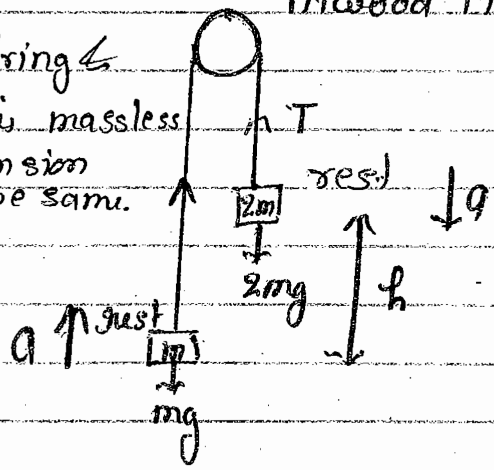
$$\boxed{N = 48\,kg}$$ Ans.

Q.7 In the fig. shown the two blocks are released from the position shown. After what time the two will cross each other. [Assume pulley & string to be light & smooth].

Sol^n

Equation of motion for m:

Atwood Machine.



$$T - mg = ma \quad \text{--- (1)}$$

{ If string & pulley is massless then tension will be same.

for 2m:

$$2mg - T = 2ma \quad \text{--- (2)}$$

adding (1) & (2)

$$\cancel{m}g = 3\cancel{m}a$$

$$\boxed{a = \frac{g}{3}}$$

$$S = ut + \frac{1}{2}at^2$$

$$\frac{h}{2} = 0 + \frac{1}{2}\frac{g}{3}t^2$$

{ The distance for crossing will be $h/2$

$$t^2 = \frac{3h}{g}$$

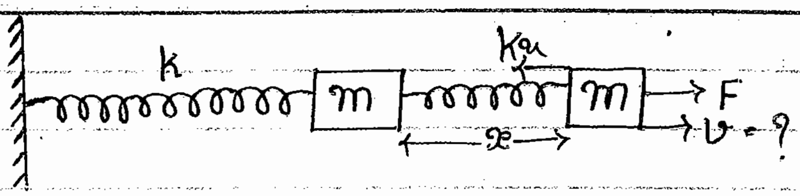
$$\Rightarrow \boxed{t = \sqrt{\frac{3h}{g}}} \text{ Ans}$$

A-3

Q.12 A spring mass system (Spring constant k & mass m) lies on a smooth horizontal surface with one end of spring being rigidly fixed. At t = 0 the mass is pulled with a constant horizontal force F. Speed of the mass after time t is ?

Sol^n

Equation of motion -



$$\vec{F} - kx = m\vec{a}$$

$$\boxed{\vec{a} = \frac{F}{m} - \frac{k}{m}x}$$ $\because$ $\vec{a}$ is variable

$\because$ We want to calculate - $v = f(t)$

So $F - kx = m\frac{dv}{dt}$

Steps :-

$$\boxed{\vec{a} = v\frac{dv}{dx} \xrightarrow{\text{integrating}} v = f(x) \xrightarrow{\text{integrate}} x = f(t) \xrightarrow{\text{differentiating}} v = f(t)}$$

So

$$F - kx = mv\frac{dv}{dx}$$

$$\frac{(F - kx)}{m}dx = v\,dv$$

$$\int \frac{F}{m}dx - \int \frac{k}{m}x\,dx = \int v\,dv$$

$$\frac{Fx}{m} - \frac{kx^2}{2m} = \frac{v^2}{2}$$

$$v^2 = \frac{2Fx}{m} - \frac{kx^2}{m} = \frac{2Fx - kx^2}{m} = \frac{2}{m}\left[Fx - \frac{kx^2}{2}\right]$$

$$v = \sqrt{\frac{2x[F - kx/2]}{m}}$$

$$\frac{dx}{dt} = \sqrt{\frac{2x[F - kx/2]}{m}}$$

$$\frac{dx}{\sqrt{2Fx - \frac{2kx^2}{2}}} = \frac{1}{\sqrt{m}}\int dt$$

Now integrating

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}$$

$$\Rightarrow \int \frac{dx}{\sqrt{[2Fx - kx^2]}} = \frac{1}{\sqrt{m}}\int dt$$

let $2Fx - kx^2 = -[kx^2 - 2Fx]$

$$= -\left[(\sqrt{k}x)^2 - 2\sqrt{k}x \cdot \frac{F}{\sqrt{k}} + \left(\frac{F}{\sqrt{k}}\right)^2 - \left(\frac{F}{\sqrt{k}}\right)^2\right]$$

$$= -\left[\left(\sqrt{k}x - \frac{F}{\sqrt{k}}\right)^2 - \left(\frac{F}{\sqrt{k}}\right)^2\right]$$

$$= \underbrace{\left(\frac{F}{\sqrt{k}}\right)^2}_{(a)^2} - \underbrace{\left(\sqrt{k}x - \frac{F}{\sqrt{k}}\right)^2}_{(y)^2}$$

$\sqrt{k}x - \frac{F}{\sqrt{k}} = y$

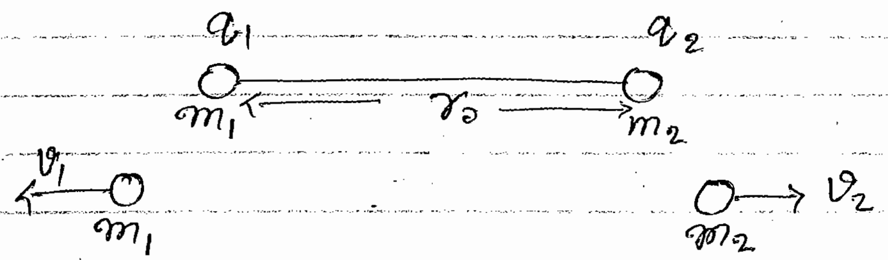
$\sqrt{k}\,dx = dy$

$dx = \frac{dy}{\sqrt{k}}$

Q.5 Two particles of masses m_1, m_2 and charges q_1, q_2 are placed r_0 distance apart on a smooth horizontal surface. Due to electrostatic repulsion they move away from each other. Ratio of their kinetic energy at a later time is -

Sol^n

$\vec{F}_{ext} = 0$



$$0 = m_2V_2 - m_1V_1$$

$$m_2V_2 = m_1V_1$$

$$\boxed{\frac{V_1}{V_2} = \frac{m_2}{m_1}} \text{ --- (1)}$$

$$\frac{K_1}{K_2} = \frac{\cancel{\frac{1}{2}} m_1 V_1^2}{\cancel{\frac{1}{2}} m_2 V_2^2}$$

$$\frac{K_1}{K_2} = \frac{m_1}{m_2}\left(\frac{V_1}{V_2}\right)^2$$

$$\frac{K_1}{K_2} = \frac{m_1}{m_2}\left(\frac{m_2}{m_1}\right)^2$$

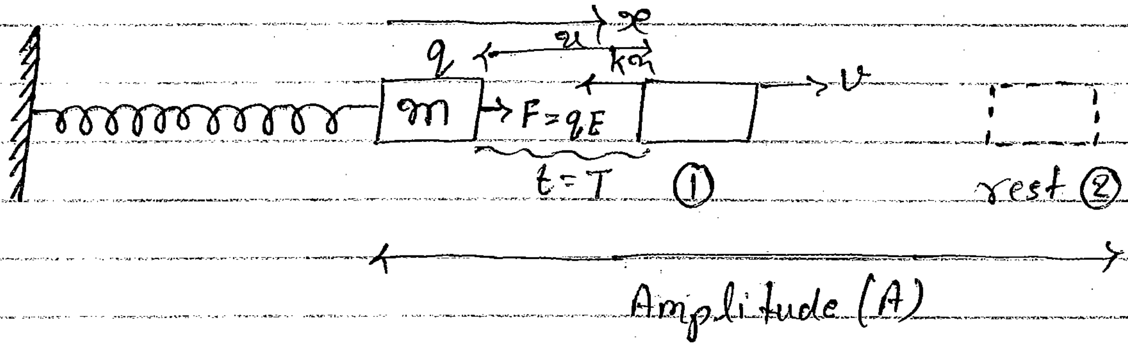
$$\boxed{\frac{K_1}{K_2} = \frac{m_2}{m_1}}$$

Ans

Net - 2011

2.10 A point particle of mass m carrying an electric charge q is attached to a spring of stiffness constant K. A constant electric field E along the direction of spring is switched on for a time interval T (where $T \ll \sqrt{\frac{m}{K}}$). Neglecting radiation loss, the amplitude of oscillation after the field is switched off is -

Sol^n



$$v = \frac{qE}{\sqrt{mk}} \sin\sqrt{\frac{K}{m}}\, T$$

$$\frac{dx}{dt} = \frac{qE}{\sqrt{mK}} \sin\sqrt{\frac{K}{m}}\, t$$

$$[x]_0^x = \frac{qE}{\sqrt{mK}} \left[-\cos\sqrt{\frac{K}{m}}\, t\right]_0^T \times \sqrt{\frac{m}{K}}$$

Elongation $\boxed{x = \frac{qE}{K}\left[1 - \cos\sqrt{\frac{K}{m}}\, T\right]}$

Conservation of energy :-

$$(K.E. + P.E.)_{I} = (K.E. + P.E.)_{II}$$

$$\left(\cancel{\tfrac{1}{2}} m v^2 + \cancel{\tfrac{1}{2}} k x^2\right) = \left(0 + \cancel{\tfrac{1}{2}} k A^2\right)$$

Date
31/07/2019

$$\Rightarrow m.\frac{q^2E^2}{mk}\sin^2 + \frac{kq^2E^2}{k^2}(1-\cos)^2 = kA^2$$

$$\Rightarrow \frac{q^2E^2}{k}\left[\sin^2 + 1 + \cos^2 - 2\cos\right] = kA^2$$

$$\Rightarrow 2\frac{q^2E^2}{k^2}\left[1-\cos\sqrt{\frac{k}{m}}\,T\right] = A^2$$

$$\Rightarrow \frac{4q^2E^2}{k^2}\left[\sin^2\sqrt{\frac{k}{m}}\cdot\frac{T}{2}\right] = A^2$$

$$\Rightarrow A = \frac{2qE}{k}\sin\sqrt{\frac{k}{m}}\frac{T}{2}$$

$$\Rightarrow A = \frac{2qE}{k}\sin\frac{T}{2\sqrt{\frac{m}{k}}} \longrightarrow \text{Very small}$$

$$\Rightarrow A = \frac{\cancel{2}qE}{k}\frac{T}{\cancel{2}\sqrt{\frac{m}{k}}}$$

$$= \frac{qE}{\cancel{k}}\sqrt{\frac{\cancel{k}}{m}}\,T$$

$$= \frac{qE}{\sqrt{km}}T$$

Ans

~~Ans 3 In previous question if m = Me, then time of fall will be -~~

A-3

Q.7 A small object of mass m falls from a height equal to radius of earth Re. If Me be mass of earth time taken by the particle to reach the earth's surface is (take $m << Me$).

(a) $\left(\frac{\pi}{2}+1\right)\sqrt{\frac{2R_e^3}{GM_e}}$ (b) $\left(\frac{\pi}{2}+1\right)\sqrt{\frac{R_e^3}{GM_e}}$

(c) $\frac{\pi}{2}\sqrt{\frac{R_e^3}{GM}}$ (d) $\sqrt{\frac{2R_e^3}{GM}}$

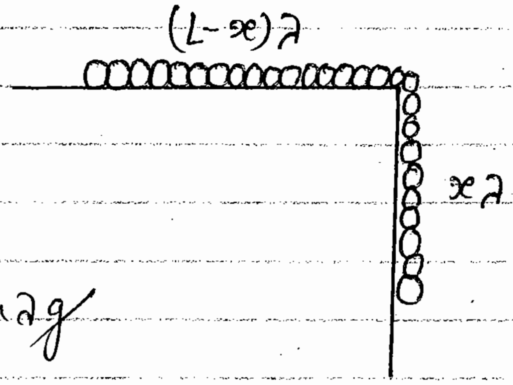
Sol^n

Basic

Ques A uniform rope of length 'L' is hanging off the edge of a rough table having coefficient of static friction μ. What should be minimum length of hanging hanging part so that the rope starts sliding down.

(a) $\frac{\mu L}{2}$ (b) $\frac{\mu L}{\mu+1}$ ✓ (c) $\left(\frac{1}{\mu}-1\right)L$ (d) $\frac{\mu L}{2-\mu}$

Sol^n



Equation of motion:-

$$x\lambda g = \mu(L-x)\lambda g$$

$$\Rightarrow x\lambda \cancel{g} = \mu L\lambda \cancel{g} - \mu x\lambda \cancel{g}$$

$$\Rightarrow x\cancel{\lambda} = \mu L\cancel{\lambda} - \mu x\cancel{\lambda}$$

$$\Rightarrow x + \mu x = \mu L$$

$$\Rightarrow x(1+\mu) = \mu L$$

So $\boxed{x = \frac{\mu L}{1+\mu}}$ Ans

31/July/2014

Stability Analysis

By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

Equilibrium criteria in one dimension :-

In such problem potential energy is given. If $V(x)$ be the potential under which a particle is moving then force acting on the particle is

$$\boxed{\vec{F_x} = -\frac{dV(x)}{dx}}$$

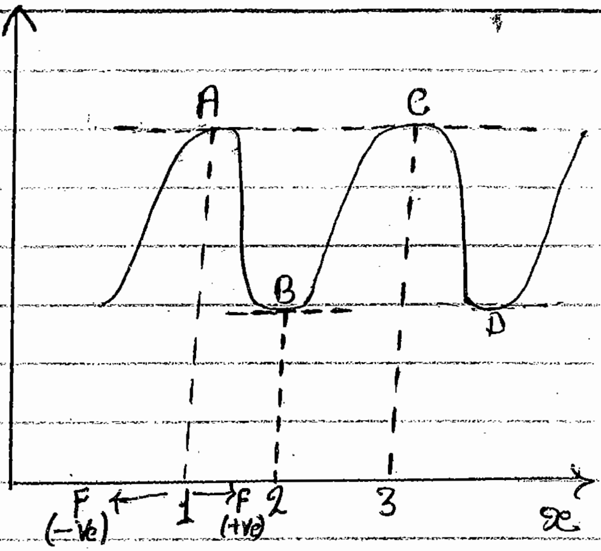
If system (Particle) is in equilibrium, then net force on particle is zero.

So, $\vec{F_n} = 0$

$$\Rightarrow \boxed{\frac{dV(x)}{dx} = 0}$$ (It gives equilibrium point (positions))

In $V(x)$ versus x graph $\frac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to x-axis. Therefore in the figure shown below point A, B, C are equilibrium points.

Point 1, 2, and 3 are equilibrium points in space.



Point A is the point where slope = 0
" B " " " = 0
" B " " " = 0

Stable Equilibrium Point :-

A point is stable equilibrium point if, a particle at this point when displaced towards right experiences force towards left and vice-versa. That is the force tries to bring it back.
Therefore at stable equilibrium point -

$$\frac{dF_x}{dx} < 0$$

or

$$\boxed{\frac{dF_x}{dx} = -ve}$$

$$\therefore \frac{d^2V(x)}{dx^2} > 0 \quad \text{(Condition for minimum)}$$

Thus, a stable equilibrium point is a minimum on V(x) versus x plot.
Therefore points B and D are stable point.

Unstable Equilibrium point :-

In this case a particle at equilibrium point is when displaced ~~towards~~ towards right it experiences force also in rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point,

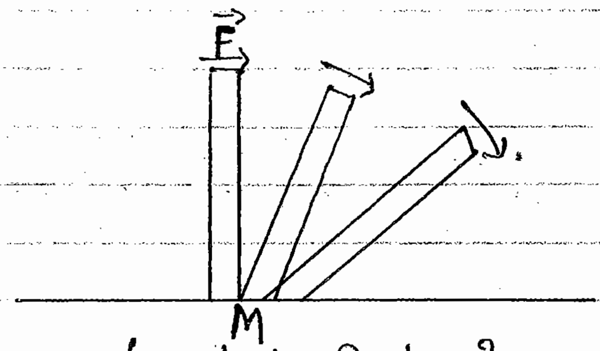
$$\frac{dF_x}{dx} > 0 \quad \text{or}$$

$$\boxed{\frac{dF_x}{dx} = +ve}$$

$\therefore \quad \frac{d^2V(x)}{dx^2} < 0 \quad$ (Condition of maximum)

Thus, an unstable point is a maximum on $V(x)$ versus x plot. Therefore points A and C in the above figure are unstable points.

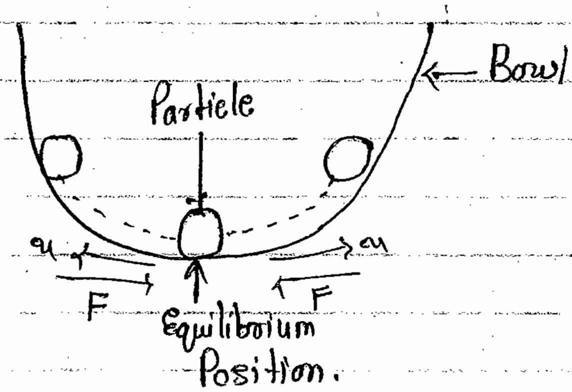
Example :-



{Unstable Position}

To understand equilibrium more closely we consider another example we take a hemispherical bowl and a marble like small sphere as shown in figure.

He we can see that the direction of force is opposite to the direction of displacement.



Now from $V(x)$ v/s x graph :-

⇒ At point 1 :-

Rightwards displacement :-

$$dx = +ve \quad , \quad dV = -ve$$

So $\vec{F} = -\frac{dV}{dx} = +ve$

So force is positive.

Leftwards displacement :-

$$dx = -ve \quad , \quad dV = -ve$$

$$\vec{F} = -\frac{dV}{dx} = \cancel{+}ve$$

So leftward displacement $= -ve$

Here

~~Rightwards Force ≠ Leftwards Force~~

Therefore we conclude that point 1 is unstable point.

∴ At unstable point $\frac{d^2V}{dx^2} < 0$ {for max^m V}

A condition of maxima.

⇒ At point 2 in $V(x)$ V/s x graph :-

Rightwards displacement :-

$$dV = +ve$$
$$dx = +ve$$

$$\vec{F} = -\frac{dV}{dx} = -ve$$

So for +ve displacement force is negative.

Leftwards Displacement :-

for $dx = -ve$

$dv = +ve$

$$\vec{F} = -\frac{dV}{dx} = +ve$$

So for negative displacement we get force is +ve

So we conclude that point 2 is stable point.

i.e. At stable point we must have -

$$\boxed{\frac{d^2V}{dx^2} > 0} \qquad \{\text{for minimum } V\}$$

* Special Condition:-

If $\frac{d^2V}{dx^2} = 0$ then check higher even derivative. Then -

If,

$$\frac{d^nV}{dx^n} > 0 \longrightarrow (\text{Stable})$$

when

$$\frac{d^nV}{dx^n} < 0 \longrightarrow (\text{Unstable})$$

Here n is even, $n = 2, 4, 6, \ldots$ (any even value).

⇒ If all even higher derivatives are zero then the point neither "stable" nor "Unstable."

Ex-1 $V(x) = \alpha x^4$ when $\alpha > 0$

find equilibrium point and check it is stable or unstable ?

Sol^n Let x_0 is equilibrium point.

$$\left(\frac{dV(x)}{dx}\right)_{x = x_0} = 0$$

$$4\alpha x_0^3 = 0$$

$$\Rightarrow x_0 = 0$$

So x_0 is a equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 12\alpha x_0^2$$

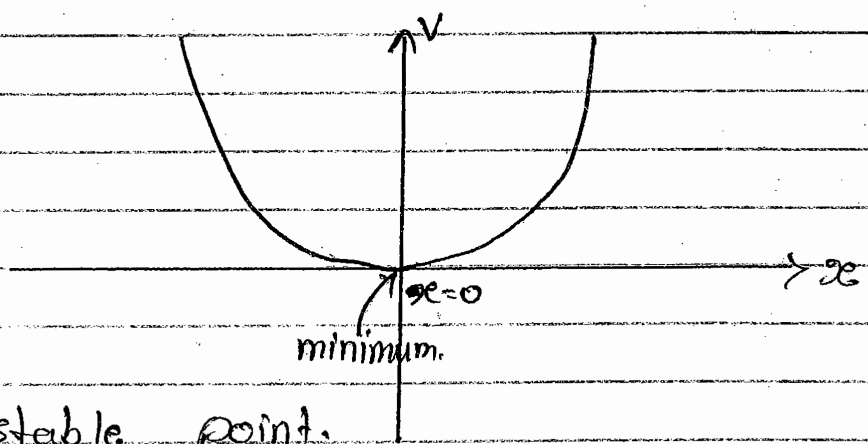
$$= 0 \quad \{\because x_0 = 0\}$$

ferther we check higher derivative.

$$\left.\frac{d^4V}{dx^4}\right|_{x=x_0} = 24\alpha > 0$$

So $x = x_0$ is stable point.

Graph of αx^4:-

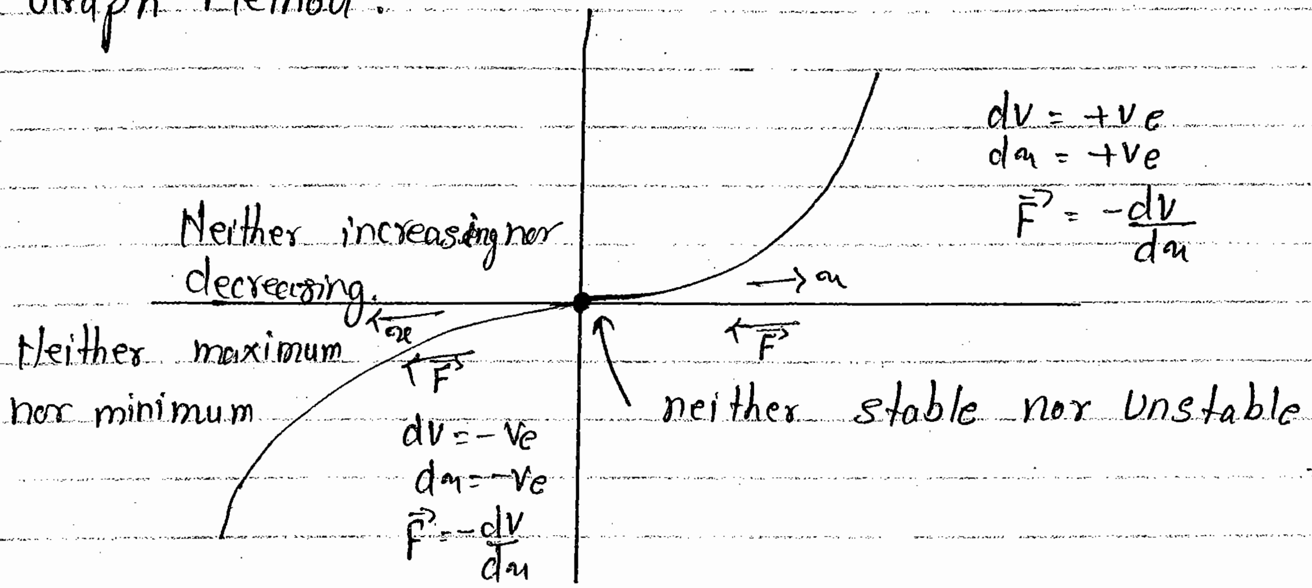


So,

$x = 0$ is a stable point.

Q. $V = \alpha x^3$, $x > 0$ find equilibrium point and check it is stable or unstable.

Sol. Graph Method :-



Result :- Here for +ve displacement force is −ve but for −ve displacement it is also negative. So it is neither stable nor unstable.

Second Method :-

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 3\alpha x^2$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6\alpha x$$

$$\left.\frac{d^3V}{dx^3}\right|_{x=x_0} = 6\alpha$$

$$\left.\frac{d^4V}{dx^4}\right|_{x=x_0} = 0$$

So $x = x_0$ neither stable point nor unstable

Q. If $V(x) = x(x-2)^2$, how many stable and unstable points are there?

Sol^n

Condition of equilibrium (x_0) :-

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$\Rightarrow (x_0-2)^2 + 2x_0(x_0-2) = 0$$

$$\Rightarrow (x_0-2)\left[(x_0-2) + 2x_0\right]$$

$$\Rightarrow (x_0-2)\left[3x_0-2\right] = 0$$

$$\Rightarrow \left.\begin{array}{l} x_0 = 2 \\ x_0 = 2/3 \end{array}\right\} \text{Two equilibrium point.}$$

at $x_0 = 2$:-

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=2} = 12 - 8 = 4 > 0$$

So $x_0 = 2$ is stable point.

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=\frac{2}{3}} = 6 \times \frac{2}{3} - 8 = -4 < 0$$

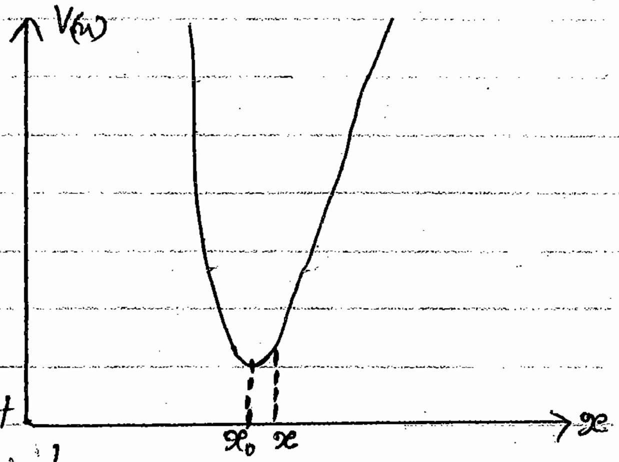
So $x_0 = \frac{2}{3}$ is a unstable point.

* Frequency of Oscillation about stable equilibrium point {for small oscillation}:-

We know a particle of mass m moving under potential $V(x) = V_0 + \frac{1}{2}kx^2$ has freq. of oscillation $\omega = \sqrt{\frac{k}{m}}$, about stable equilibrium point, then we can expand $V(x)$ about x_0 using Taylor's series expansion as,

$$V(x) = V(x_0) + (x-x_0)\left.\frac{dV}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

$$+ \frac{(x-x_0)^3}{3!}\left.\frac{d^3V}{dx^3}\right|_{x=x_0} + \cdots\cdots$$



Let a particle of mass m is moving under a potential $V(x)$. Let x_0 be a stable point.

In the above expansion- $(x-x_0)$ is a displacement from stable equilibrium point.

At the equilibrium position -

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

therefore, if $(x-x_0)$ be small then higher power terms are neglected.

$$V(x) = V(x_0) + \frac{(x-x_0)^2}{2!}\left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

it is greater than 0 so let it is = K.

$$\boxed{V(x) = V(x_0) + \frac{1}{2} K (x - x_0)^2}$$

So force :-

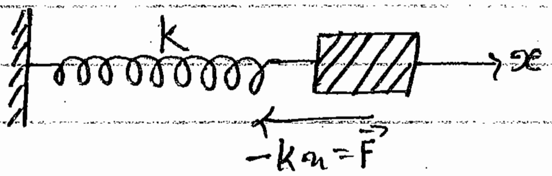
$$\vec{F} = -\frac{dV}{dx}$$

$$\vec{F} = -(x - x_0) K$$

$$\boxed{\vec{F} = -K(x - x_0)}$$

Here force is $(-k)$ times displacement and this is a case of S.H.M.

An example of S.H.M :-



So in the case of S.H.M.

$$\Rightarrow -Kx = ma$$

$$\Rightarrow \boxed{\ddot{x} = -\left(\frac{K}{m}\right) x}$$

$$\omega^2 = \frac{K}{m}$$

So $$\boxed{\omega = \sqrt{\frac{K}{m}}}$$

ω : Freq. of oscillation
m : mass of particle
k : Force Constant.

Here $$\boxed{K = \left.\frac{d^2V}{dx^2}\right|_{x = x_0}}$$ x_0 : stable point.

Here frequency of oscillation is independent of displacement so it is true for small oscillation ω is independent of amplitude. If oscillation is not small then ω is dependent of amplitude (energy).

Conclusions :-

⇒ $$\omega = \sqrt{\frac{k}{m}} \quad \text{where } k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

x_0 is stable point.

⇒ It is not applicable in the case of -

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 0$$

⇒ If $\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 0$. Then we will write -

$$V(x) = V(x_0) + \frac{(x-x_0)^3}{\lfloor 3} \left.\frac{d^3V}{dx^3}\right|_{x=x_0}$$

$$V(x) = V(x_0) + 3\frac{(x-x_0)^2}{\lfloor 3} . k$$

$$\boxed{V(x) = V(x_0) - \frac{(x-x_0)^2}{\lfloor 2} . k}$$

$$\boxed{\vec{F} = -\frac{dV}{dx}}$$

$$\boxed{\vec{F} = -\frac{(x-x_0)^2}{\lfloor 2} . k}$$

Here $\vec{F} \propto (x-x_0)^2$ So it is not in S.H.M.

⇒ If $V = V(\theta)$ then -

$$\omega = \sqrt{\frac{K}{I}} \quad , \quad K = \left.\frac{d^2V}{d\theta^2}\right|_{x=\theta_0}$$

Where θ_0 = Stable point.

I = Moment of Inertia.

CSIR - 2012
A-4

Ques 9 Consider the motion of a classical particle in a one dimensional double-well potential $V(x) = \frac{1}{4}(x^2-2)^2$. If the particle is displaced infinitesimally from the minimum of the positive X-axis (and friction is neglected), then-

(a) the particle will execute simple harmonic motion in the right well with an angular freq. $\omega = \sqrt{2}$

(b) the particle will execute simple harmonic motion in the right with an angular frequency $\omega = 2$.

(c) the particle will switch between the right and left well. {Not possible in classical mechanics.}

(d) the particle will approach the bottom of the right well and settle there {Here no settle because there is no friction}.

Sol^n

$$V(x) = \frac{1}{4}(x^2-2)^2$$

$$\therefore \omega = \sqrt{\frac{K}{m}}, \quad K = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

At equilibrium point -

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$x_0(x_0^2-2) = 0$$

$$\Rightarrow \left.\begin{array}{l} x_0 = 0 \\ x_0 = \pm\sqrt{2} \end{array}\right\} \text{there is 3 equilibrium points.}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 3x_0^2 - 2$$

at $x = x_0 = 0$

$$\frac{d^2V}{dx^2} = -2 < 0 \quad \left(\text{So } x_0 = 0 \text{ is unstable point}\right)$$

at $x = x_0 = \pm\sqrt{2}$

$$\left.\frac{d^2V}{dx^2}\right|_{x_0 = \pm\sqrt{2}} = 6 - 2 = 4 > 0 \left\{ \begin{array}{l} x_0 = \pm\sqrt{2} \text{ is} \\ \text{stable point} \end{array} \right.$$

So there are two stable points $(\pm\sqrt{2})$.

$\therefore$ $\boxed{k = 4}$

$\therefore$ $\omega = \sqrt{\frac{4}{1}}$ $\left\{ \begin{array}{c} \text{take } m = 1 \\ \text{Unit mass} \end{array} \right\}$

Angular freq. $\boxed{\omega = 2}$

---

* How to Plot a graph {V(x) v/s x}:-

Steps :-

(i) Put $V(x) = 0$ and find the value of x.

(ii) Put $dV/dx = 0$ and find the value of x.

(iii) Find position of maxima and minima i.e. check second derivative.
If second derivative is zero then check higher derivative.

(iv) Connect all the points where $V(x) = 0$ by making appropriate maxima or minima.

(V) Consider $x \longrightarrow \pm\infty$

Ex- $V(x) = \frac{1}{4}(x^2-2)^2$

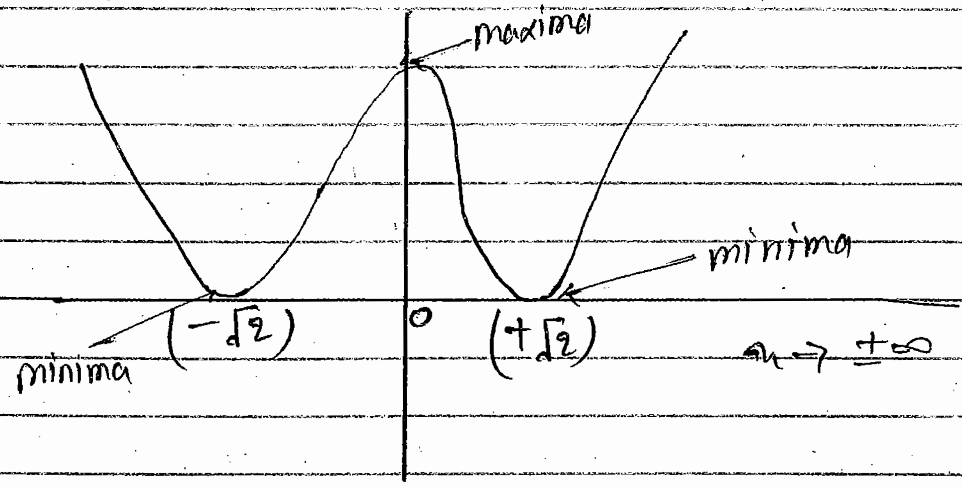
Sol^n $V(x) = 0 \Rightarrow (x^2-2)^2 = 0$

$x = \pm\sqrt{2}$

$\frac{dV}{dx} = 0$ find $x = 0, \pm\sqrt{2}$

$\frac{d^2V}{dx^2} > 0$ at $x = \pm\sqrt{2}$ i.e. minima

$\frac{d^2V}{dx^2} < 0$ at $x = 0$ } i.e. it is maxima



A-4
Q.14 A particle of mass 'm' is moving under potential $V(x) = ax^3 - bx$, frequency of oscillation about the stable equilibrium posion is -

(a) $\left[\frac{12ab}{m^2}\right]^{1/4}$ ✓ (b) $\left[\frac{6ab}{m^2}\right]^{1/4}$ (c) $\left[\frac{4ab}{m^2}\right]^{1/4}$ (d) $\left[\frac{3ab}{m^2}\right]^{1/4}$,

Sol^n

$V(x) = ax^3 - bx$

let x_0 is equilibrium point then

$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 3ax_0^2 - b = 0$$

$$3ax_0^2 = b$$

$$x_0^2 = \frac{b}{3a}$$

$$\boxed{x_0 = \pm\sqrt{\frac{b}{3a}}}$$

Now $\frac{d^2V}{dx^2} = 6ax$

$$= \sqrt{\frac{b}{3a}} \cdot 6a > 0 \quad \text{stable (minima)}$$

$$= -6a\sqrt{\frac{b}{3a}} < 0 \quad \text{Unstable (maxima).}$$

$$\therefore \left.\frac{d^2V}{dx^2}\right|_{x=x_0} = K = 6a\sqrt{\frac{b}{3a}} = \left[\frac{36a^2b}{3a}\right]^{1/2}$$

So angular freq (ω) about stable point. $= \sqrt{\frac{K}{m}}$

So $$\omega = \sqrt{\left(\frac{\cancel{36}^{12}a^{\cancel{2}}b}{\cancel{3}\cancel{a}}\right)^{1/2} \cdot \frac{1}{m}}$$

$$\omega = \sqrt{\left(\frac{12ab}{m^2}\right)^{1/2}}$$

$$\boxed{\omega = \left(\frac{12ab}{m^2}\right)^{1/4}}$$

Ans

Note:-

In a particle in which speed is to be calculated & Potential energy is given.
Then we apply conservation of energy.

Ex- $V(x) = A - Bx + Cx^3$

For equilibrium point -

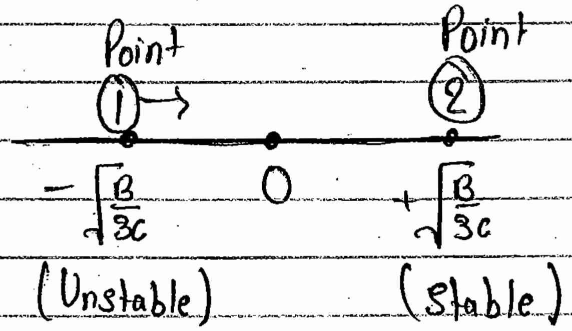
$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$-B + 3Cx_0^2 = 0$$

$$\boxed{x_0 = \pm\sqrt{\frac{B}{3C}}}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6Cx_0$$

If system is conservative then conservation of energy.



$$(P.E. + K.E.)_{\text{at point 1}} = (P.E. + K.E.)_{\text{at point 2}}$$

$$\cancel{A} + B\sqrt{\frac{B}{3C}} - \frac{CB}{3C}\sqrt{\frac{B}{3C}} + 0 = \cancel{A} - B\sqrt{\frac{B}{3C}} + \frac{CB}{3C}\sqrt{\frac{B}{3C}} + \frac{1}{2}mv^2$$

$$2B\sqrt{\frac{B}{3C}} - \frac{2CB}{3C}\sqrt{\frac{B}{3C}} = \frac{1}{2}mv^2$$

$$\sqrt{\frac{B}{3C}}\left[2B - \frac{2\cancel{C}B}{3\cancel{C}}\right] = \frac{1}{2}mv^2$$

$$2B\sqrt{\frac{B}{3C}}\left[1-\frac{B}{3}\right] = \frac{1}{2}mv^2$$

$$\boxed{\left\{\frac{4B}{m}\sqrt{\frac{B}{3C}}\left[1-\frac{B}{3}\right]\right\}^{1/2} = V}$$

Ans

June 2013

Q. There is a particle of mass m moving with potential $V(x) = ax - bx^3$ initially particle is at rest at stable point what is minimum speed should be given to it. so that its motion becomes unstable.

Sol^n

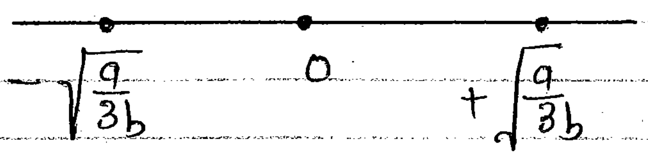
$$V(x) = ax - bx^3$$

$$\left.\frac{dV(x)}{dx}\right|_{x_0} = a - 3bx_0^2 = 0$$

$$\Rightarrow 3bx_0^2 = a$$

$$\Rightarrow x_0 = \pm\sqrt{\frac{a}{3b}}$$

So that equilibrium point is, $+\sqrt{\frac{a}{3b}}$ and $-\sqrt{\frac{a}{3b}}$



$$\frac{d^2V(x)}{dx^2} = -6bx_0 < 0$$

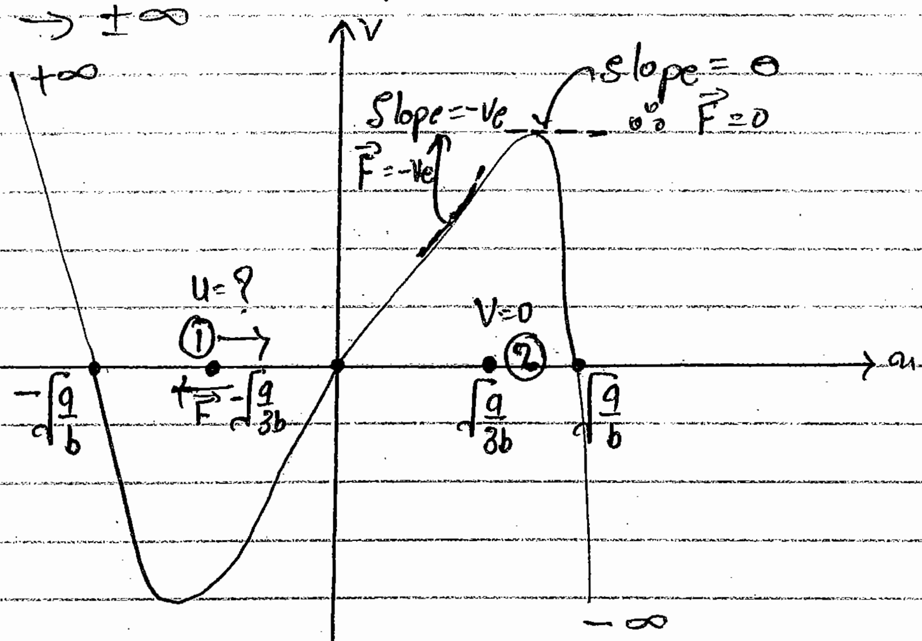
So $-\sqrt{\frac{a}{3b}}$ is stable point (min^m potential point.

$V(x) = 0$

$\Rightarrow \quad x(a - bx^2) = 0$

$$x = 0, \quad x = \pm\sqrt{\frac{a}{b}}$$

and $x \to \pm\infty$



for minimum speed that particle reaches unstable point and stop there.

Applying Energy Conservation at point ① and ②:

$$(P.E. + K.E.)_I = (P.E. + K.E.)_{II}$$

$$\Rightarrow \frac{1}{2}mu^2 - a\sqrt{\frac{a}{3b}} + b\left(\frac{a}{3b}\right)^{3/2} = 0 + a\sqrt{\frac{a}{3b}} - b\left(\frac{a}{3b}\right)^{3/2}$$

$$\Rightarrow \frac{1}{2}mu^2 = 2a\sqrt{\frac{a}{3b}} - 2b\left(\frac{a}{3b}\right)^{3/2}$$

$$\frac{1}{2} mu^2 = 2\sqrt{\frac{a}{3b}}\left\{a - \cancel{b}\left(\frac{a}{3\cancel{b}}\right)\right]$$

$$mu^2 = 4\sqrt{\frac{a}{3b}}\left\{\frac{3a-a}{3}\right\} = 4\sqrt{\frac{a}{3b}}\left\{\frac{2a}{3}\right\}$$

$$mu^2 = \sqrt{\frac{a}{3b}}\left\{\frac{8a}{3}\right\} = \left\{\frac{64a^2 \cdot a}{27b}\right\}^{1/2}$$

$$u^2 = \left\{\frac{64a^3}{27m^2b}\right\}^{1/2}$$

$$\boxed{u = \left\{\frac{64a^3}{27m^2b}\right\}^{1/4}}$$

Q.6 A particle of mass $m = 4$ moves along the x-axis under the influence of the potential $V(x) = 2\left(e^{-\frac{3x}{2}} - 2e^{-\frac{3}{4}x}\right)$, if the particle oscillates with small amplitude around the minimum potential, what is the period of the oscillation.

(a) 0.12 (b) 1.33 (c) 8.37 ✓ (d) 11.17

Sol^n

$$V(x) = 2\left[e^{-\frac{3}{2}x} - 2e^{-\frac{3}{4}x}\right]$$

$$\omega = \sqrt{\frac{k}{m}}, \quad k = \left.\frac{d^2V}{dx^2}\right|_{x = x_0} \text{ (stable)}$$

$$\left.\frac{dV}{dx}\right|_{x = x_0} = 0$$

$$2\left[\frac{-3}{2}e^{-\frac{3}{2}x_0} + \frac{3}{2}e^{-\frac{3}{4}x_0}\right] = 0$$

$$e^{-\frac{3}{2}x_0} = e^{-\frac{3}{4}x_0}$$

Taking ln on both side.

$$\cancel{-}\frac{\cancel{3}}{2}x_0 = \cancel{-}\frac{\cancel{3}}{4}x_0$$

$$\left(\frac{1}{2}-\frac{1}{4}\right)x_0 = 0$$

$$\because \left(\frac{1}{2}-\frac{1}{4}\right) \neq 0$$

$$\therefore \boxed{x_0 = 0}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 2\left[\frac{9}{4}e^{-\frac{3}{2}x_0} - \frac{9}{8}e^{-\frac{3}{4}x_0}\right]$$

$$= 2\left[\frac{9}{4}-\frac{9}{8}\right]$$

$$= 2\times\frac{9}{8}$$

$$= \frac{9}{4} = k = +ve$$

So $x_0 = 0$ is stable point.

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\frac{2\times 3.14}{T} = \sqrt{\frac{9}{4\times 4}}$$

$$\frac{2\times 3.14}{T} = \frac{3}{4}$$

$$T = \frac{8\times 3.14}{3} = \frac{25.12}{3}$$

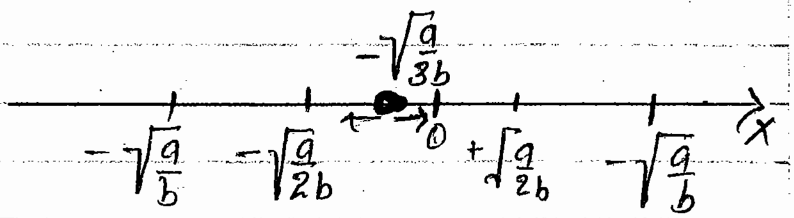
$$\boxed{T = 8.3733}$$

Q.7 Force acting on a particle along x-axis is given to be $F(x) = 2bx^3 - ax$. If the particle is released from point $x = -\sqrt{\frac{a}{3b}}$ it will move towards which of the following points.

(a) 0 ✓ (b) $+\sqrt{\frac{a}{3b}}$ (c) $-\sqrt{\frac{a}{2b}}$ (d) $-\sqrt{\frac{a}{b}}$

Sol^n

$\vec{F}(x) = 2bx^3 - ax$



$$f(x) = -2\cancel{b} \cdot \frac{a}{3\cancel{b}}\sqrt{\frac{a}{3b}} + a\sqrt{\frac{a}{3b}}$$

$$= \left(\frac{-2a}{b} + a\right)\sqrt{\frac{a}{3b}}$$

$$= \frac{a}{3b}\sqrt{\frac{a}{3b}} \quad (+ve)$$

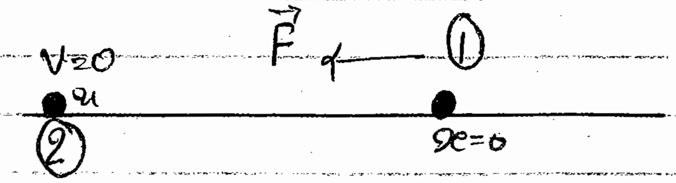
So it move in the direction of 0.

Q.12 A particle of mass 2kg is released at $x = 0$ m. The particle is acted upon by a force whose potential is expressed as $V(x) = 2x - 4x^3$ joule. The force on the particle at the point where it will again come to rest for the first time is -

(a) -4 Newton (b) $+4$ Newton ✓ (c) zero Newton
(d) $+2$ Newton.

Sol^n

$V(x) = 2x - 4x^3$



$\vec{F} = -\frac{dV}{dx} = 12x^2 - 2$

Put $x = 0$

$F = -2$ so F is $-ve$

Apply conservation of energy at ① & ②

$$(K.E. + P.E.)_I = (K.E. + P.E.)_{II}$$

$$(0+0) = (0 + 2x - 4x^3) \quad \left(\begin{matrix}\text{Put } x=0 \\ \text{in } V(x)\end{matrix}\right)$$

$$x = 0, \pm \frac{1}{\sqrt{2}}$$

$-\frac{1}{\sqrt{2}}$ is the point where particle is in rest for first time.

So $\vec{F} = 12x^2 - 2 = 12\left(-\frac{1}{\sqrt{2}}\right)^2 - 2$

$= 6 - 2 = 4$ Newton. Ans

A-4

Q.10 The total energy E of the particle of mass m executing small oscillations about the origin along on the x-direction is given by, $E = \frac{1}{2}mv^2 + V_0 \cosh\left(\frac{x}{L}\right)$, where V_0 and L are positive constants. The time period T of oscillation is -

(a) $T = \frac{1}{2\pi}\sqrt{\frac{m}{V_0}}$ (b) $T = 2\pi\sqrt{\frac{L}{m}}$ (c) $T = \pi L\sqrt{\frac{m}{E}}$

(d) $T = 2\pi\sqrt{\frac{mL^2}{V_0}}$ ✓

Sol^n

$$E = \underbrace{\frac{1}{2} m v^2}_{K.E.} + \underbrace{V_0 \cosh\left(\frac{X}{L}\right)}_{P.E.}$$

$$V(x) = V_0 \cosh\left(\frac{X}{L}\right)$$

$\because$ Origin is stable point (given)

$$x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$k = \left.\frac{d^2V}{dx^2}\right|_{x=0} = \frac{V_0}{L^2}\left(\cosh\frac{x}{L}\right)_{x_0}$$

$$\boxed{K = \frac{V_0}{L^2}}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{V_0}{L^2 m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{V_0}{L^2 m}}$$

$$\boxed{T = 2\pi\sqrt{\frac{L^2 m}{V_0}}}$$ Ans

Q.11 A particle of mass 'm' is moving under a one dimensional potential $V(x) = ax^3 - bx$. Due to the force acting on the particle its kinetic energy changes as the particle moves from one point to the other. What can be maximum change in K.E. of the particle in this case -

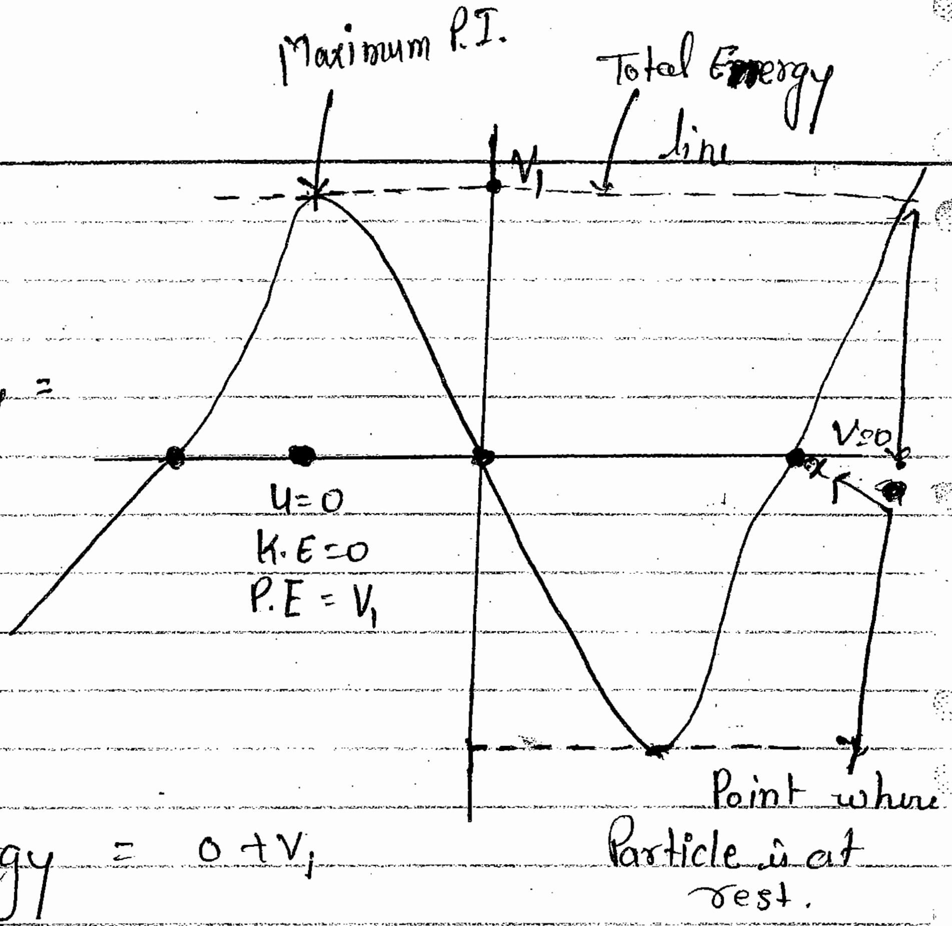
(a) $\sqrt{\frac{16b^3}{3a}}$ (b) $\sqrt{\frac{2b^3}{3a}}$ (c) $\sqrt{\frac{4b^3}{3a}}$ (d) $\sqrt{\frac{8b^3}{3a}}$

Sol^n

$V(x) = ax^3 - bx$

$\because$ Total Energy =

K.E + P.E.



So

Total Energy $= 0 + V_1$

Total Energy $= V_1$

Maximum Change in K.E. = Max^m change in P.E.

$= V_{max} - V_{min}$

Q.14 A particle of mass 'm' is moving under potential $V(x) = ax^3 - bx$, frequency of oscillation about the stable equilibrium position is -

(a) $\left[\frac{12ab}{m^2}\right]^{1/4}$ (b) $\left[\frac{6ab}{m^2}\right]^{1/4}$ (c) $\left[\frac{9ab}{m^2}\right]^{1/4}$ (d) $\left[\frac{3ab}{m^2}\right]^{1/4}$

Sol^n $V(x) = ax^3 - bx \Rightarrow \left.\frac{dV}{dx}\right|_{x=x_0} = 3ax_0^2 - b$

$3ax_0^2 - b = 0 \Rightarrow 3ax_0^2 = b \Rightarrow x_0^2 = \frac{b}{3a}$ $\boxed{x_0 \pm\sqrt{\frac{b}{3a}}}$

So $\pm\sqrt{\frac{b}{3a}}$ are two equilibrium point.

$\left.\frac{d^2V}{dx^2}\right|_{x_0=\sqrt{\frac{b}{3a}}} = 6ax_0 = 6a\sqrt{\frac{b}{3a}} > 0$

So $x_0 = \sqrt{\frac{b}{3a}}$ is minima or stable point.

$\left.\frac{d^2V}{dx^2}\right|_{x_0=-\sqrt{\frac{b}{3a}}} = 6ax_0 = -6a\sqrt{\frac{b}{3a}} < 0$

So $x_0 = -\sqrt{\frac{b}{3a}}$ is maxima or unstable point.

$\because$ $K = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$ where x_0 is stable point

So $K = 6a\sqrt{\frac{b}{3a}} = \cancel{36}\sqrt{\frac{a^2b}{\cancel{3a}}} = \sqrt{\frac{36a^2b}{3a}}$

$= \sqrt{\frac{36ab}{3}}$

$\because$ $\omega = \sqrt{\frac{K}{m}} = \sqrt{\left(\frac{\cancel{36}ab}{\cancel{3}m^2}\right)^{1/2}}$

$\omega = \sqrt{\left(\frac{12ab}{m^2}\right)^{1/2}} = \left(\frac{12ab}{m^2}\right)^{1/4}$

$\boxed{\omega = \left(\frac{12ab}{m^2}\right)^{1/4}}$ Ans

Q.15 In the previous question force on the particle is maximum at

(a) $x = 0$ (b) $x = \sqrt{\frac{b}{a}}$ (c) $x = \sqrt{\frac{b}{3a}}$ (d) $x = -\sqrt{\frac{b}{3a}}$

Sol^n

$$V(x) = ax^3 - bx$$

$$\vec{F} = -\frac{dV}{dx} = -(3ax^2 - b) = b - 3ax^2$$

for $\vec{F}$ to be max^m or min^m

$$\frac{dF}{dx} = 0$$

$$-6ax = 0$$

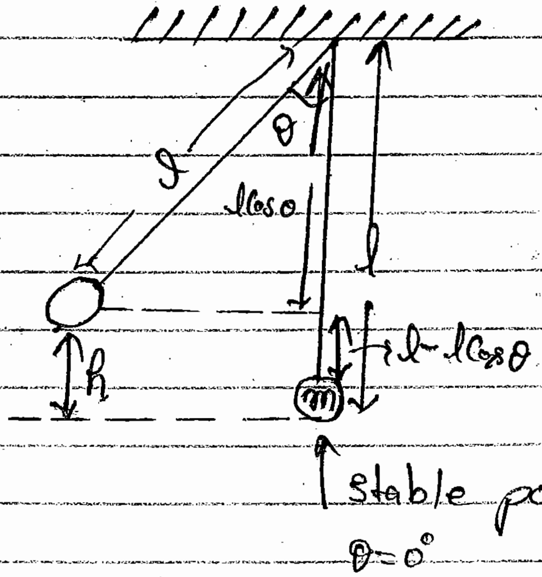
$$\because -6a \neq 0$$

$$\therefore \boxed{x = 0}$$

$$\frac{d^2F}{dx^2} = -6a \quad (\text{min}^m)$$

$$\therefore F_{max} = b - 0 = b \quad \text{at } \boxed{x = 0}$$

* Time Period of Simple Pendulum :-



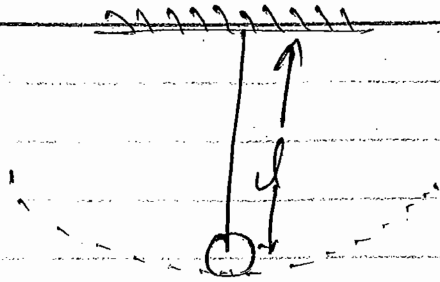
$$P.E = mgh$$

$$\boxed{P.E. = mgl(1 - \cos\theta)}$$

$$K = \left.\frac{d^2V}{d\theta^2}\right|_{\theta=\theta_0}$$

$$= mgl\cos\theta \Big|_{\theta=0}$$

$$\boxed{K = mgl}$$



$$I = \sum_i m_i r_i^2 = ml^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{\cancel{m}g\cancel{l}}{\cancel{m}l^{\cancel{2}}}}$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Note:-

⇒ K in Linear Case :-

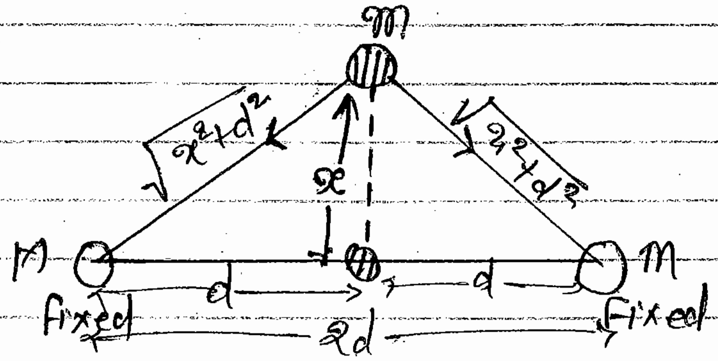
$$K = \left.\frac{d^2 V}{dx^2}\right|_{x=0} \qquad \omega = \sqrt{\frac{K}{m}}$$

⇒ K in Angular Case :-

$$K = \left.\frac{d^2 V}{d\theta^2}\right|_{\theta=\theta_0} \qquad \omega = \sqrt{\frac{K}{I}}$$

Ques: If m is slightly displaced along perpendicular bisector, what is its frequency of oscillation (Use P.I. method.)

Sol^n



P.E of the system $(V_x) = -\dfrac{2GMm}{\sqrt{x^2+d^2}} - \dfrac{GM^2}{2d}$

Stable point is $x = 0$

$$K = \left.\frac{d^2V}{dx^2}\right|_{x=0}$$

$$\frac{dV}{dx} = -2GMm \frac{d}{dx}(x^2+d^2)^{-1/2}$$

$$= \frac{2GMm}{(x^2+d^2)^{3/2}}$$

$$\frac{d^2V}{dx^2} = 2GMm \frac{d}{dx} \frac{x}{(x^2+d^2)^{3/2}}$$

$$= 2GMm \left[\frac{1.(x^2+d^2)^{3/2} - x.\frac{3}{2} \times (x^2+d^2).2x}{(x^2+d^2)^3}\right]$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=0} = \frac{2GMm\cancel{d^3}}{d^{\cancel{6}}} = \frac{2GMm}{d^3}$$

$$\omega = \sqrt{\frac{k}{m}}$$

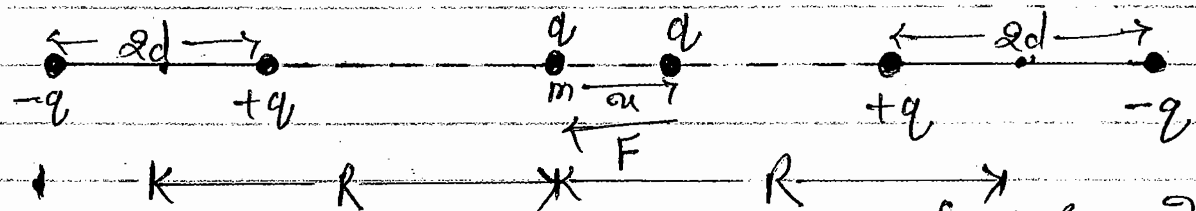
$$\omega = \sqrt{\frac{2GM\cancel{m}}{d^3\cancel{m}}} = \sqrt{\frac{2GM}{d^3}}$$

Angular frequency $\boxed{\omega = \sqrt{\frac{2GM}{d^3}}}$ Ans

CSIR - 2013 (June)

Q. If particle of mass m and charge +q is slightly displaced from given position what will be its frequency of oscillation $(d <<< R)$.

Sol^n



$$V(x) = \frac{1}{4\pi\epsilon_0} Q.q \left[\frac{1}{(R-d+x)} - \frac{1}{(R+d+x)} + \frac{1}{R-d-x} - \frac{1}{R+d-x} \right]$$

{+q & -q are fixed}

$$\frac{d^2V}{dx^2} = \frac{Qq}{4\pi\epsilon_0} \left[\frac{2}{(R-d+x)^3} - \frac{2}{(R+d+x)^3} + \frac{2}{(R-d-x)^3} - \frac{2}{(R+d-x)^3} \right]$$

$\therefore$ $x = 0$ is stable point -

$$\left.\frac{d^2V}{dx^2}\right|_{x=0} = \left[\frac{qQq}{4\pi\epsilon_0}\left\{\frac{1}{(R-d)^3} - \frac{1}{(R+d)^3}\right\}\right]$$

$$\because \; d <<< R$$

$$\text{so } K = \frac{qQq}{4\pi\epsilon_0 R^3}\left[\left(1-\frac{d}{R}\right)^{-3} + \left(1+\frac{d}{R}\right)^{-3}\right]$$

$$= \frac{\cancel{q}Qq}{\cancel{4}\pi\epsilon_0 R^3}\left[\cancel{1}+\frac{3d}{R} - \cancel{1} + \frac{3d}{R}\right]$$

$$= \frac{6Qqd}{\pi\epsilon_0 R^4}$$

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{6dqQ}{\pi\epsilon_0 R^4}}$$

$$\boxed{\omega = \sqrt{\frac{6dQq}{\pi\epsilon_0 R^4}}} \text{ Ans}$$

06/Aug/2014

Non-Inertial Frames of Reference And Pseudo Forces

* Non-Inertial Frames :-

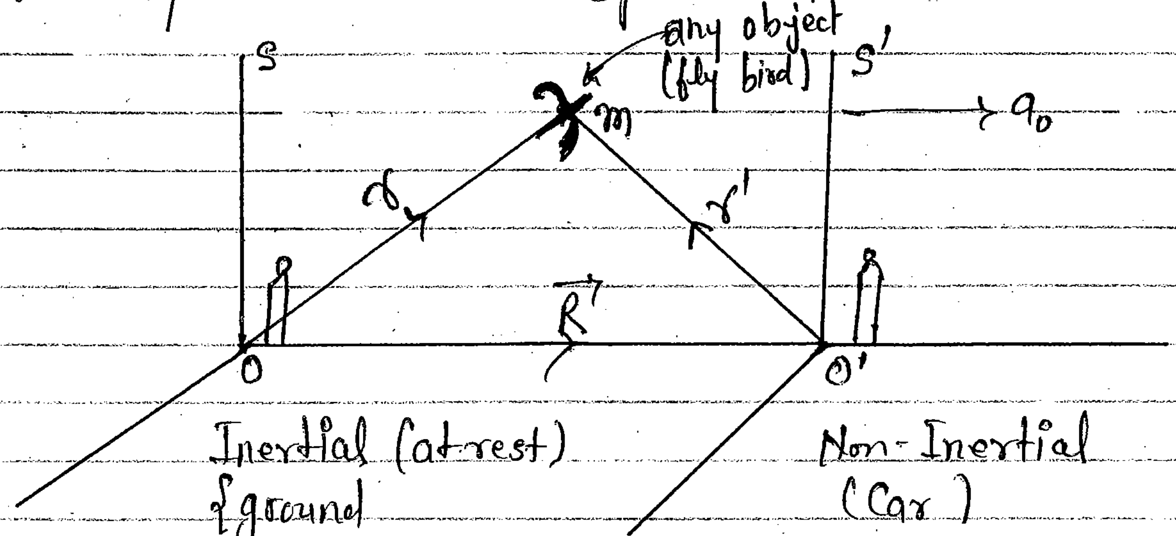
An accelarated frame is called Non-Inertial frames.

Accelarated frames are of two types-

1. Linearly Accelarated frame
2. Uniformly Rotating frame (ω = constt) OR Rotating frame.

1 Linearly Accelarated Frame :-

Let us consider two frame S and S' where S is at rest and S' is moving with constant velocity 'v' with respect to S.



Let a_0 = accelaration of S'.

$$\vec{R} + \vec{r}' = \vec{r}$$
$$\vec{r}' = \vec{r} - \vec{R} \quad \text{— (1)}$$

Differentiate twice with respect to time -

$$\frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \frac{d^2\vec{R}}{dt^2}$$

$$\vec{a}' = \vec{a} - \vec{a_0}$$

both side multiplying by m :-

$$m\vec{a}' = m\vec{a} - m\vec{a_0}$$

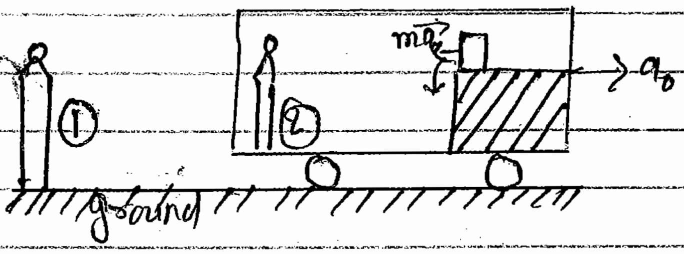
$$\boxed{\vec{F}' = \vec{F} - m\vec{a_0}}$$ ← Pseudo Force.

force on object at seen from s'   force on object at seen from s

$$\boxed{\vec{F}' = \vec{F} + (-m\vec{a_0})}$$

Note :-

Cause of fall of body in the transparent bus :-
Explanation of :-
first Person :- (which is at ground) :-

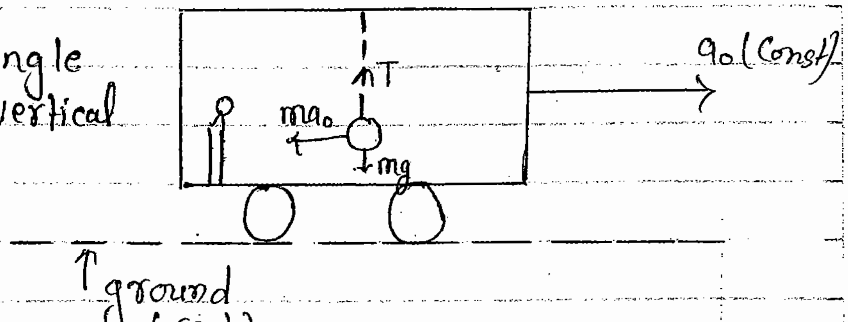


Cause of fall of body in bus is ; There is no sufficient friction.
Second Person :-
Due to pseudo force.

⇒ The origin of pseudo force is not known.
⇒ Concept of pseudo force is used when observation is made from non-inertial frame.

* Simple Pendulum in linearly accelerated frame :-

Let string makes angle θ with downward vertical in equilibrium position.

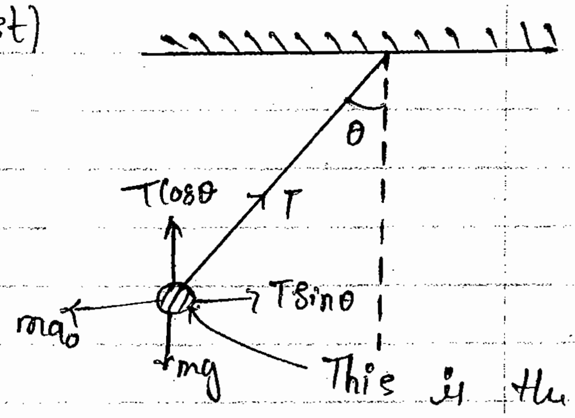


$$T\sin\theta = ma_0 \quad \text{—(I)}$$
$$T\cos\theta = mg \quad \text{—(II)}$$

(I) / (II)

$$\tan\theta = \frac{a_0}{g}$$

$$\boxed{\theta = \tan^{-1}\frac{a_0}{g}}$$

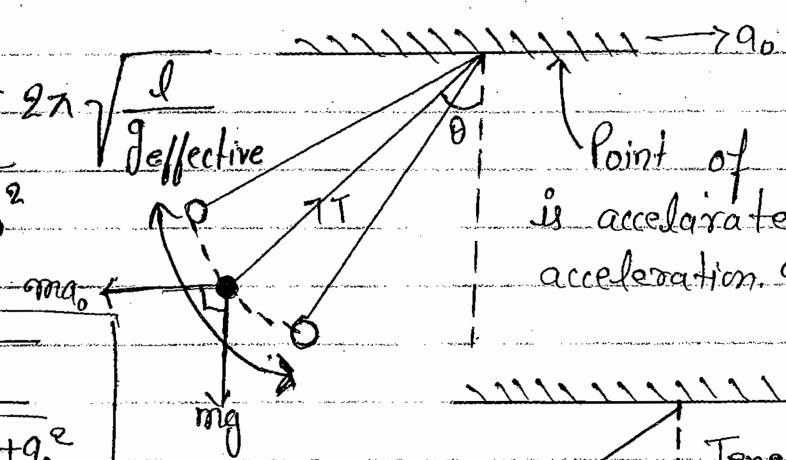


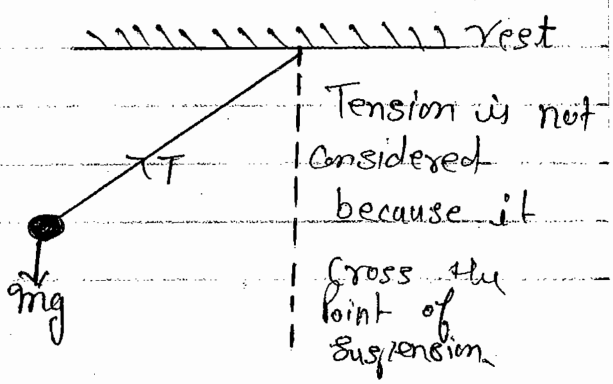
* Time Period of oscillation in accelarated frame :-

Time Period (T) $= 2\pi\sqrt{\frac{l}{g_{effective}}}$

$\therefore g_{eff} = \sqrt{g^2 + a_0^2}$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a_0^2}}}}$$

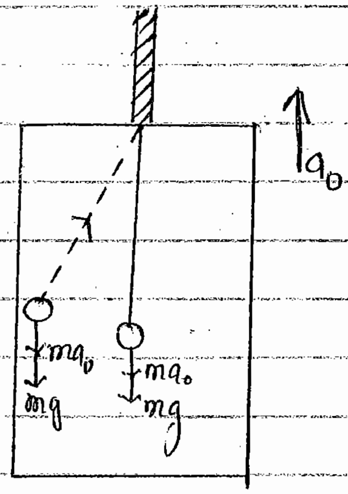




* Pendulum in a Lift :-

In this case :-

$$g_{eff} = a_0 + g$$

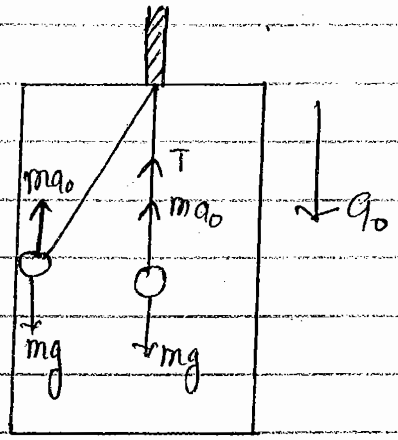


$$\therefore \; T = 2\pi\sqrt{\frac{l}{g_{effective}}}$$

So

$$\boxed{T = 2\pi\sqrt{\frac{l}{g+a_0}}}$$

In this case :-

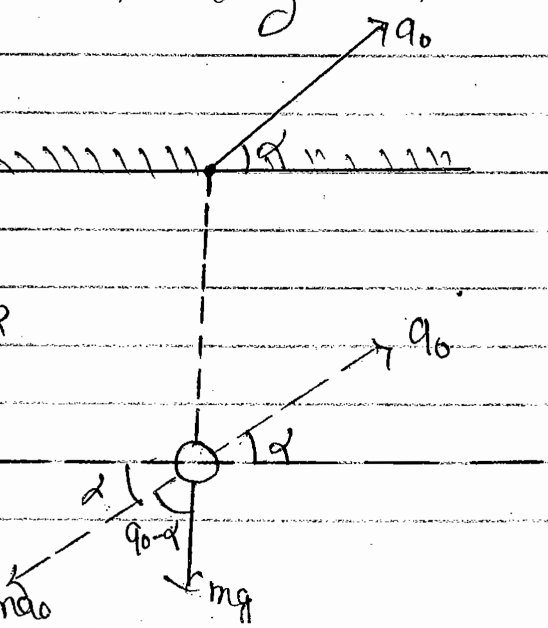


$$g_{eff} = g - a_0$$

$$a_0 < g$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g-a_0}}}$$

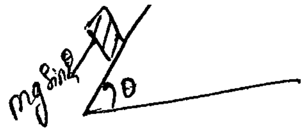
* When point of suspension is accelarated at an angle α ~~ver~~ with horizontal :-



$$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\cos(90-\alpha)}$$

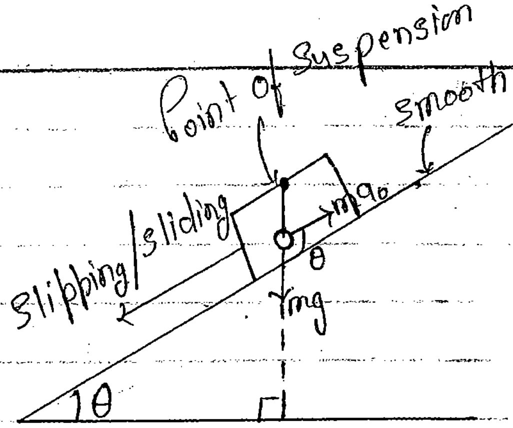
$$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\sin\alpha}$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g_{eff}}}}$$



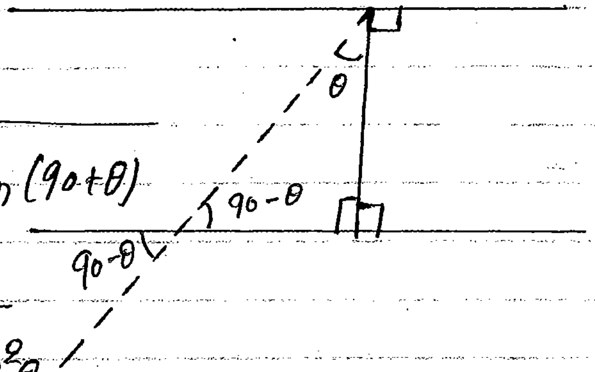
$v = mg\sin\theta$ $\quad a = \frac{mg\sin\theta}{m} \Rightarrow \boxed{a = g\sin\theta}$

Q. What is the time period of pendulum.



$a_0 = g\sin\theta$

$\therefore \quad \theta = 90 + \theta$



$$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\sin(90+\theta)}$$

$$= \sqrt{g^2 + (g\sin\theta)^2 - 2g(g\sin\theta)\sin(90+\theta)}$$

$$= \sqrt{g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta}$$

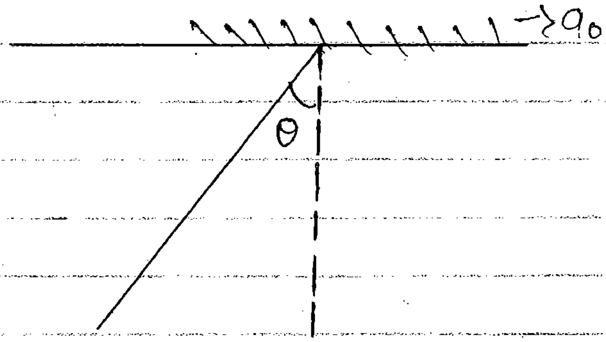
$$= g\sqrt{1 + \sin^2\theta - 2\sin^2\theta} = g\sqrt{1 - \sin^2\theta}$$

$$g_{eff} = g\cos\theta$$

So $\boxed{T = 2\pi\sqrt{\frac{l}{g\cos\theta}}}$

Q. Acceleration of point of suspension is $\sqrt{3}g$ in horizontal direction.

Sol^n



$$\theta = \tan^{-1}\left(\frac{a_0}{g}\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$\boxed{\theta = 60°}$ Ans

Given -

In stationary case time period $= T$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

then

$$T' = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

$$g_{eff} = \sqrt{a_0^2 + g^2}$$

$$= \sqrt{3g^2 + g^2} = \sqrt{4g^2}$$

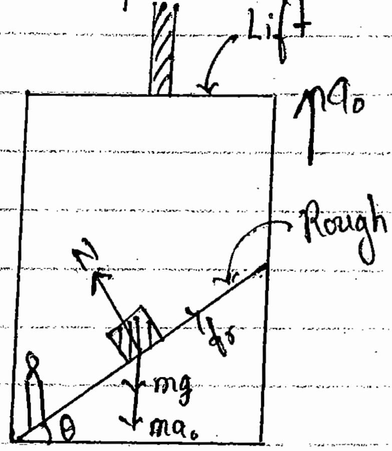
$$g_{eff.} = 2g$$

$$\boxed{T' = 2\pi\sqrt{\frac{l}{2g}}}$$

$$\boxed{T' = \frac{T}{\sqrt{2}}}$$ Ans

13/08/2014

Q. Block is not slidding on inclined plane what work done by the friction on Block. During the time lift move up by distance 'h'



Sol^n As seen from inside the lift, block is at rest.

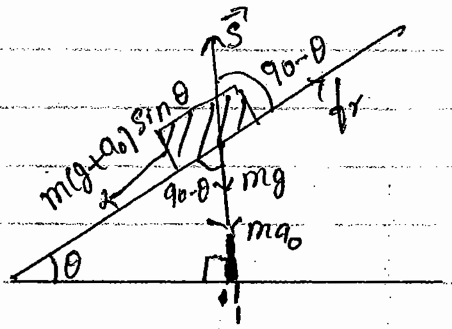
Therefore -

$f_r = m(g+a_0)\sin\theta$

Work done = force . Displacement

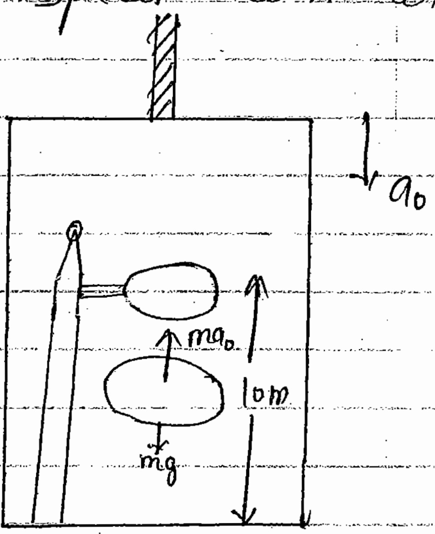
$= \vec{F}.\vec{S}$

$= f_r h \cos(90-\theta)$

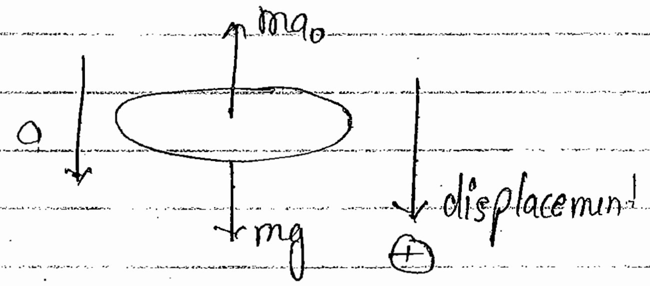


$$\boxed{W.D. = m(g+a_0)h\sin^2\theta}$$

Q. A lift is going downward direction with $a_0 = 4\ m/s^2$. A person drops a coin at 10m distance from initial speed 0 with initial velocity. With what speed coin will hit the flour of lift.



Sol^n



Let a = acc^n of coin as seen by person -

$mg - ma_0 = ma$

$$a = (g - a_0)$$

$$v^2 = u^2 + 2as$$

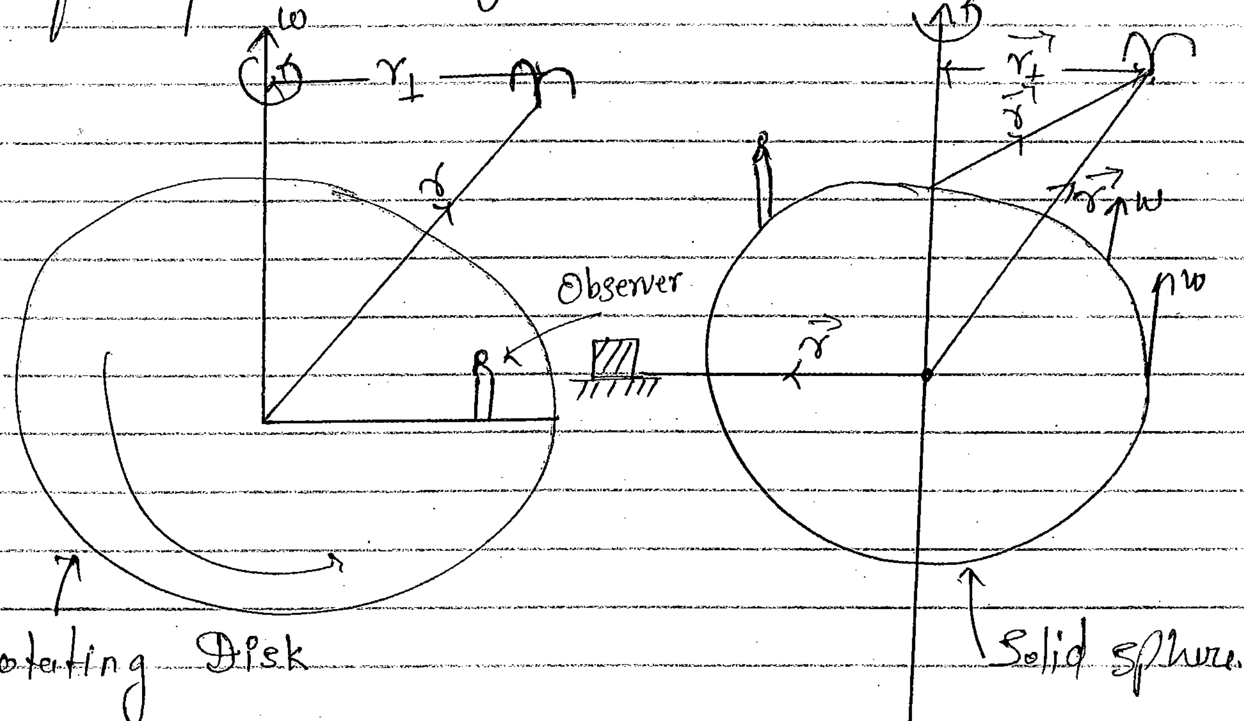
$$v^2 = 0 + 2(g - a_0) \times 10$$

$$= 2 \times 6 \times 10$$

$$v^2 = 120$$

$$\boxed{v = \sqrt{120} \text{ m/sec}}$$ Ans

* Uniformly Rotating frame :- { ω = Constt } :-

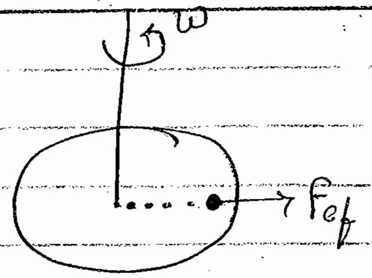


Rotating frame is example of Non-Inertial frame.

When observation is made from rotating frame on all objects (irrespective of their location). following pseudo forces appears to act on them.

1 Centrifugal force :-

$$\boxed{F_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})}$$



m = mass of object.
$\vec{\omega}$ = angular velocity of rotating frame
$\vec{r}$ = position vector of object (with respect to a stationary point of frame).

Magnitude of Centrifugal Force :-

$$\boxed{\vec{F}_{cf} = m\omega^2 r_\perp}$$

$r_\perp$ = $\perp^{or}$ distance of object from axis of rotation.

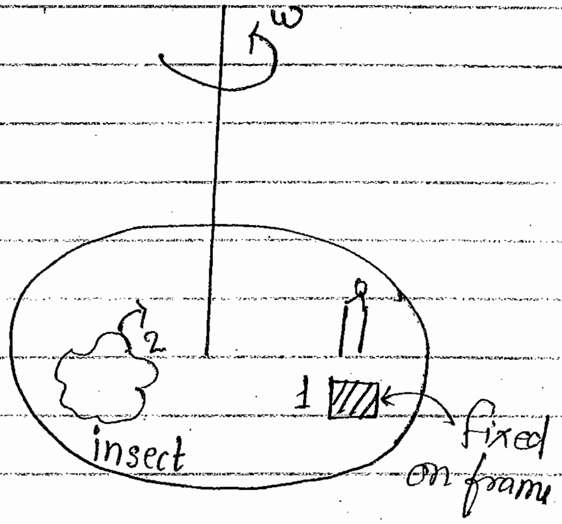
⇒ Centrifugal force acts on all object irrespective of their state of motion.

2. Coriolis Force :-

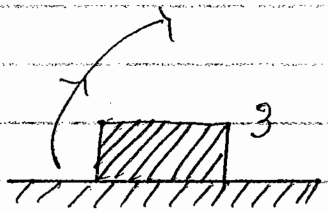
Coriolis :- A name of Mechanical Engineer
Coriolis ~~was~~ a mechanical engineer. He is observing force on the liquid which is in rotating arm of the machine and unfortunately he discover a different type of force which is after his name Klat coriolis force.

"This force is appears to act on all object which appear to be moving when seen from the rotating frame (or object have speed with respect to rotating frame.)"

Here Coriolis force act on object 2 & 3 but not on 1.



Expression of Coriolis force :-



$$\vec{F}_{cor.} = -2m(\vec{\omega} \times \vec{v}_r)$$

v_r = Velocity of object with respect to rotating frame.

$$\boxed{\vec{F}_{cor} = 2m(\vec{v}_r \times \vec{\omega})}$$

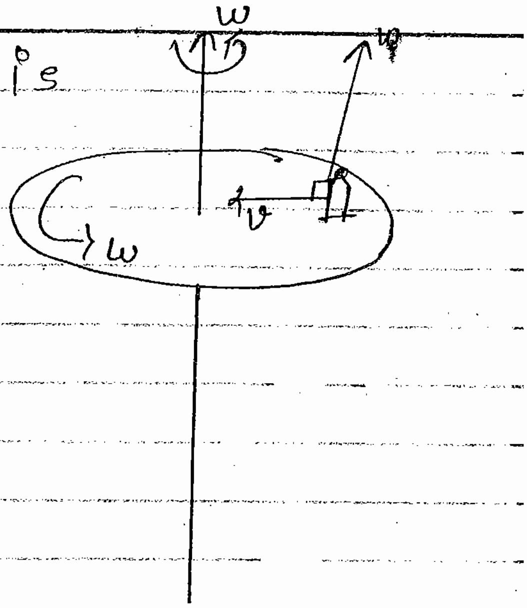
* Some Basic Problems Based on Coriolis force :-

A-5
Q.22 A circular platform is rotating with a uniform angular speed ω counterclockwise about an axis passing through its center and perpendicular to its plane as shown in the figure. A person of mass m walks radially inwards with a uniform speed v on the plateform. The magnitude and the direction of the coriollis force (with respect to the direction along which the person walks is ?

Curl the finger v to ω to get direction of

$v_r = v$

Axis is parallel to plane of paper.



$|\vec{F}_{cor}| = 2 m v_r \omega \sin 90^\circ$

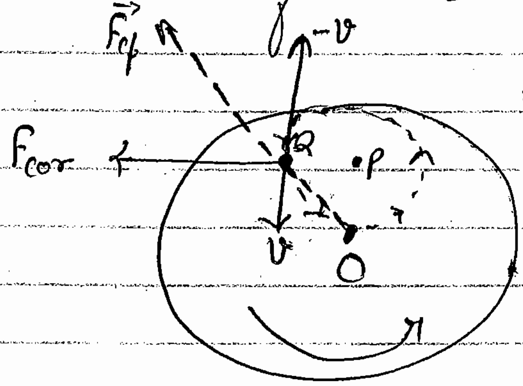
$= 2 m v \omega \quad \{\sin 90^\circ = 1\}$

$\boxed{F_{cor} = 2 m v \omega}$ towards his Right

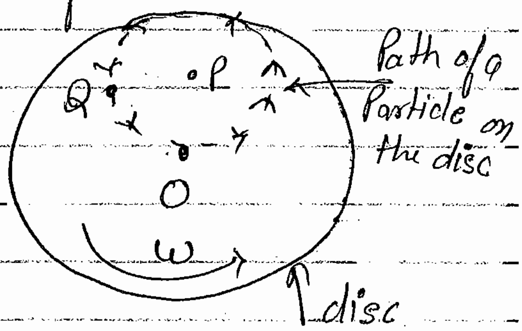
So option (C) is correct.

Q-5
Q.4 A circular disc is rotating in anticlockwise sence as shown in the figure. On the disc a particle moves in anticlockwise circle with center at P. At the instant particle at Q, which of the following options correctly represents directions of centrifugal and coriolis forces ['O' is center of the disc].

Sol^n



Axis is ⟂ to the Plane of Paper.



When particle (insect) is at Q show direction of centrifugal and Coriolis force.

"Direction of centrifugal force ($\vec{F}_{cf}$) is always perpendicular to axis from object of rotation and away from it."

So option (d) is correct.

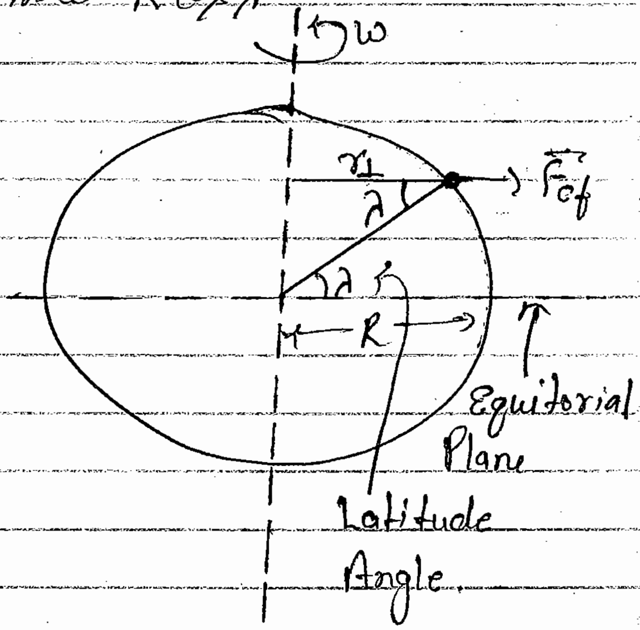
A-5

Q.5 A particle of mass m is lying on earth's surface at a location where latitude is λ. If ω be angular velocity of earth's spining motion and R be the radius of the earth then, centrifugal force on the particle is - ?

(a) $m\omega^2 R \sin\lambda$ (b) $m\omega^2 R \cos\lambda$ ✓

(c) $2m\omega^2 R \sin\lambda$ (d) $2m\omega^2 R \cos\lambda$

Sol:



$$\vec{F}_{cf} = m\omega^2 r_\perp$$

$$\cos\lambda = \frac{r_\perp}{R}$$

$$r_\perp = R\cos\lambda$$

So $\boxed{\vec{F}_{cf} = m\omega^2 R \cos\lambda}$

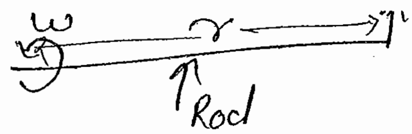
At equator (at $\lambda = 0$) :-

$$\boxed{\vec{F}_{cf} = m\omega^2 R = max^m}$$

At poles (At $\lambda = 90^\circ$)

$$\boxed{\vec{F}_{cf} = 0 = min^m}$$

Imp. Q. A disc is rotating in horizontal plane with uniform angular velocity ω an object on mass m is lying on the ground in the same plane at a distance r from the center what is centrifugal and

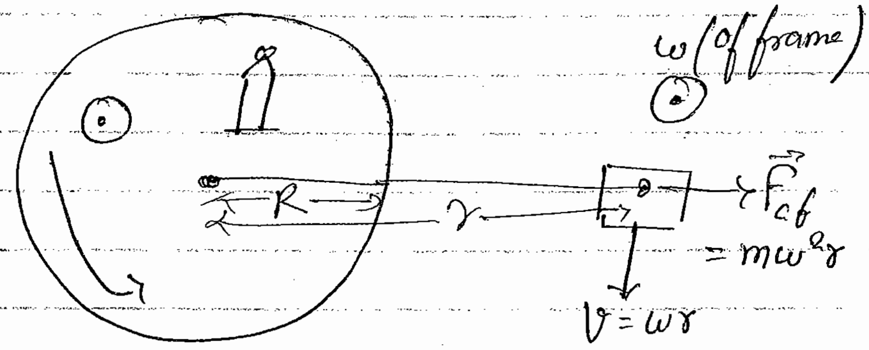


This velocity is due to rotation of rod.

Coriolis force on the object as seen by a person, standing on the disc.

Sol^n

$v = \omega r$] → Velocity arising out of rotation is given by $\omega r_\perp$



$\vec{F}_{cor} = 2m\cancel{v_r} \cdot \omega \sin 90°$

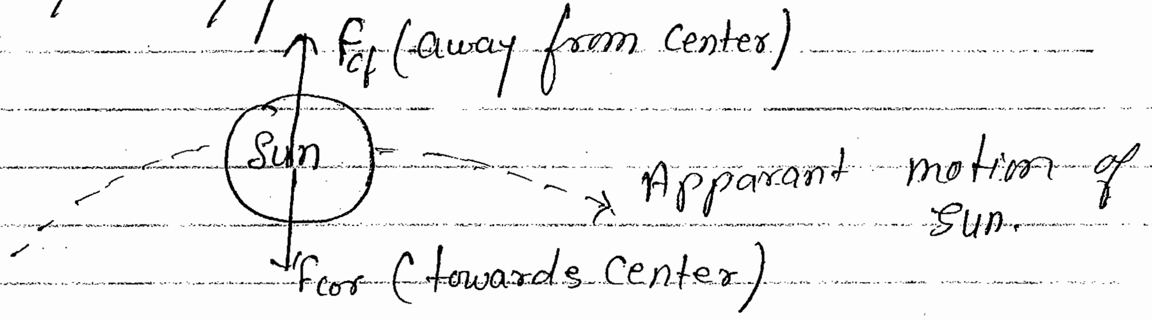
$= 2mV_r \cdot \omega \cdot 1$

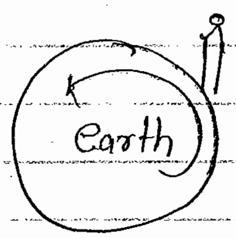
$= 2m\omega r \cdot \omega$

$$\boxed{\vec{F}_{cor} = 2m\omega^2 r}$$ Towards center.

$$\boxed{\vec{F}_{cf} = m\omega^2 r}$$ (away from center)

* Earth Sun System :-





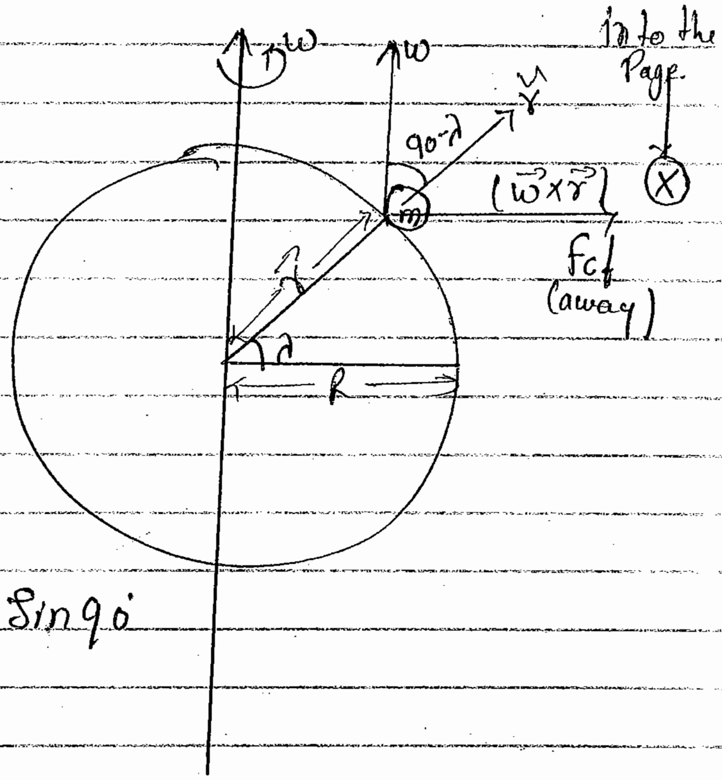
A-5
Q.5

2nd Second Method :-

$$\vec{F}_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{F}_{cf} = m(\vec{\omega} \times \vec{r}) \times \vec{\omega}$$

$$|\vec{F}_{cf}| = |m(\underline{\vec{\omega} \times \vec{r}})\ \underline{\times \vec{\omega}}|$$



$$\vec{F}_{cf} = m|\vec{\omega} \times \vec{r}|\,|\vec{\omega}| \sin 90^\circ$$

$$= m|\vec{\omega} \times \vec{r}| . \omega . 1$$

$$= m\omega |r| \sin(90-\lambda) . \omega . 1$$

$$\boxed{|\vec{F}_{cf}| = m\omega^2 R \cos\lambda}$$

Ans

* Earth: A non-Inertial (rotating) frame:

Earth spins about its axis, therefore, therefore it is a non-inertial frame.
Angular velocity of earth's spining motion is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24h} = 7.27 \times 10^{-5} \text{ rad/sec.}$$

Due to rotation of earth following effects arise.

(a) All objects on earth's surface experience centrifugal force which is directed away from the axis of rotation.

(b) All objects moving on earth's surface experience coriolis force in addition to centrifugal force.

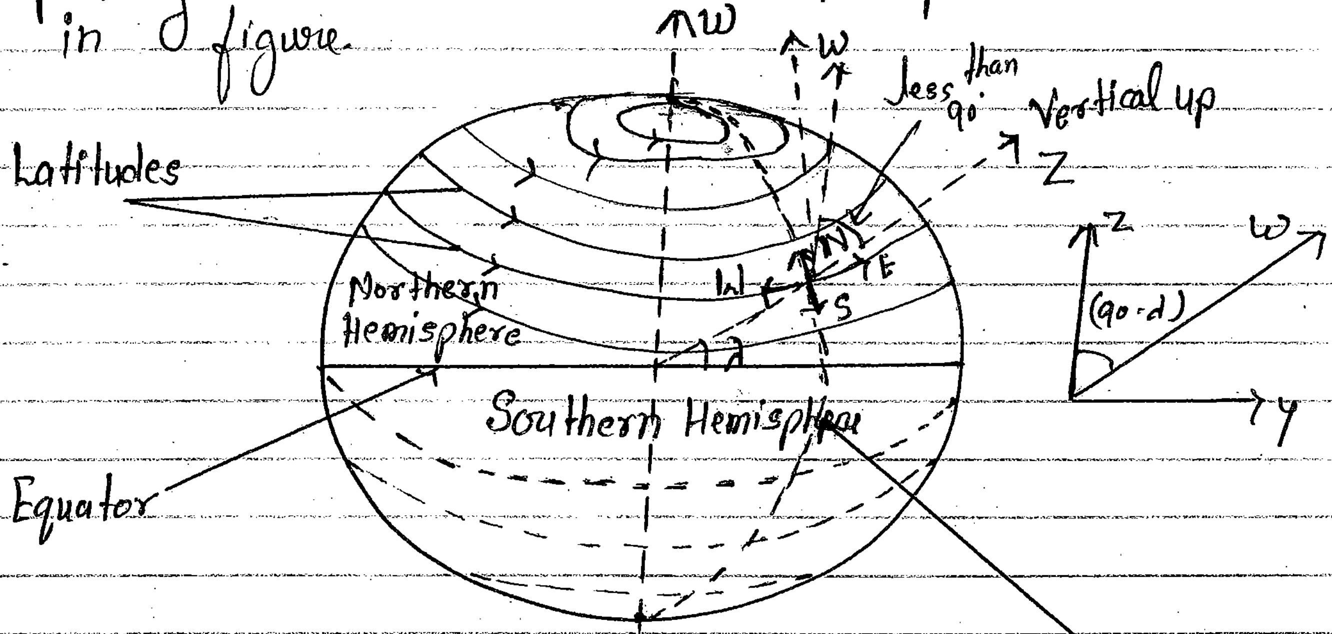
(C) An observer standing on earth's surface notices that on all objects a centrifugal and a coriolis force (if $v' \neq 0$) act irrespective of the location of the object. ~~This~~ That is the object which is being observed by the observer may or may not be on the earth's surface. For example we on the earth, see that the ~~Sun~~ Sun rises and sets. If we analyse the dynamics of the sun we will find that sun experiences a centrifugal force away from the earth and a coriolis force towards the earth.

Coordinate System on Earth's Surface:-

To analyse dynamics of objects from earth's surface we define local cartesian co-ordinate system as follows-

(i) +X-axis : Along local east, −X axis : Along local west
(ii) +Y-axis : Along local north, −Y axis : Along local south
(iii) +Z-axis : Along local vertical upward
−Z-axis : Along local vertical downward.

Therefore angular velocity vector of earth's spining motion lies in Y-Z plane as shown in figure.



A point in northern Hemisphere.

Longitude

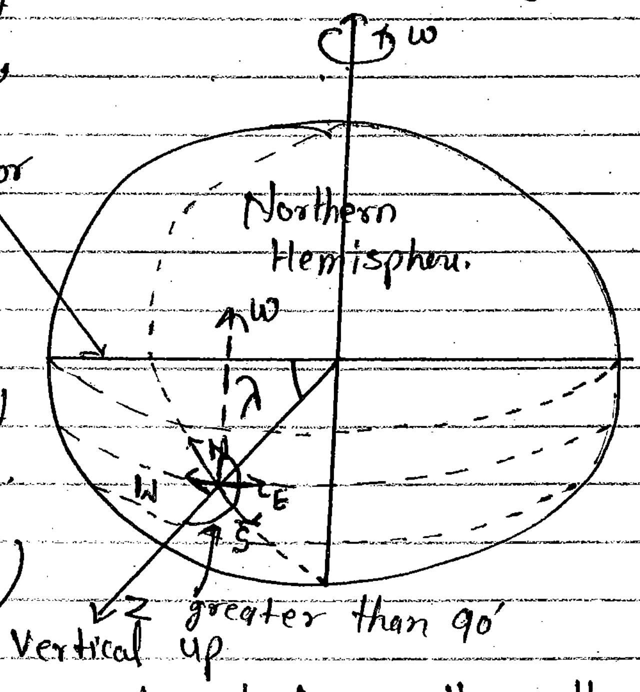
Therefore, at any point in northern hemisphere,

$\vec{w} = w(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$

When λ is latitude of the point.

And at any point in southern hemisphere.

$\vec{w} = w(\cos\lambda\,\hat{j} - \sin\lambda\,\hat{k})$



A point in southern Hemisphere.

A-5

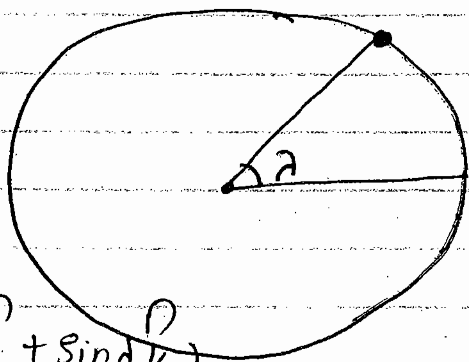
Q.6 In the previous question if the particle starts moving along the latitude from east to west with constant speed V_0, coriolis force on the particle will be (magnitude).

(a) $2m\omega V_0$ (b) $2mV_0\omega\cos\lambda$

(c) $2mV_0\omega\sin\lambda$ (d) $2mV_0\omega\sin\lambda\cos\lambda$

Soln.

$$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$



$$\vec{v} = -V_0\,\hat{i}$$

$$\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$$
$$= -2mV_0\omega\,\hat{i}\times(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

$$\vec{F}_{cor} = -2mV_0\omega(\cos\lambda\,\hat{k} - \sin\lambda\,\hat{j})$$

$$|\vec{F}_{cor}| = 2mV_0\omega\sqrt{\cos^2\lambda + \sin^2\lambda}$$

$$\boxed{|\vec{F}_{cor}| = 2mV_0\omega}$$

* Eastward deviation of a freely falling object in northern hemisphere:-

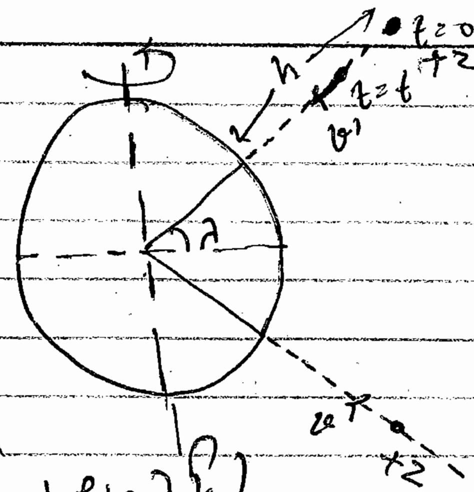
Suppose a particle falls from a height h at a place where latitude is λ.

Here we assume that $h << R_e$

Velocity of particle at time t is -

$$\vec{v}' = -gt\,\hat{k}$$

Coriolis force on the particle -



$$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

So $\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$

$$= -2m\omega g t\hat{k}\times(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

$$\vec{F}_{cor} = 2m\omega g\cos\lambda\, t\,\hat{i}$$

Force is towards east so particle will be deviate towards east, in both hemisphere.

Equation of motion of the particle in eastward direction is -

$$\vec{F}_{cor} = \frac{md^2x}{dt^3} \quad \text{or} \quad \frac{d^2x}{dt^2} = 2\omega g\cos\lambda t$$

We have following initial condition which we use to solve above equation -

at, $t=0$, $x=0, y=0, z=h$ —— (a)

at, $t=0$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$, $\frac{dz}{dt}=0$ —— (b)

Integrating the above differential equation we get, $\frac{dx}{dt} = \omega g\cos\lambda t^2 + c_1$

Using (b) we get $c_1 = 0$. Therefore $\frac{dx}{dt} = \omega g\cos\lambda t^2$

Integrating again we get, $x = \frac{1}{3}\omega g\cos\lambda\, t^3 + c_2$

Using (a) we get $c_2 = 0$

$$\therefore \boxed{x = \frac{1}{3}\omega g \cos\lambda\, t^3}$$

Where t is total time taken to reach earth surface.

$$\therefore t = \sqrt{\frac{2h}{g}}$$

$$\therefore \boxed{x = \frac{1}{3}\omega g \cos\lambda \left(\frac{2h}{g}\right)^{3/2}}$$

If a particle falls from a height of 100 metre at equator ($\lambda = 0$) then using above equation we get.

$$x = \frac{1}{3} \times 7.27 \times 10^{-5} \times 9.8 \times \left(\frac{2 \times 100}{9.8}\right)^{3/2}$$

$$= 2.19 \approx 2.2\,\text{c.m.}$$

Note :-

The expression derived for eastward deviation is an approximate one. In the derivation we have assumed that particle is uniformly accelerated in downward direction which is not true, when particle attains a velocity in X (each direction) there arises a component of coriolis force in vertical direction, due to which downward acceleration becomes a function of V_x'. Because of V_x' component of velocity, particle also experiences a force in southward direction. Therefore a freely falling object in northern hemisphere actually deviates in south-east direction.

$E \to \hat{\varphi}$, $N \to -\hat{\theta}$, $S \to \hat{\theta}$ | $E \to \hat{x}$, $N \to \hat{y}$, $W \to -\hat{x}$, $S \to -\hat{y}$ $\quad \vec{v} = v\hat{i}$

21/Aug/2014

* **Coriolis and Centrifugal forces :-**

$$\vec{F}_{cor} = 2m\,\vec{v}_r \times \vec{\omega} \qquad \text{Earth} \quad \omega = \frac{2\pi}{T}$$

$$\vec{F}_{cf} = m\,\omega^2 r_\perp \qquad\qquad = \frac{2\pi}{24}$$

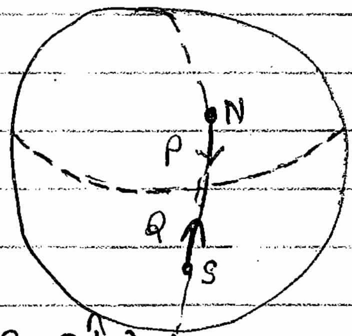
Since $\omega = 7.25 \times 10^{-5}$ rad/sec $\qquad \omega = 7.25 \times 10^{-5}$ rad/sec

In the Earth's case

$$|\vec{F}_{cf}| <<< |\vec{F}_{cor}|$$

So in the case of daviation we can not consider the Centrifugal force.

A-5
Q.20



Sol^n for P :-

$$v = v(-\hat{j})$$

$$\vec{F}_{cor} = 2m\,\vec{v} \times \vec{\omega}$$

$$= 2m\,v\omega\,(-\hat{j}) \times (\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

$$\vec{F} = 2mv\omega \sin\lambda\,(-i) \qquad \left\{ \because \omega = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k}) \right.$$

So force is in $-i$ direction or $-x$ direction
so daviation in $-x$ direction or West direction.

For Q :-

$$\vec{v} = v\,\hat{j}$$

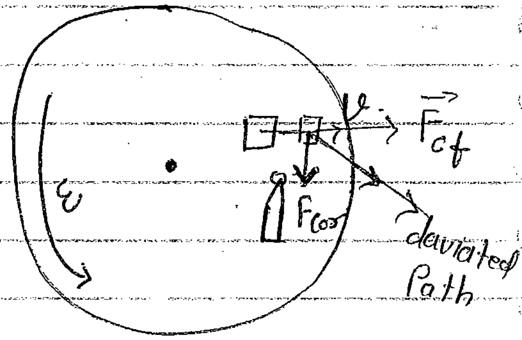
$$\vec{F}_{cor} = 2m v\omega\, \hat{j} \times (\cos\lambda\, \hat{j} - \sin\lambda\, \hat{k})$$

$$\vec{F}_{cor} = 2m v\omega \sin\lambda\, (-\hat{i})$$ {West direction}

So both P & Q daviate in West direction.

Q.21

$$\vec{F}_{CF} = m\omega^2 r_\perp$$



option ①.

y
x

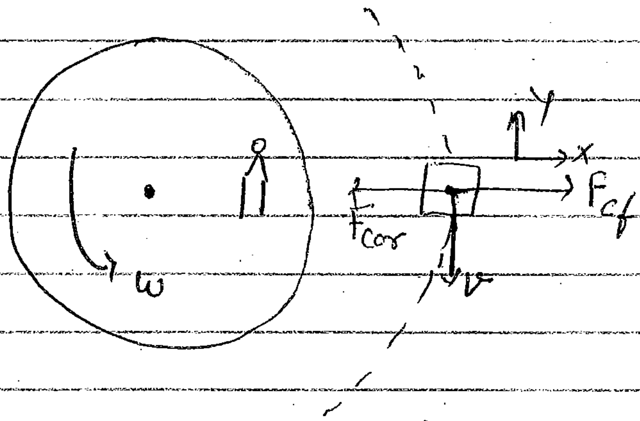
$\vec{v} \times \vec{\omega}$

$\hat{i} \times \hat{k} = -\hat{j}$

So F_{cor} is in $-y$ direction

Q.1 :-

Ⓒ

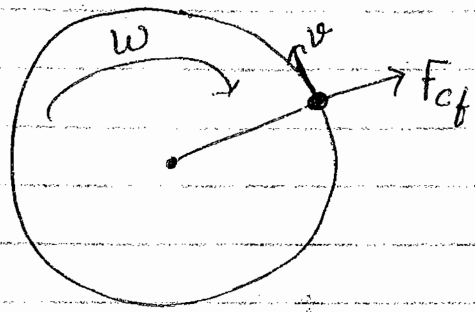


$v \times \omega$

$= (-\hat{j}) \times \hat{k}$

$= -\hat{i}$

Q.2



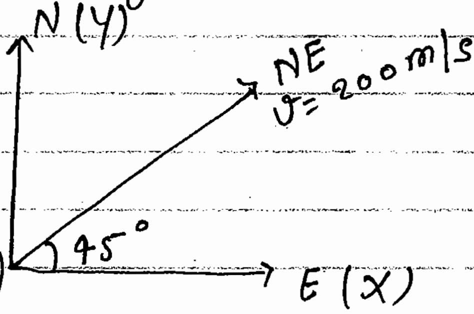
Direction of centrifugal force is independent of velocity

$$\vec{F}_{cf} = m\omega^2 r_\perp$$
$$= m\omega^2 R$$

Q.11

Sol^n

$\lambda = 45°$ N , $m = 2$ kg



$$\vec{F}_{cor} = 2m\,\vec{v} \times \vec{\omega}$$

0.05 $= 2 \times 2 \{200 (\cos 45\,\hat{i} + \sin 45\,\hat{j})$

$\times\ 7.25 \times 10^{-5} (\cos 45\,\hat{j} + \sin 45\,\hat{k})$

$= 4 \{200 \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right) \times 7.25 \times 10^{-5} \left(\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right)\}$

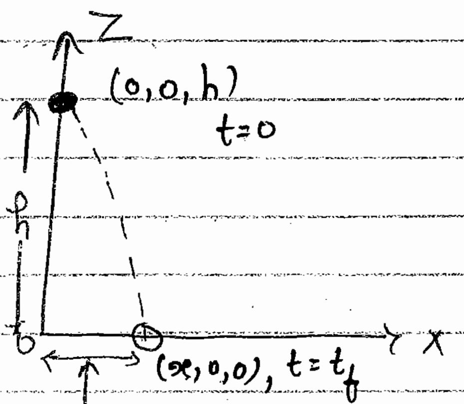
$=$

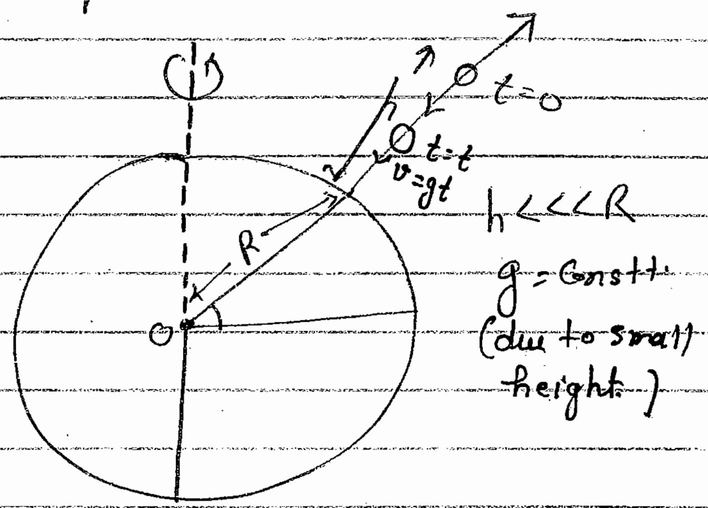
$\vec{v} = 200 (\cos 45\,\hat{i} + \sin 45\,\hat{j})$

$\vec{\omega} = \omega (\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$

$\vec{\omega} = 7.25 \times 10^{-5} (\cos 45\,\hat{j} + \sin 45\,\hat{k})$

* Value of East Ward deviation of an object falling on Earth Surface :-





Eastward daviation or distance coverd in East direction.

$$\vec{F}_{cor} = 2mv(-\hat{k}) \times \omega(\cos\lambda\,\hat{j} \pm \sin\lambda\,\hat{k})$$

$$\boxed{\vec{F}_{cor} = 2mv\cos\lambda\,\hat{i}}$$

So the direction of force is $\hat{i}$ or $+x$ direc^n. or in East direction.

Accelaration in East direction -

$$a_x = \frac{F_x}{m} = \frac{2\cancel{m}\,v\,\omega\cos\lambda}{\cancel{m}}$$

$$a_x = 2v\omega\cos\lambda$$

$$\boxed{a_x = 2gt\omega\cos\lambda} \neq \text{Constt.}$$

Velocity at $t = t$

$v = u + gt$

$v = 0 + gt$

$v = gt.$

∵ Acceleration is not constt. so we use defination.

$$a_x = \frac{dv_x}{dt}$$

$$\frac{dv_x}{dt} = 2gt\omega \cos\lambda$$

$$v_x = 2g\omega\cos\lambda \cdot \frac{t^2}{2}$$

$$\frac{dx}{dt} = g\omega t^2 \cos\lambda$$

$$\int_0^x dx = g\omega\cos\lambda \int_0^t t^2\,dt$$

$$\boxed{x = \frac{1}{3} g\omega\cos\lambda\, t_f^3}$$ $\left\{ t_f = \text{time of fall} \right\}$

⇒ To write eastward deviation in terms of height, Consider vertical motion :-

$$s = ut + \frac{1}{2}at^2$$

$u = 0,\ t = 0$

$a = g = \text{Constt.}$

$t = t_f$

$$h = 0 + \frac{1}{2} g t_f^2$$

$$\boxed{t_f = \sqrt{\frac{2h}{g}}}$$

$$\boxed{x = \frac{1}{3} g\omega\cos\lambda \left(\frac{2h}{g}\right)^{3/2}}$$ → Memorise.

This is the final expression for eastward deviation.

At equator $\lambda = 0$

If $h = 100$ meter

So $x = 2.2$ cm.

Helix

Eastward deviation is maximum at equator and minimum at poles {∵ at poles $\theta = 90°$ and $\cos 90° = 0$ }.

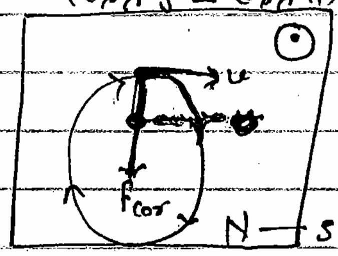
* Direction of deviation of horizontally thrown object on earth surface :-

Note :-

Relation b/w Magnetic force and Coriolis force :-

$$\vec{F}_{cor} = 2m\,\vec{v} \times \vec{\omega}$$

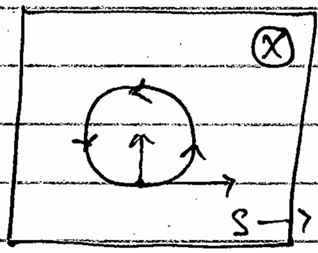
$\omega = \omega(\cos\lambda\,\hat{j} \pm \cos\lambda\,\hat{k})$



$\omega \perp$ (due to +ve sign)

Clockwise

N → sphere

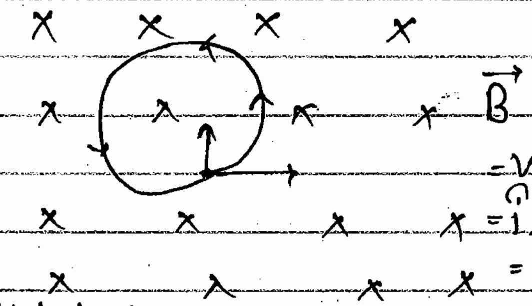


$\omega \perp$ (due to negative sign)

Anti-Clockwise

S → sphere

$$\vec{F}_m = q\,\vec{V} \times \vec{B} \quad \{B = \text{Const}\}$$



$= \vec{V} \times \vec{B}$

$= \hat{i} \times (-\hat{k})$

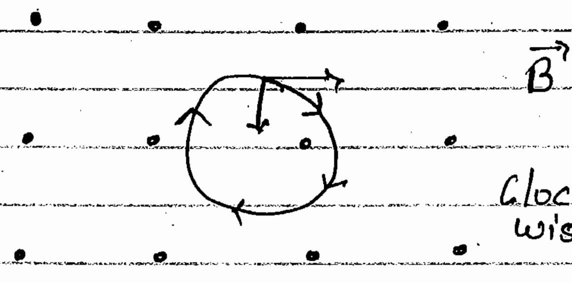
$= \hat{j}$

Anticlockwise Path

$\vec{V} \perp \vec{B}$

If we ~~throw~~ thrown a charge perpendicular to the direction of field the charge move in a anti-clockwise circle.

*



Clock wise

If $\vec{V} \perp \vec{B}$ → Circle path

$\vec{V}$ is not perpendicular $\vec{B}$ → Helix

Conclusion :-

"Due to Coriolis force a horizontally thrown object will move clockwise in N-sphere and Anticlockwise in S-sphere."

* Radius of path followed by particle due to Coriolis force :-

In such cases :-

Centipital force = Coriolis force

$$\frac{\cancel{m}v^{\cancel{2}}}{r} = 2\cancel{m}\cancel{v}\omega_{\perp} \sin 90^\circ$$

$$r = \frac{v}{2\omega_{\perp}} = \frac{v}{2m \sin\lambda}$$

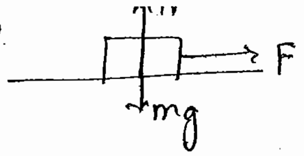
$$\boxed{r = \frac{v}{2\omega \sin\lambda}}$$

Deviation is minimun at equator for a vertically thrown particle.

* Coriolis and Centrifugal forces :-

Net 2011
Q.24

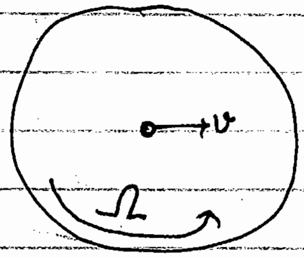
Normal force arises due to responce of perpendicular force



Ans

$\vec{F}_{cf} = m\omega^2 r_\perp$ ← Just after firing.

$r_\perp = 0$ { bcoz initialy is at centre of rotating frame }



So $\boxed{\vec{F}_{cf} = 0}$

$\vec{F}_{cor} = 2m\vec{v} \times \vec{\omega}$

$|\vec{F}_{cor}| = 2mv\Omega \sin 90° = 2mv\Omega$

$$a = \frac{\text{Force}}{\text{Mass}}$$

$$= \frac{2\cancel{m}v\Omega}{\cancel{m}}$$

$= 2v\Omega$ to his right.

Ques In Q. N. 24 if the person is standing on the edge of the plateform and fires the bullet in radially inward direction what is the acc^n of prevous frame just after the firing, if radius of the plateform is R ?

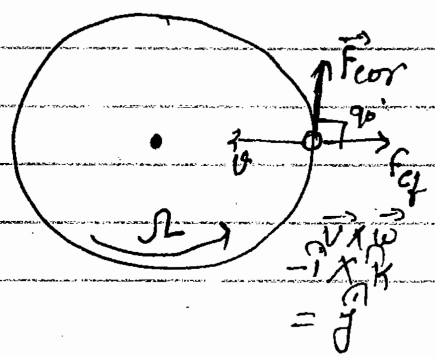
Sol^n

Angular velocity $= \Omega$
initial velocity $= v$

$r_\perp = R$

So

$f_{cf} = m\Omega^2 R$



Speed produced by rotation $\omega = \omega \times r$

$$\vec{F}_{cor} = 2m\,\vec{\omega} \times \vec{v}$$
$$= 2mv\Omega \sin 90^\circ = 2mv\Omega$$
$$\vec{F}_{cor} = 2mv\Omega$$

So Resultant force :-

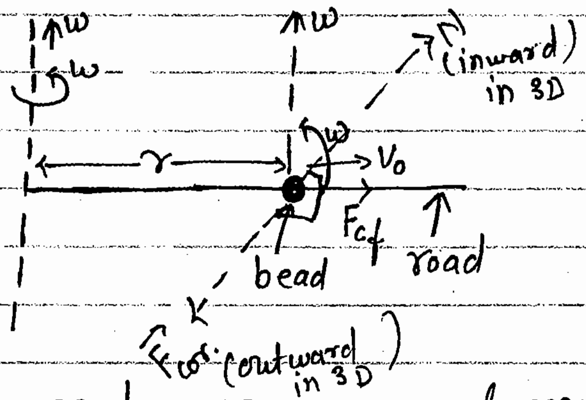
$$= \sqrt{\vec{F}_{cf}^2 + \vec{F}_{cor}^2}$$
$$= m\sqrt{\Omega^4 R^2 + 4\Omega^2 V^2}$$

$$\vec{a} = \frac{\text{resultant force}}{\text{mass}}$$

$$\boxed{\vec{a} = \sqrt{\Omega^4 R^2 + 4\Omega^2 V^2}}$$

Ans

A-5
Q.16



$\vec{f}_{cor} = 2mV_0\omega$

$\because$ Here normal reaction arises due to coriolis forces.

Here F_{cor} is $\perp$ to rod so normal reaction will arise in response to $\vec{F}_{cor}$.

$$\therefore N = \vec{F}_{cor}$$

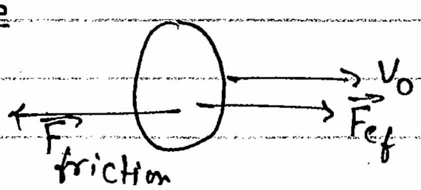
$$\boxed{N = 2mV_0\omega}$$

Ans

A-5
Q. 17.

Soln

Here speed v must be constant

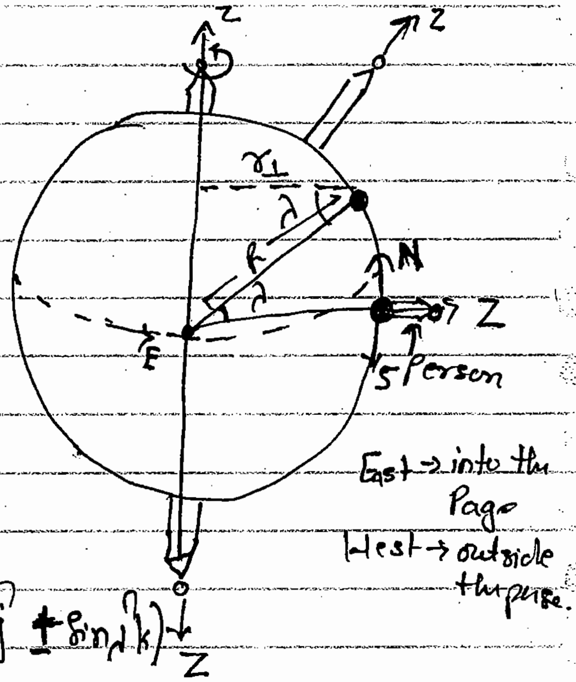


$\therefore$ $\vec{F}_{cf}$ is cancelled (balanced by) another force which is friction force.

$$\boxed{\vec{F}_r = \vec{F}_{cf} = m\omega^2 r}$$

A-5
Q. 9

Soln



$r_\perp = R\cos\lambda$

$\lambda = 0$ (at equator)

$r_\perp = R$

$\vec{F}_{cf} = m\omega^2 R$

$\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$

$= 2m(\pm v\hat{i}) \times \omega(\cos\lambda\,\hat{j} \pm \sin\lambda\,\hat{k})$

$= 2m(\pm v\hat{i}) \times \omega\,(\cos 90\,\hat{j} \pm \sin 90\,\hat{k})$

$\because$ $\lambda = 90$ at equator

$$\boxed{\vec{F}_{cor} = 2m\omega v\,(\pm\hat{k})}$$

quator is a latitude. Train moving
Along the latitude means train. move either East or West
for East +V for West -V.

If object moving towards east then $(+\hat{k})$ &
if moving toward west then $(-\hat{k})$.
the F_{cor} is radially outward and radially inward respectivily.

$$\vec{F}_{cf} = \vec{F}_{cor}$$

$$m\omega^2 R = 2m\omega V$$

$$\boxed{V = \frac{\omega R}{2}}$$

Ans

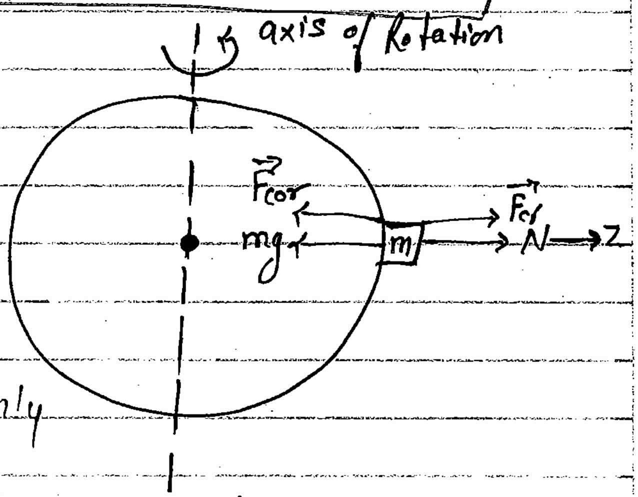
A-5
Q.10

Soln

$$\boxed{\text{Apparant Weight} = \text{Normal Reaction}}$$

OR

$$\boxed{\text{Reading of machine} = \text{Normal Reaction}}$$



In order to $N = mg$
then $\vec{F}_{cf} = \vec{F}_{cor}$
So $mg \Rightarrow$ true weight
is equal to normal reaction.

In previous ques $\vec{F}_{cor} = \vec{F}_{cf}$ only when
Person is running from East to west.
so option (c) is correct.

Qus 12

Meridian Plane

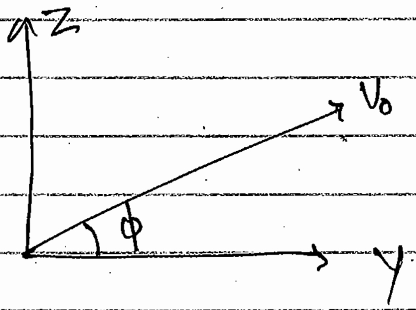
Magnetic Maridian Plane | Geographic Meridian Plan.

Meridian Plane = A vertical plane.

When we take a horizontal plane in East - West direction then the meridian plane is a vertical plane $\perp^{or}$ to the horizontal plane in North - South direction.

$\vec{v} = v_0 (\cos\phi\,\hat{j} + \sin\phi\,\hat{k})$

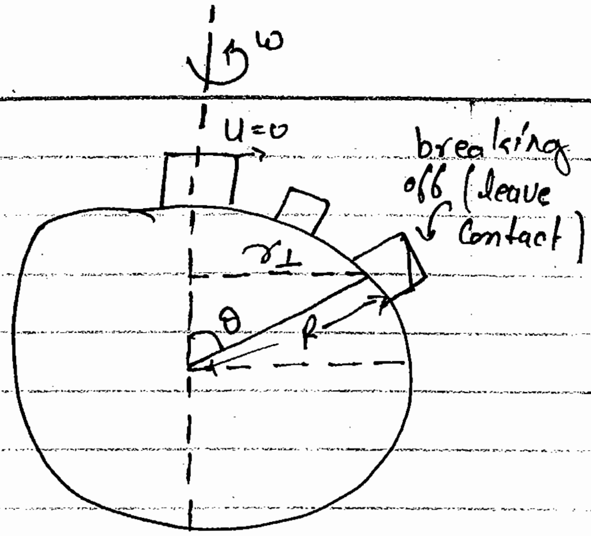
$\vec{\omega} = \omega (\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$



(if not given hemisphere then always take Northarn hemisphere.

$\vec{F}_{cor} = 2m\vec{v} \times \vec{\omega}$

Q.14



$\boxed{\theta = \cos^{-1}\frac{2}{3}}$ → see the booklet.

Sol^n

$$\vec{F}_{cf} = m\omega^2 r_\perp$$

$$r_\perp = R\sin\theta$$

$$\vec{F}_{cf} = m\omega^2 R\sin\theta$$

$$= m\omega^2 R\sqrt{1-\cos^2\theta}$$

$$\vec{F}_{cf} = m\omega^2 R\sqrt{1-\frac{4}{9}}$$

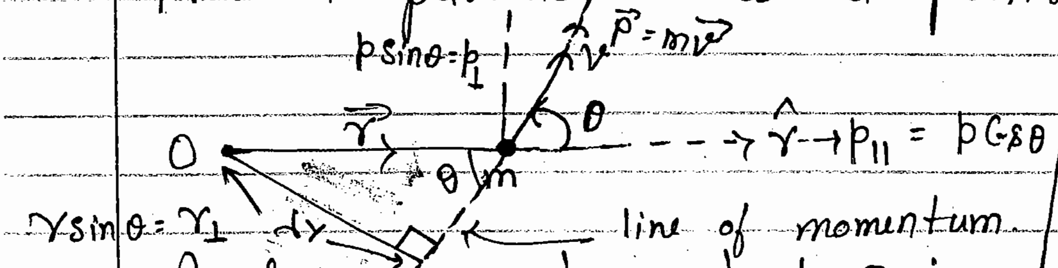
$$\boxed{\vec{F}_{cf} = \frac{\sqrt{5}}{3}\, m\omega^2 R}$$ Ans

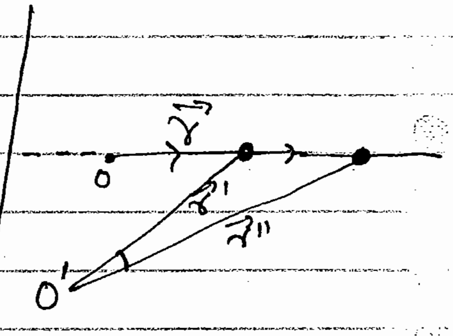
{ it is a famous prob. and for θ see booklet }

[Rigid body dynamics is important topic for NEET]

* Angular Momentum And Torque :-

{For a particle about a point} :-





Angular momentum about O :-

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} \Rightarrow \text{Vector form}$$

$\vec{r}$ = position of particle from point O.

Use right hand thumb rule to find the direction of $\vec{L}$.

Magnitude form :-

$$\boxed{|\vec{L}| = rp\sin\theta = rp_\perp}$$

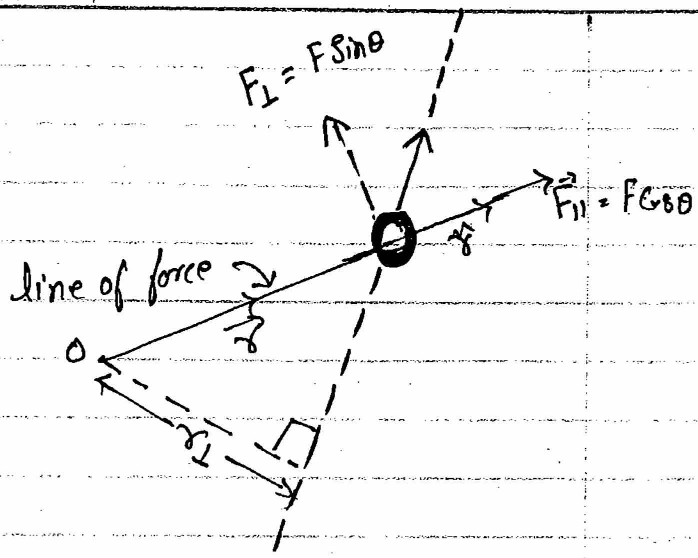
$$(\because p_\perp = p\sin\theta)$$

$$\therefore \boxed{|\vec{L}| = r_\perp p}$$

$$(\because r_\perp = r\sin\theta)$$

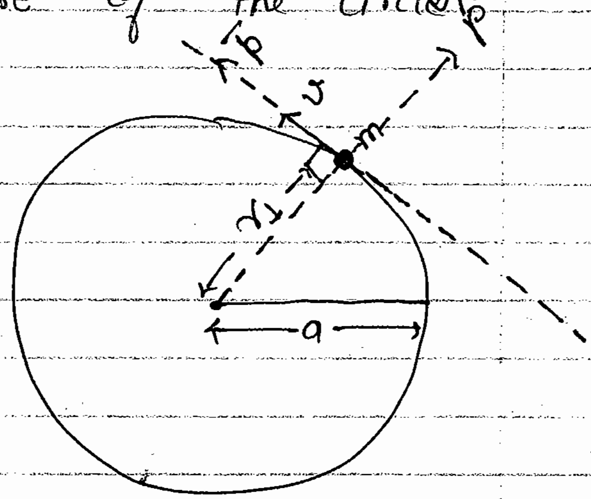
$$\therefore \boxed{L = rp\sin\theta = rp_\perp = pr_\perp}$$

* Torque :-



Qu. A particle is moving in a circle of radius 'a' with constant speed v calculate its angular momentum about its centre of the circle.

Solⁿ



$$L = p\, r_\perp$$

$$\boxed{L = mva}$$

$(\because p = mv)$ $r_\perp = a$

for Direction -

$$\vec{L} = \vec{r} \times \vec{p} \quad \{\text{finger curl from } r \text{ to } p\}$$

~~So direction is up the plane of paper.~~ Up the plane of paper.

Qu. A particle of mass m is moving with speed v_0 along a straight line $y = (\alpha x + \beta)$ $\alpha > 0$ $\beta > 0$ In x-y plane. Calculate angular momentum about origin.

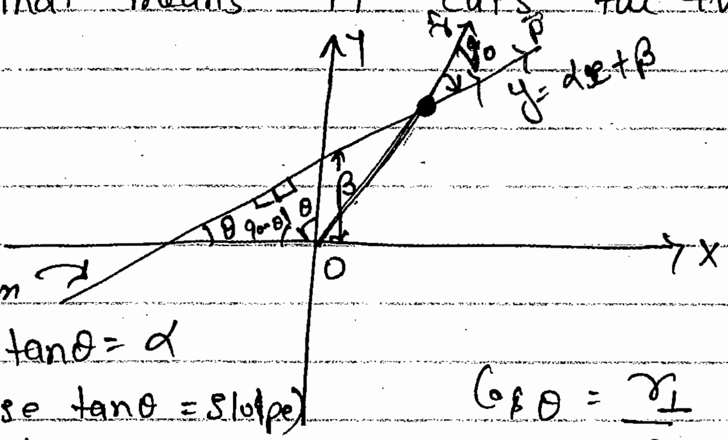
Sol^n

$$y = \alpha x + \beta$$

α ↓ Slope β ↓ intersept

Slope is +ve that means angle from +ve x direction is less than 90°.

Intersept is +ve that means it cuts the +ve Y-axis.



$$L = p r_\perp \quad \text{—①}$$

→ line of momentum (Particle moving about this line)

$\tan\theta = \alpha$ (becouse $\tan\theta$ = slope which is α.)

$$\cos\theta = \frac{r_\perp}{\beta}$$

$$\because \tan\theta = \frac{\alpha}{1} = \frac{\text{Perpendicular}}{\text{Base}} \quad \because r_\perp = \beta\cos\theta$$

$$\cos\theta = \frac{r_\perp}{\beta} = \frac{\text{Base}}{\text{Hypoten.}}$$

So $$r_\perp = \beta\cdot\frac{\text{Base}}{\text{Hypoten.}} = \beta\cdot\frac{1}{\sqrt{\alpha^2+1}}$$

So $$\boxed{r_\perp = \frac{\beta}{\sqrt{\alpha^2+1}}}$$

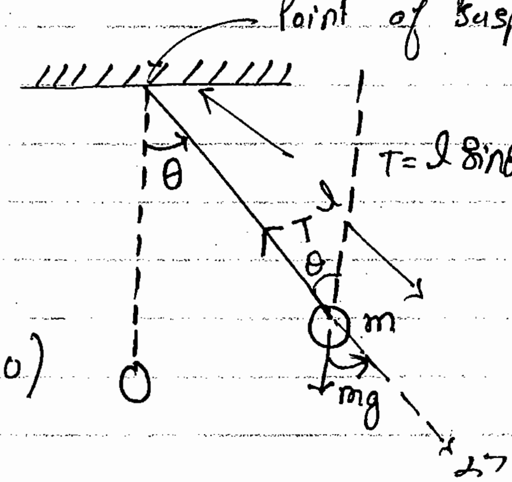
So $$\boxed{L = m v_0 \cdot \frac{\beta}{\sqrt{\alpha^2+1}}}$$ {from ①}

direction of angular momentum is into the page or $(-z)$ axis.

$r \times p$

Que A simple pendulum consisting of a small bob of mass m and light string of length l is deflected by an angle θ from lowest position. Calculate torque about point of suspention in this position.

Sol^n



$\tau = F r_{\perp}$

① Due to tension :-

$\tau = T \cdot 0 = 0 \quad (\because r_{\perp} = 0)$

$\boxed{\tau = 0}$

If a force passes through a point so that force will not produce any torque about point.

② Due to mg :-

$\tau = mg \cdot l\sin\theta$

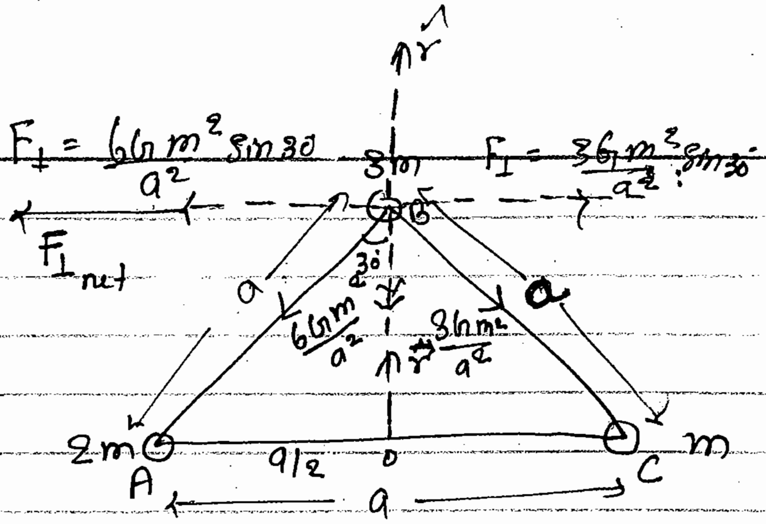
$\boxed{\tau = mgl\sin\theta} \quad (r_{\perp} = l\sin\theta)$

Direction of τ is into the plane of paper.

Que What is torque on mass 3m due to forces of other masses about point O ?

Sol^n

(1) $\tau = F_{\perp} r$



$r = \frac{\sqrt{3}}{2} a$

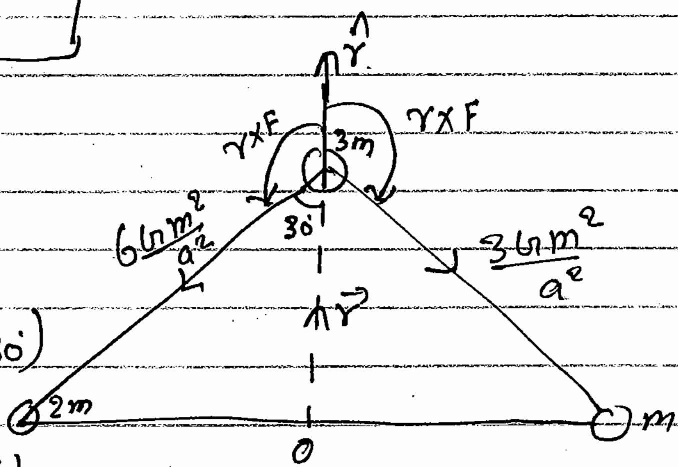
$F_{\perp net} = \frac{6m^2G}{a^2}\sin 30° - \frac{3Gm^2}{a^2}\sin 30°$

$F_{\perp net} = \frac{3Gm^2}{a^2}\sin 30°$

So $\tau = \frac{3Gm^2}{a^{\cancel{2}}} \cdot \frac{\sqrt{3}}{2}\cancel{a} \cdot \frac{1}{2}$

$$\boxed{\tau = \frac{3\sqrt{3}Gm^2}{4a}}$$

Second Method :-



$\tau = rF\sin\theta$

$= \frac{\sqrt{3}}{2} a \frac{6Gm^2}{a^2}\sin(\pi - 30°) - \frac{\sqrt{3}}{2} a \frac{3Gm^2}{a^2}\sin(\pi - 30°)$

$\tau = \frac{\sqrt{3}}{2} a \frac{3Gm^2}{a^2}\{2 - 1\}\sin(\pi - 30°)$

$= \frac{3\sqrt{3}}{2}\frac{Gm^2}{a}\sin 30° = \frac{3\sqrt{3}}{2}\frac{Gm^2}{a}\cdot\frac{1}{2}$

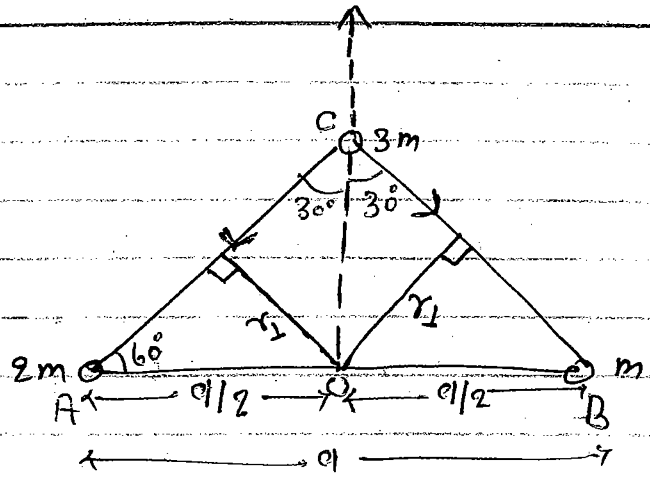
$$\boxed{\tau = \frac{3\sqrt{3}}{4}\frac{Gm^2}{a}}$$ up the plane of paper.

Ans

Third Method :-

$$\vec{\tau} = \vec{F} r_{\perp}$$

$$\sin 60^\circ = \frac{r_{\perp}}{a/2}$$



$$\frac{\sqrt{3}}{2} = \frac{2r_{\perp}}{a}$$

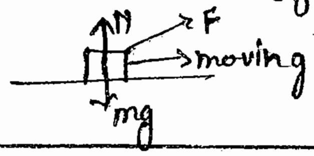
$$\boxed{r_{\perp} = \frac{\sqrt{3}a}{4}} \quad , \quad \vec{F} =$$

$$\vec{F} = \frac{6Gm^2}{a^2} \sin 30^\circ$$

$$\tau = \vec{F} r_{\perp}$$

$$\tau =$$

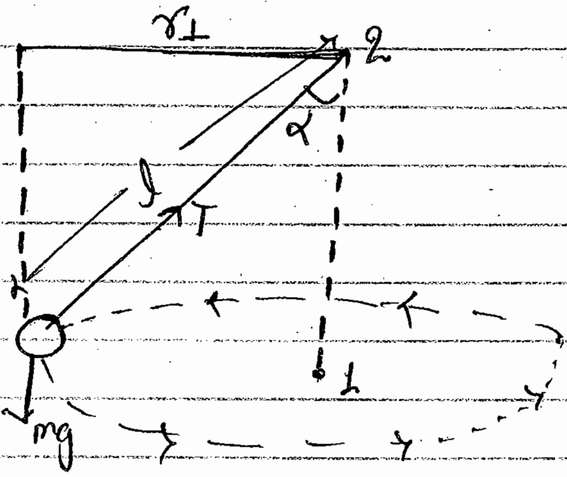
force is resolve in the direction of motion.



28/Aug/2014

Q. A conical pendulum consist of a bob of mass m and light string of length l. If string of pendulum makes an angle α with downward vertical.

Sol^n Torque about point 2:-



Here tension is not produces Torque because it passes through the point of suspension.

So $\tau_2 = mg\, r_\perp$

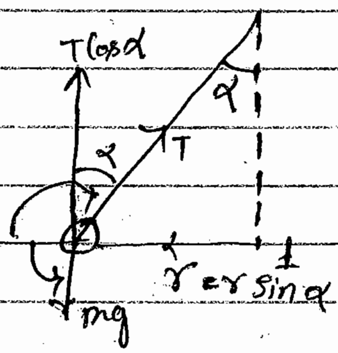
$$\boxed{\tau_2 = mgl\sin\alpha}$$

About point 1:-

$\tau = \vec{F}_\perp r$, $\tau = F r_\perp$

$\tau_1 = mgl\sin\alpha - T\cos\alpha \cdot l\sin\alpha$

$$\boxed{\tau_1 = l\sin\alpha\,(mg - T\cos\alpha)}$$

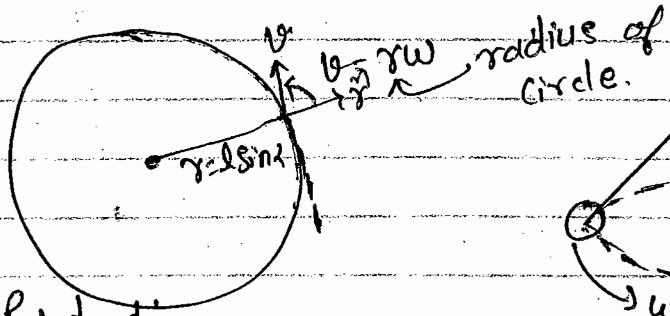


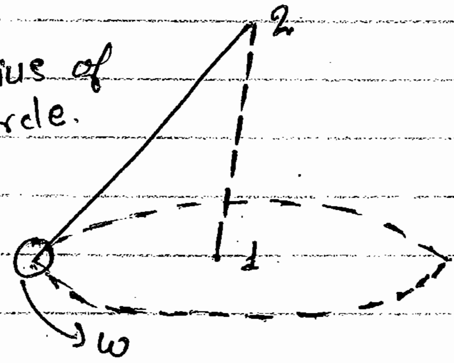
∵ Here there is vertical equilibrium.

So ∵ $T\cos\alpha = mg$.

$\vec{L} = \vec{r} \times \vec{p}$

Ques In previous question if angular speed of conical pendium is ω Calculate angular momentum about points 1 and 2.





About Point 1 :-

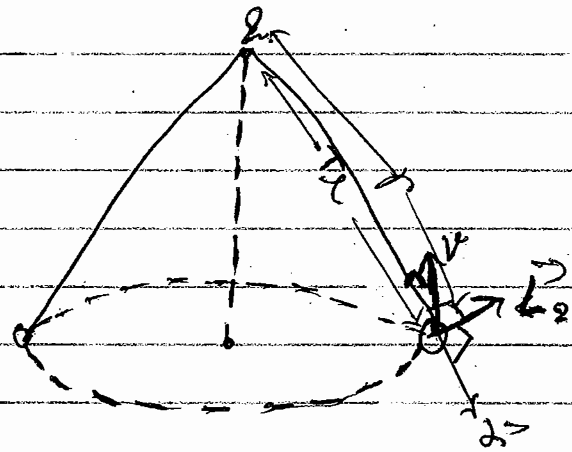
$L = p \cdot r_{\perp}$

$= mv \cdot r$

$= m\omega r^2$

$\boxed{L_1 = m\omega l^2 \sin^2\alpha}$ (And direction of angular momentum along the vertical line.)

About point 2 :-



$L_2 = p r_{\perp}$

$= mv r_{\perp}$

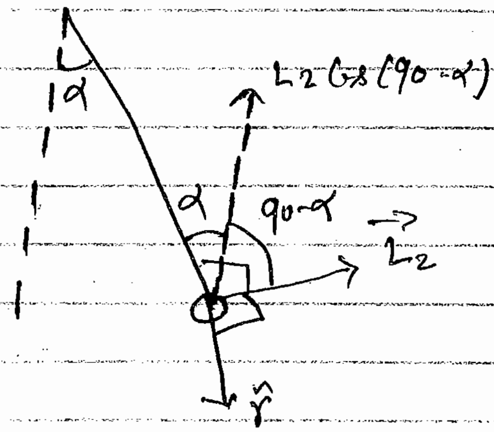
$= m\omega l \sin\alpha \cdot r_{\perp}$

$= m\omega l \sin\alpha \cdot l$

$\boxed{L_2 = m\omega l^2 \sin\alpha}$

It is not along the vertical line. It is perpendicular to string.

Let us take component of L_2 along vertical line,



L_2 (along vertical line) =

$= L_2 \sin\alpha$

$= m\omega l^2 \sin\alpha . \sin\alpha$

$$\boxed{L_2 = m\omega l^2 \sin^2\alpha = L_1}$$

Conclusion :-

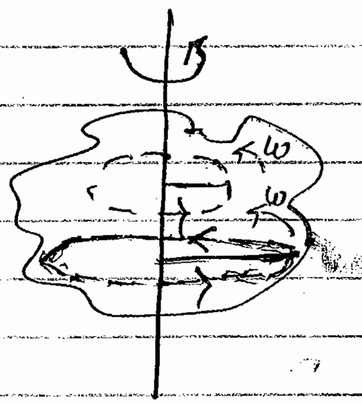
If we calculate angular momentum about different points of axis of rotation then magnitude as well as direction will be different. However value of angular momentum about or along the axis [i.e. Component along the axis is always same].

And its value is given as –

$$\boxed{L_{axis} = m\omega r_\perp^2} \quad \left\{ \text{for Single Particle.} \right\}$$

Where $r_\perp$ = perpendicular distance from axis of Rotation

* If there is a rigid body :-



So every particle moves in a circle of different radius. but same angular speed.

For a rigid body :-

$$L_{axis} = \sum_i m_i \omega r_{\perp i}^2$$

$$\boxed{L_{axis} = \omega \sum_{i=1}^{n} m_i r_{\perp i}^2}$$ for system of particles.

This may or may not be total angular momentum.

$$\boxed{L_{axis} = \omega I}$$

$$I = \sum_{i=1}^{N} w_i r_{\perp i}^2 = \text{Moment of Inertia}$$

* Moment of Inertia about an axis:-

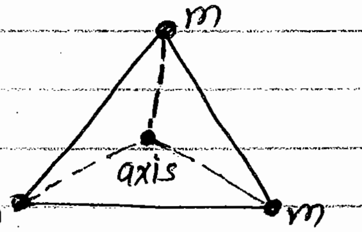
$$\boxed{I = \sum_{i=1}^{N} m r_{\perp i}^2} \rightarrow \text{for descreate case}$$

$$\boxed{I = \int dm\, r_{\perp}^2} \longrightarrow \text{Continuous case}$$

$r_{\perp}$ = $\perp^r$ distance of elementary part dm from axis.

Q Three particles of mass m are placed at the corners of an equilateral triangle of side 'a'. Calculate M.I. about axis passes through centroide and $\perp^r$ to its plant.

Sol^n

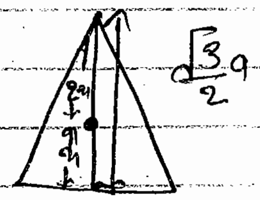


$$I = \sum_{i=1}^{3} m_i r_{\perp i}^2$$

$$I = m\frac{a^2}{3} + m\cdot\frac{a^2}{3} + m\cdot\frac{a^2}{3}$$

$$= \frac{\cancel{3}ma^2}{\cancel{3}}$$

$$\boxed{I = ma^2}$$



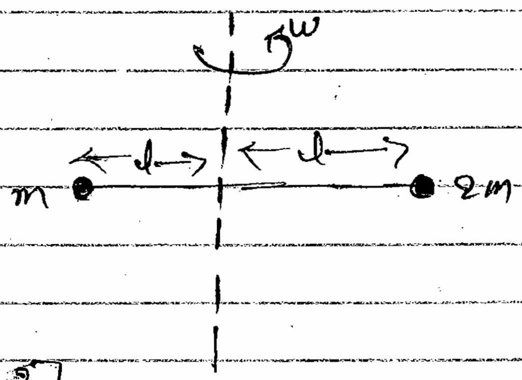
$$3x = \frac{\sqrt{3}}{2}a$$

$$x = \frac{a}{2\sqrt{3}}$$

$$2x = \frac{a}{\sqrt{3}}$$

Jul-2011

Q What is angular momentum about axis of rotation?



$$L_{axis} = I\omega$$

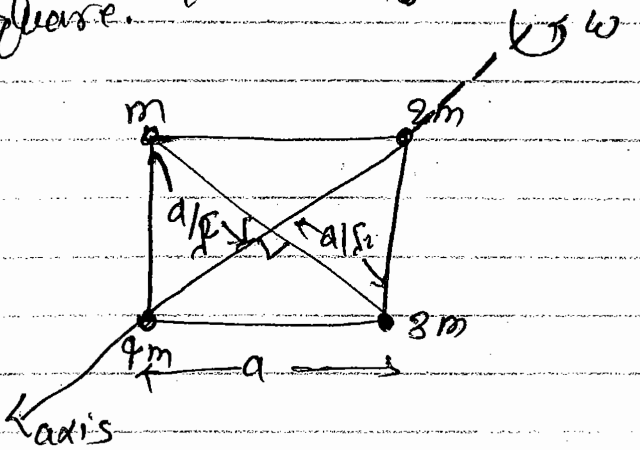
$$= \omega\left[\sum_{i=1}^{2} m_i r_{\perp i}^2\right]$$

$$= \omega\left[m \times l^2 + 2ml^2\right]$$

$$= 3ml^2\omega$$

Q. Four particles are placed at corners of square as shown in fig. Calculate M.I. of system about diagonal of square.

Sol^n



$$I = \sum_{i=1}^{4} m_i r_{\perp i}^2$$

$$= m \times \frac{a^2}{2} + 2m \times 0 + 3m \times \frac{a^2}{2} + 4m \times 0$$

$$= 4m \times \frac{a^2}{2} = 2ma^2$$

$$\boxed{I = 2ma^2}$$

Note :-

<u>Mass density</u> :-

Linear $(\lambda) = \frac{\text{mass}}{\text{length}} = \frac{m}{l} = \frac{dm}{dl}$ → non-Uniform

Surface $(\sigma) = \frac{\text{mass}}{\text{area}} = \frac{dm}{dA} = \frac{m}{A}$ → Uniform

Volume $(\rho) = \frac{\text{mass}}{\text{Volume}} = \frac{dm}{dV} = \frac{m}{V}$

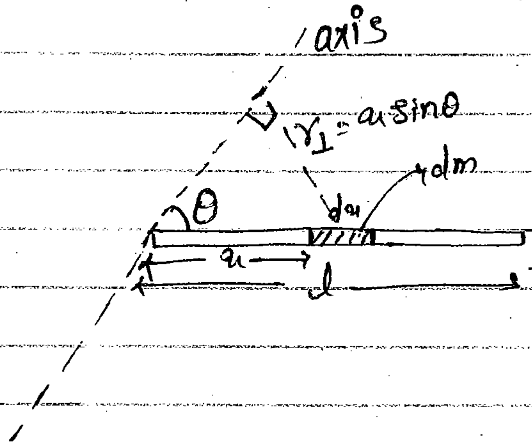
* M.I. of a Continuous object :-

A thin rod {Uniform} :-

What is moment of Inertia about axis shown in figure.

Mass of rod = m

length of rod = L



$$I = \int dm\, r_\perp^2$$

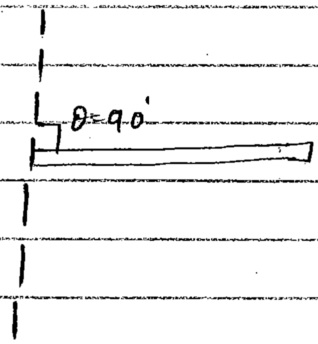
$$= \int_0^L \frac{M}{L} dx \cdot x^2 \sin^2\theta$$

$$= \frac{M}{L}\sin^2\theta \int_0^L x^2\, dx = \frac{ML^2}{3}\sin^2\theta$$

Case I :-

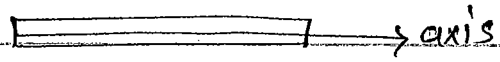
If $\theta = 90°$

then $\boxed{I = \frac{ML^2}{3}}$



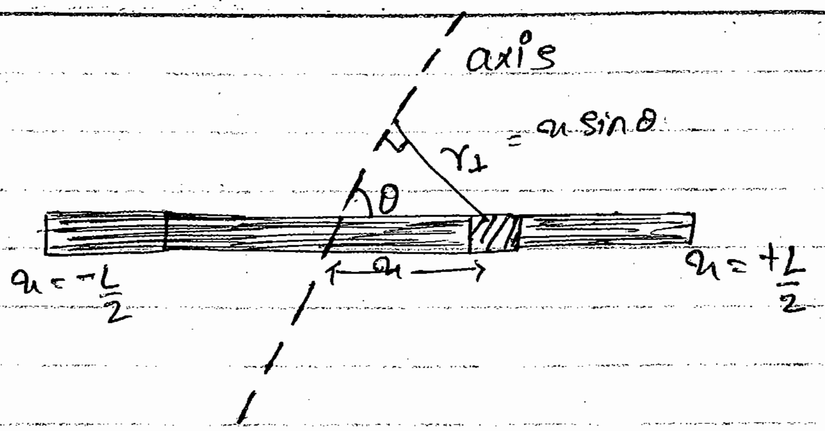
Case II :-

If $\theta = 0°$



then $\boxed{I = 0}$

Case III :-



$$I = \int_{-\frac{L}{2}}^{+\frac{L}{2}} dm \; r_\perp^2$$

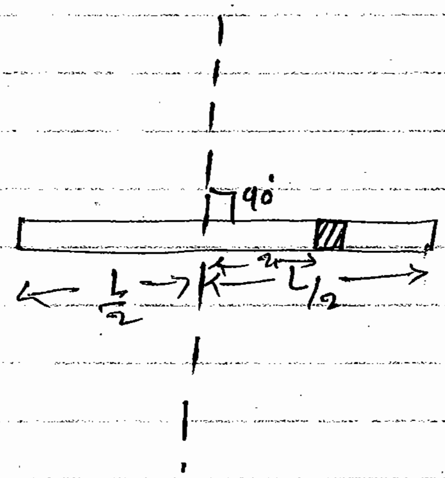
$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} dx \cdot x^2 \sin^2\theta$$

$$= \frac{M}{L} \sin^2\theta \int_{-\frac{L}{2}}^{+\frac{L}{2}} x^2 dx = \frac{M}{L} \sin^2\theta \left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{+\frac{L}{2}}$$

$$= \frac{1}{3}\frac{M}{L}\sin^2\theta \left[\frac{L^3}{8} + \frac{L^3}{8}\right] = \frac{1}{3}\frac{M}{\cancel{L}}\sin^2\theta \left[\frac{2L^{\cancel{3}}}{\cancel{8}4}\right]$$

$$\boxed{I = \frac{ML^2 \sin^2\theta}{12}}$$

Case - IV :-



Q. Four thin rods adjoit to get to form a square loop. If mass of thin rod is M and length is L. What is moment of inertia of square plan about it ~~square~~ diagonal plan.

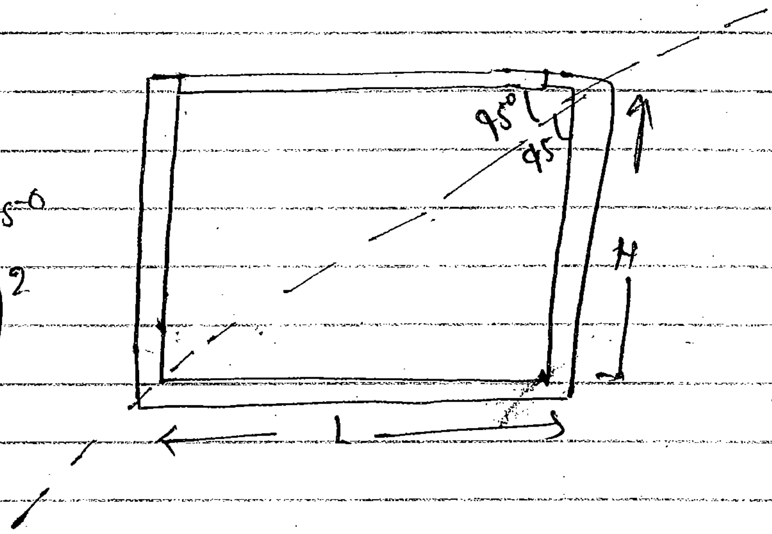
Sol

$$I = \frac{ML^2}{3}\sin^2\theta$$

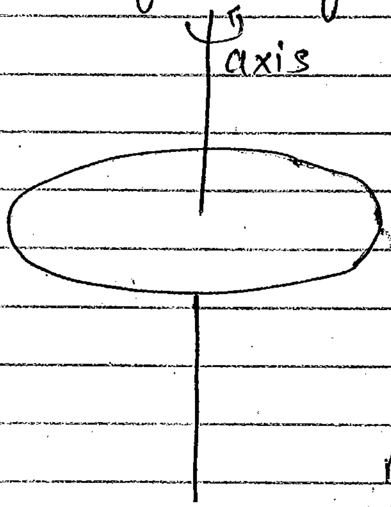
$$= 4 \times \frac{ML^2}{3}\sin^2 45^\circ$$

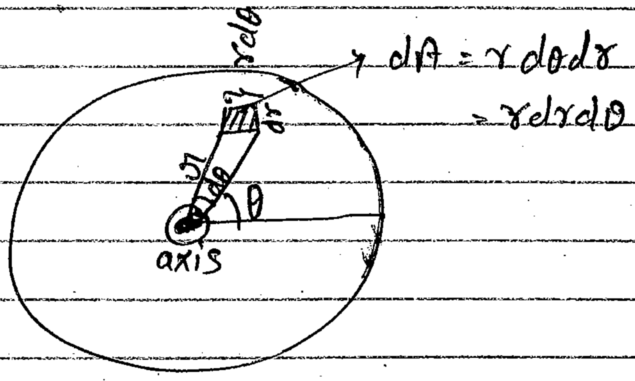
$$= 4 \times \frac{ML^2}{3} \times \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\boxed{I = \frac{2ML^2}{3}}$$



Q. M.I. of Uniform disc :-





Mass = M
Radius = R

$$\sigma = \frac{M}{\pi R^2}$$

$$\sigma = \frac{dm}{dA}$$

$$\frac{M}{R^2} = \frac{dM}{dA}$$

$$\boxed{dm = \frac{M}{R^2} \cdot dA}$$

$$I = \int dm\, r_\perp^2$$

$$= \int \frac{M}{\pi R^2} dA \cdot r^2$$

$$= \frac{M}{\pi R^2} \iint r\, dr\, d\theta \cdot r^2$$

$$= \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\theta$$

$$= \frac{\cancel{2\pi} M}{\cancel{\pi} \cancel{R^2}} \cdot \frac{R^{\cancel{4}}}{\cancel{4}}$$

$$\boxed{I = \frac{MR^2}{2}}$$

{ Memorize } Read list of booklet.

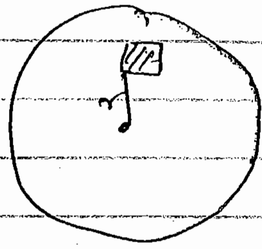
Que A circular disc of mass M and Radius R have non-uniform density varying a distance from center as $\sigma = \sigma_0\left(1 - \frac{r}{R}\right)$ Calculate moment of inertia of disc in terms of A and R about $\perp^{or}$ axis through it centre.

Sol^n

$$\because \sigma = \frac{dM}{dA}$$

$$\sigma_0\left(1 - \frac{r}{R}\right) = \frac{dM}{dA}$$

$$dM = \sigma_0\left(1-\frac{r}{R}\right)dA$$



$$I = \int dM\, r_{\perp}^2$$

$$= \int \sigma_0\left(1-\frac{r}{R}\right)\cdot r^2 \cdot dA$$

$$= \iint \sigma_0\left(1-\frac{r}{R}\right) r\,dr\,d\theta \cdot r^2$$

$$= \sigma_0 \int_0^R \left(1-\frac{r}{R}\right) r^3 dr \int_0^{2\pi} d\theta$$

$$= 2\pi\sigma_0\left[\int_0^R r^3 dr - \int_0^R \frac{r^4}{R} dr\right]$$

$$= 2\pi\sigma_0\left[\frac{R^4}{4} - \frac{R^5}{5\cancel{R}}\right]$$

$$= 2\pi\sigma_0\left[\frac{R^4}{4} - \frac{R^4}{5}\right]$$

$$= \cancel{2}\pi\sigma_0\left[\frac{R^4}{\cancel{20}}\right]$$

$$\boxed{I = \frac{\pi\sigma_0 R^4}{10}} \quad \text{—— (A)}$$

To eliminate σ_0 integrate dM :-

$$\int dM = \sigma_0 \int_0^R \left(1-\frac{r}{R}\right) r\,dr \int_0^{2\pi} d\theta$$

$$= 2\pi\sigma_0\left[\frac{R^2}{2} - \frac{R^3}{3\cancel{R}}\right]$$

$$= \sigma_0 \, 2\pi \cdot \frac{R^2}{6}$$

$$M = \frac{\sigma_0 \pi R^2}{3}$$

$$\boxed{\sigma_0 = \frac{3M}{\pi R^2}}$$

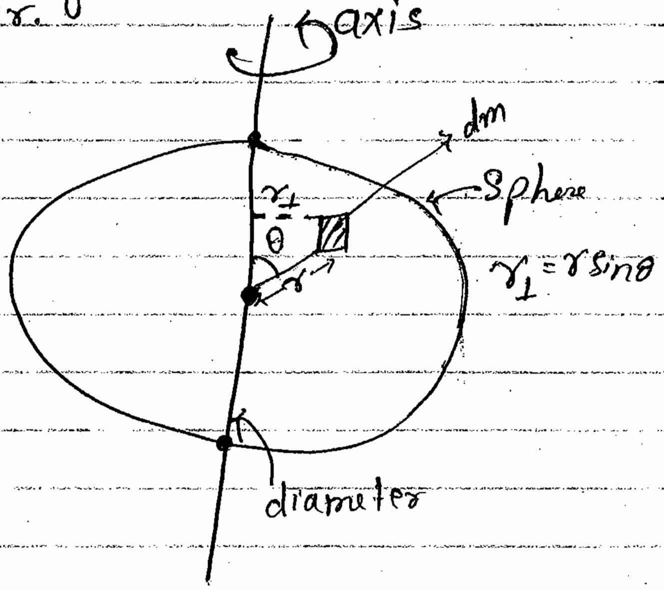
Put in (A)

$$I = \frac{3M}{\cancel{\pi R^2}} \cdot \frac{\cancel{\pi} R^{4}}{10}$$

$$\boxed{I = \frac{3MR^2}{10}}$$

Ans

Q. A solid sphere of mass M and radius R has volume mass density $\rho = kr^2$ where k is constant. r is distance from centre calculate M.I. about its diameter.



Sol^n

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\rho = \frac{dm}{dv}$$

$$kr^2 = \frac{dm}{dv}$$

$$dm = kr^2 \cdot r^2 dr \cdot \sin\theta d\theta \cdot d\phi$$

$$= kr^4 dr \, \sin\theta d\theta \cdot d\phi$$

$$I = \int dm \, r_\perp^2$$

$$= \int dm \, r^2 \sin^2\theta$$

$$I = K\int_0^R r^6\,dr\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi} d\phi$$

$$= \frac{KR^7}{7}\cdot 2\pi\cdot\int_0^\pi \sin^2\theta\cdot\sin\theta\,d\theta$$

$$\int_0^\pi \sin^3\theta \qquad \int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta \qquad \int_{-1}^{+1}(1-t^2)\,dt$$

$$= \frac{KR^7}{7}\cdot 2\pi\int_{-1}^{+1}(1-t^2)\,dt$$

$$= \frac{KR^7}{7}\cdot 2\pi\left[\left[t\right]_{-1}^{+1} - \left[\frac{t^3}{3}\right]_{-1}^{+1}\right]$$

$$= \frac{KR^7}{7}\cdot 2\pi\left[1+1\right] - \left[\frac{1}{3}+\frac{1}{3}\right]$$

$$= \frac{KR^7}{7}\,2\pi\cdot\left(2-\frac{2}{3}\right) = \frac{KR^7}{7}\,2\pi\cdot\left(\frac{6-2}{3}\right)$$

$$= \frac{8\pi KR^7}{7\times 3} = \frac{8\pi KR^7}{21}$$ Ans —— (9)

Now to remove K

$$\int dM = K\int_0^R r^4\,dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi$$

$$M = \frac{KR^5}{5}\cdot\left[-\cos\theta\right]_0^\pi\cdot 2\pi$$

$$= \frac{2\pi KR^5}{5}\cdot 2$$

$$M = \frac{4\pi KR^5}{5}$$

$$\boxed{K = \frac{5M}{4\pi R^5}}$$

Put the value of k in eqⁿ (a)

$$I = \frac{\cancel{8}^{2}\pi R^7}{21} \times \frac{5M}{\cancel{4}\pi R^5}$$

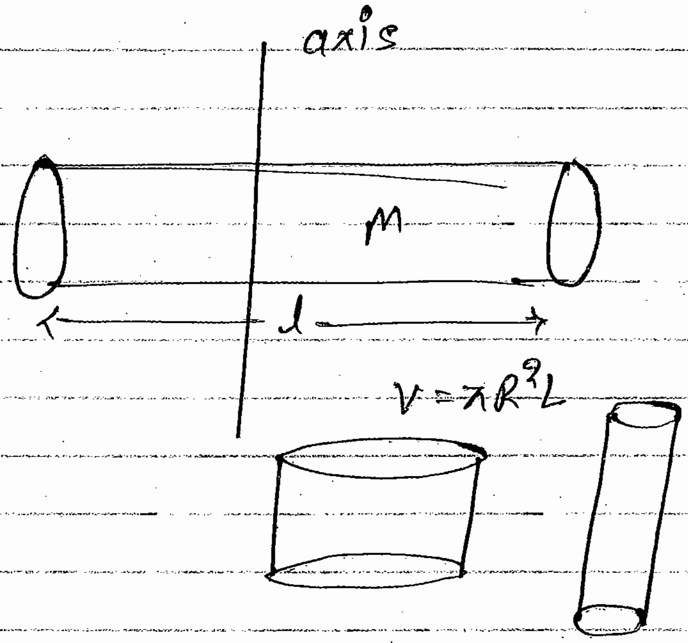
$$= \frac{10MR^{\cancel{7}}}{21R^{\cancel{5}}} = \frac{10MR^2}{21}$$

$$\boxed{I = \frac{10MR^2}{21}}$$

Ans

Q.4 What should be ratio of radius and length of a solid cylinder of uniform mass density so that its moment of inertia through the axis and ⊥ʳ to its length is minimum for given volume.

Solⁿ



$$I = \frac{MR^2}{4} + \frac{ML^2}{12}$$

given V = Constt.
M = Constt.

$$V = \pi R^2 l$$

$$R^2 = \frac{V}{\pi l}$$

$$I = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right]$$

$$I = M\left[\frac{V}{4\pi l} + \frac{l^2}{12}\right]$$

$$I = f(l)$$

for I to be minimum -

$$\frac{dI}{dl} = 0$$

$$0 = m\left[\frac{-V}{4\pi l^2} + \frac{l}{6}\right]$$

$$\frac{l}{6} = \frac{V}{4\pi l^2}$$

$$\frac{\pi R^2 l}{4\pi l^2} = \frac{l}{6}$$

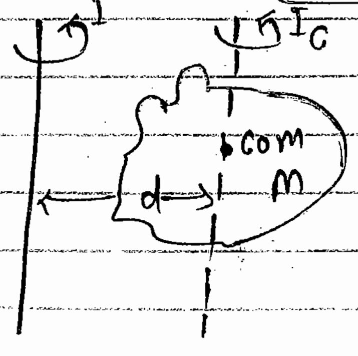
$$\left(\frac{R}{L}\right)^2 = \frac{4}{6}$$

$$\boxed{\frac{R}{L} = \frac{2}{\sqrt{6}}}$$

* Parallel Axis Theorem :-

Applicable for 1d, 2d and 3d object. We take two parallel axes one of them must pass through centre of mass.

$$\boxed{I = I_C + Md^2}$$

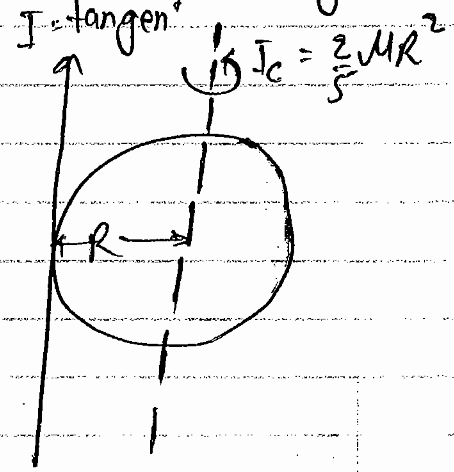


M = mass of object

d = $\perp^{r}$ distance b/w the axis.

Application :-

1 M.I of a uniform sphere about its tangent



M = mass of sphere

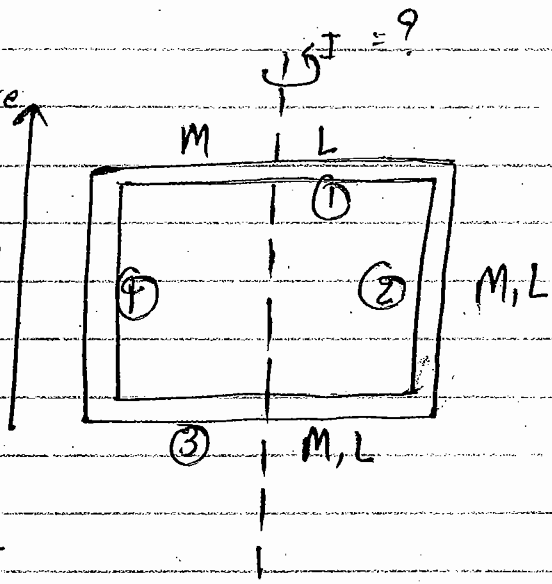
R = Radius of sphere

$$I_{tangent} = I_C + Md^2$$

$$= \frac{2}{5}MR^2 + MR^2$$

$$\boxed{I_{tangent} = \frac{7}{5}MR^2}$$

2 Calculate M.I. of square fram shown in fig. about an axis lying its plane and passing through the centre.



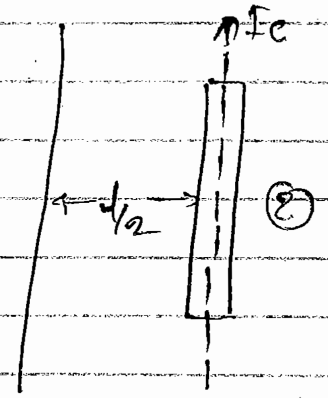
Sol^n M.I. about (1) Rod :-

$$I_{1} = \frac{ML^2}{12} \quad \text{—— (1)}$$

M. I. about (3) Rod :-

$$I_3 = \frac{ML^2}{12} \quad \text{—— (II)}$$

M.I. about (2) Rod



$$I_{(2)} = I_c + Md$$

$$= 0 + \frac{ML^2}{4}$$

$$\boxed{I_{(2)} = \frac{ML^2}{4}} \quad \text{—— (III)}$$

Similarly

$$I_4 = \frac{ML^2}{4} \quad \text{—— (IV)}$$

So $\quad I_{Net} = I_1 + I_2 + I_3 + I_4$

$$= \frac{ML^2}{12} + \frac{ML^2}{12} + \frac{ML^2}{4} + \frac{ML^2}{4}$$

$$= \frac{2ML^2}{12} + \frac{2ML^2}{4} = ML^2\left[\frac{\cancel{2}}{\cancel{12}_6} + \frac{\cancel{2}}{\cancel{4}_2}\right]$$

$$I_{net} = ML^2\left[\frac{2+6}{12}\right] = ML^2\left[\frac{\cancel{8}\,\cancel{4}_2}{\cancel{12}\,\cancel{6}_3}\right]$$

$$\boxed{I_{net} = \frac{2}{3}ML^2}$$ Ans

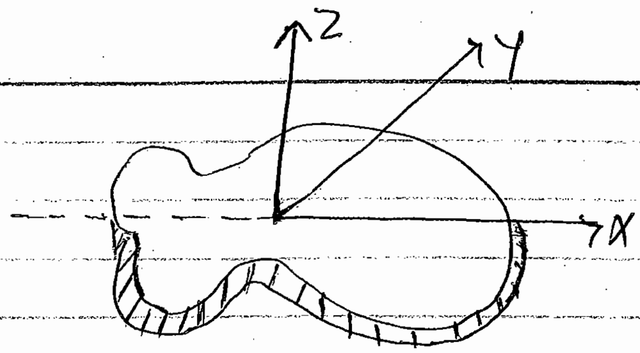
* Perpendicular Axis Theorem :-

Here we have three perpendicular axis (x, y, z). If x and y lie in the plane of object and z is perpendicular to its plane z must pass through point of intersection of x and y

⇒ It is not applicable in 3 dimension.

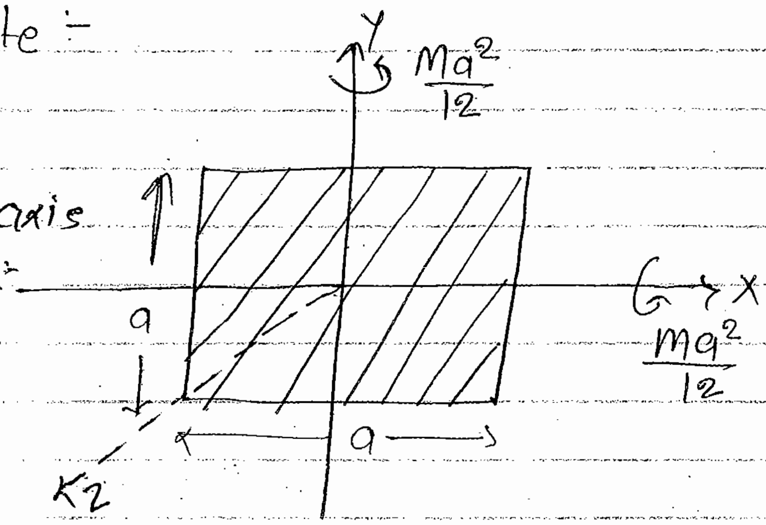
01/Sep./2014

$$\boxed{I_z = I_x + I_y}$$



* Square Plate :-
{2D object}:-

M.I about ⊥r axis through center :-



$I_z = I_x + I_y$

$$\boxed{I_z = \frac{Ma^2}{6}}$$

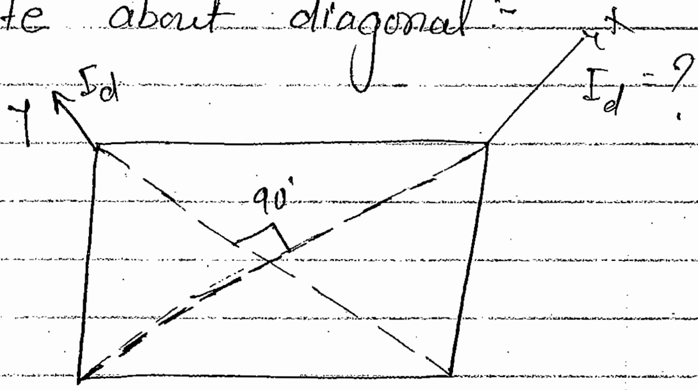
* M.I of square plate about diagonal :-

$I_z = I_x + I_y$

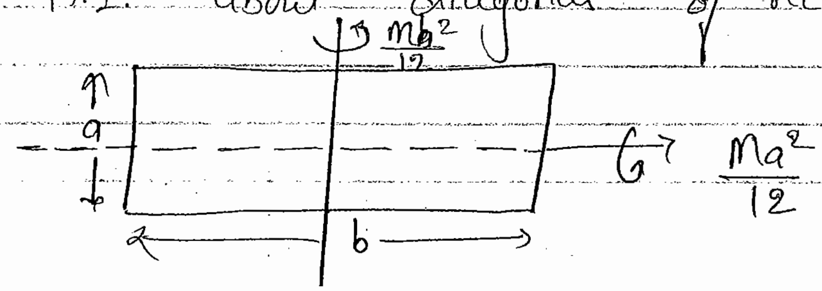
$\frac{Ma^2}{6} = I_d + I_d$

$2I_d = \frac{Ma^2}{6}$

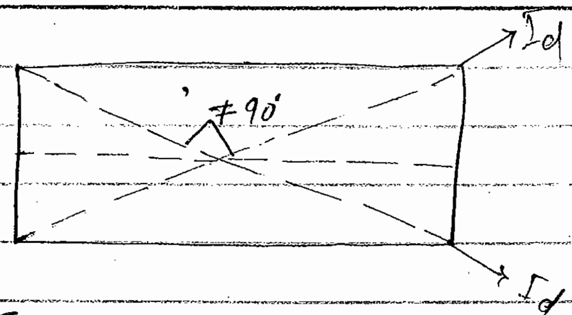
$$\boxed{I_d = \frac{Ma^2}{12}}$$



* What is the M.I. about diagonal of rectangle:-



I_d can not be calculated using $\perp^{or}$ axis theorem about its center.

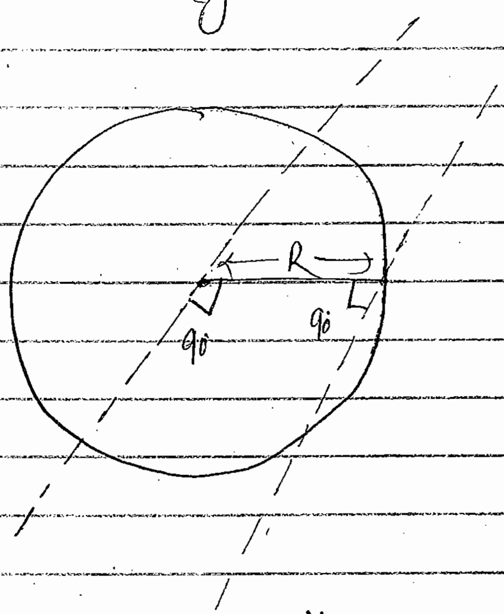


Note :- Inertia Tensor will be used.

* Disc of mass M and Radius R :-
{Assume Disc is uniform}

Q. M.I about an axis $\perp^{or}$ to its surface and passing through its edge?

Sol^n Method I :-

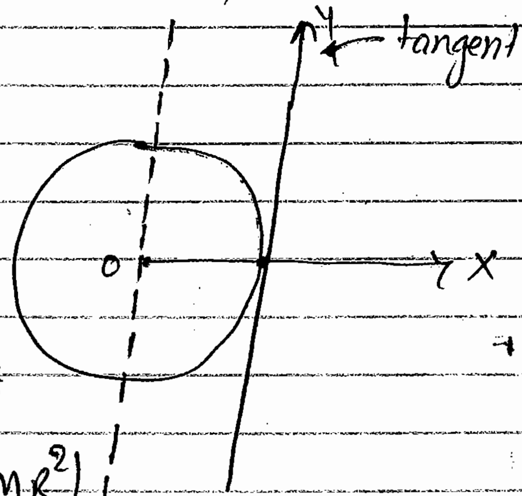


From $\parallel^{lel}$ Axis theorem :-

$$I = I_c + Md^2$$

$$= \frac{MR^2}{2} + MR^2$$

$$= \frac{3}{2} MR^2$$



Method II^{nd} :-

$$I_z = I_x + I_y$$

$$I_{required} = I_{diameter} + I_{tangent}$$

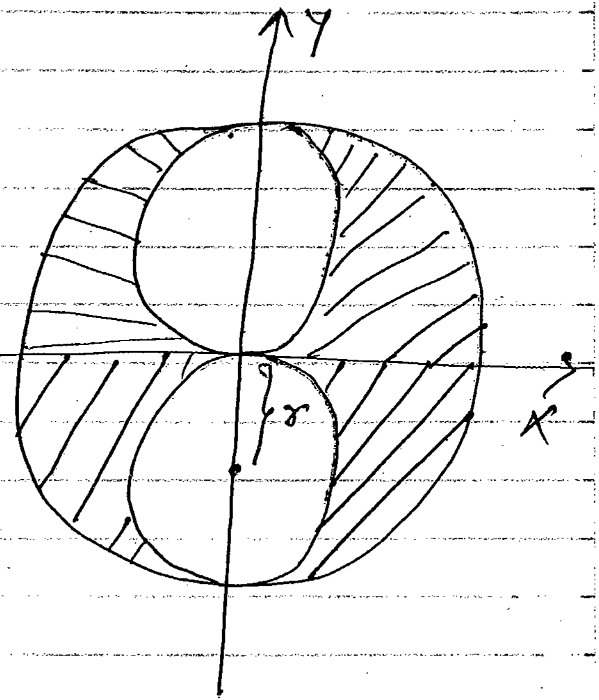
$$= \frac{MR^2}{4} + \left(\frac{MR^2}{4} + MR^2\right)$$

M.I. is a additive quantity.

$$= \frac{MR^2}{2} + MR^2 = \frac{3}{2} MR^2$$

So $\boxed{I_{required} = \frac{3}{2} MR^2}$ Ans

Gate Q.15



M.I. of remeaning object = M.I. of big spher − M.I. of two small sphere..

$$I_y = \frac{2}{5} MR^2 - \frac{2}{5} mr^2 \times 2$$

$$M = \rho \frac{4}{3} \pi R^3$$

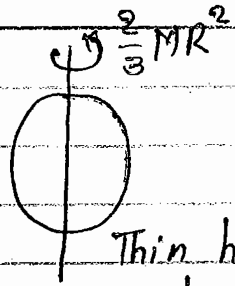
$$= \rho \frac{4}{3} \pi r^3, \quad r = \frac{R}{2}$$

$$I_y =$$

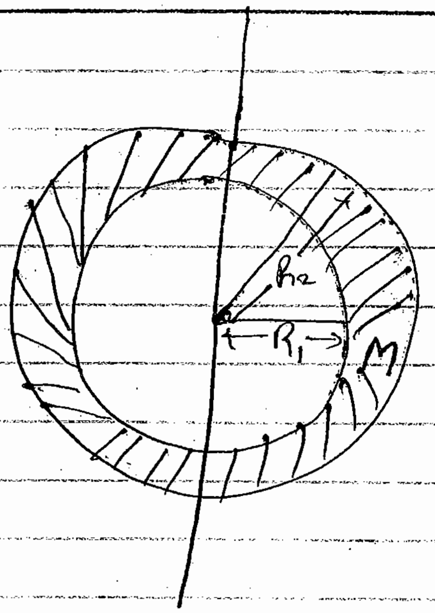
$$I_x = \frac{2}{5} MR^2 - \frac{7}{5} mr^2 \times 2$$

Q.9 A thick hollow sphere of mass 'M' has inner and outer radii R_1 and R_2. Moment of inertia of sphere about its diameter is -?

$\underline{Sol^n}$



Thin hollow sphere



To check options –

(c) $\frac{1}{3} M(R_1^2 + R_2^2)$

for the thin sphere –

$R_1 = R_2 = R$

$$\boxed{I = \frac{2}{3} MR^2}$$

But for solid sphere –

$R_1 = 0 \quad R_2 = R$

$$\boxed{I = \frac{1}{9} MR^2}$$

which is not equal to the M.I. of solid sphere

(d) $\frac{2}{5} M\left(\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right)$ $\left[\frac{0}{0} = \text{L'Hospital's rule.} \right.$ (लोपिटल))

$R_1 = R$

$R_2 = R + a \qquad \lim_{a \to 0}$

$$I = \frac{2}{5} M\left(\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right)$$

$$= \lim_{a \to 0} M \frac{2}{\cancel{5}} \frac{\cancel{5}}{3}\left(\frac{(R+a)^4 - 0}{(R+a)^2 - 0}\right)$$

$$= \frac{2}{3} M \frac{R^4}{R^2} = \frac{2}{3} MR^2$$

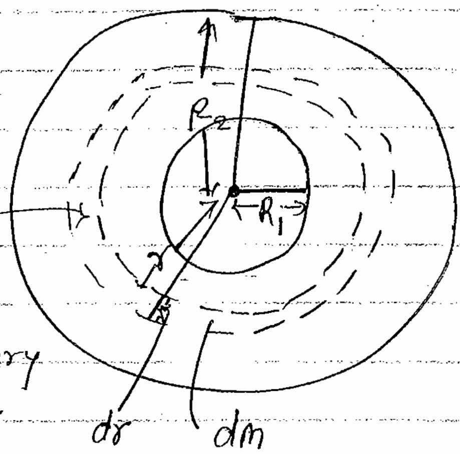
$$\boxed{I = \frac{2}{3} MR^2}$$ So it is correct.

* Moment of Inertia of a big object is to be calculated, we can first calculate M.I. of a small elementary object and integrate it.

Ex-

$$I = \int dI$$

$$dI = \frac{2}{3} dm\, r^2$$



$$\frac{dm}{dV} = \rho = \frac{M}{\frac{4}{3}\pi(R_2^3 - R_1^3)}$$

$$dm = \frac{3M \cdot dV}{4\pi(R_2^3 - R_1^3)}$$

$$\boxed{dm = \frac{3M}{4\pi(R_2^3 - R_1^3)} dV}$$

$$dV = 4\pi r^2 dr$$

$$dm = \frac{3M}{\cancel{4\pi}(R_2^3 - R_1^3)} \cancel{4\pi} r^2 dr$$

$$= \frac{3Mr^2 dr}{(R_2^3 - R_1^3)}$$

$$= \frac{2}{3} M \int_{R_1}^{R_2} \frac{3r^2 dr}{(R_2^3 - R_1^3)} r^2 = \frac{M}{(R_2^3 - R_1^3)} \times \frac{2}{\cancel{3}} \times \cancel{3} \left[\frac{r^5}{5}\right]_{R_1}^{R_2}$$

$$= \frac{2M}{5(R_2^3 - R_1^3)} \left[(R_2^5 - R_1^5)\right]$$

$$= \frac{2}{5} M \left(\frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)} \right)$$

Ans

* Dynamics of rigid body :- A rigid body can have following types of motion.

1 Pure Rotation :-
At least one point of object remains at rest.
In this case -

$$K.E \neq \frac{1}{2} m v^2$$

Here, $\boxed{K.E = \frac{1}{2} I \omega^2}$

~~2. Pure Translation :- It means there is no rotation i.e. $\boxed{\omega = 0}$~~

$$L_{axis} = I_{axis} \omega$$

$\boxed{\tau = I_{axis} \alpha}$ eq^n of motion.

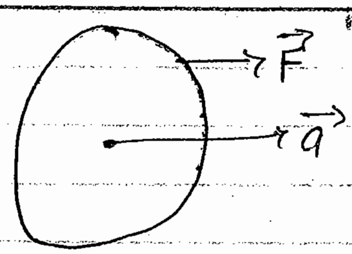
$$\boxed{\frac{dL_{axis}}{dt} = \tau_{axis}}$$

2. Pure Translation :-
It means there is no rotation i.e. $\omega = 0$

$$\boxed{K.E. = \frac{1}{2} m v^2}$$

Equation of motion:-

$$\boxed{F_{net\ force} = ma}$$



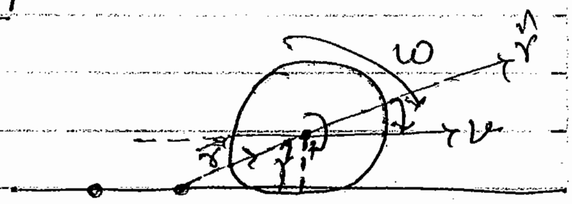
$$\boxed{L = p r_\perp}$$

3. General Motion:-

Translation + Rotation

$$\boxed{K.E. = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2}$$

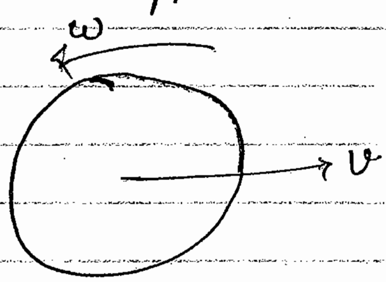
$$\boxed{L = I\omega \pm p r_\perp}$$



$r_\perp = R$ [If L is calculated about a point on path of round object]

$$\boxed{L = I\omega + p r_\perp}$$ { ∵ direction of ω and $r \times v$ is same.

$$\boxed{L = I\omega - p r_\perp}$$ { ∵ direction of ω and $r \times v$ is opposite }



$v = \omega r$ → velocity of any particle which is at the distance R at the center. of due to rotation.

---

* **Rolling Motion :-** It is a special case of general motion.

Translational velocity or translational accelaration of center of mass and rotational velocity or rotational (Angular) accelaration about center of mass are related as -

$$\boxed{V_{cm} = \omega R}$$

Diff. w.r. to t.

$$\frac{dv}{dt} = \alpha R$$

→ NET

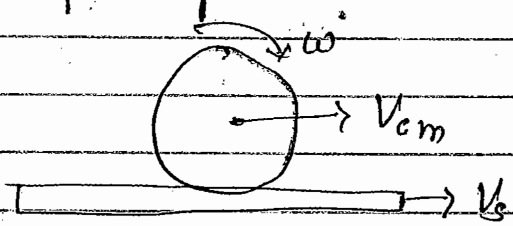
On a stationary surface.

$$V_{cm} = \omega R = V_s$$

$$a_{cm} = \alpha R = a_s$$

→ Crate

On a moving surface.

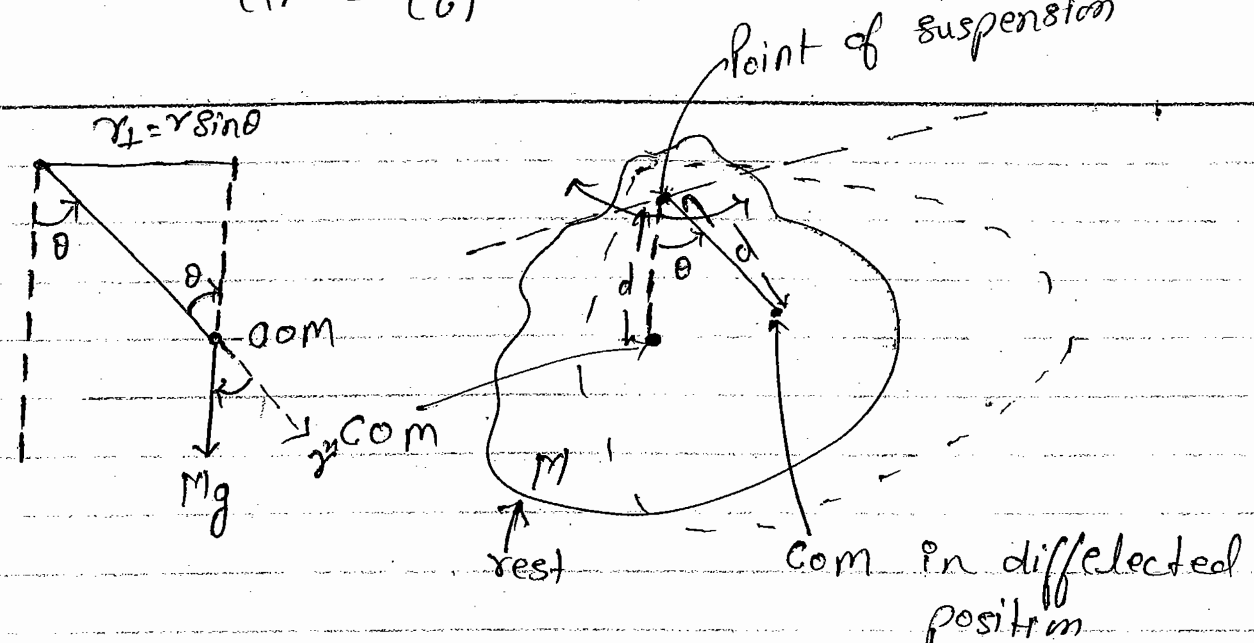


V_{cm} = Velocity of com.

* **Physical Pendulum {Compound Pendulum} :-**

It is a rigid body (rod, disc, ring etc) oscillating about a horizontal axis under the effect of gravity. Time period oscillation.

Direc^n of angular acc^n (α) = direc^n of Torque (τ).



It is a pure rotation case - So Equation of motion.

$$\boxed{\tau = I\alpha}$$

$$\boxed{Mgd\sin\theta = I\alpha}$$

I = M.I about horizontal axis through point of suspension.

If θ is small:-

$$\sin\theta \simeq \theta$$

$$Mgd\theta = I\alpha$$

$$\boxed{\alpha = \frac{Mgd}{I}\theta}$$

Torque is opposite of θ acc to direc^n

and $$\boxed{\vec{\alpha} = \frac{mgd}{I}(-\vec{\theta})}$$

This is the case of simple hormonic motion. standard eq^n of S.H.M.

$$\alpha = \omega^2 (-\vec{\theta})$$

ω → Angular frequency

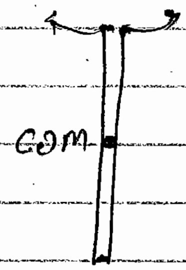
$$\omega^2 = \frac{mgd}{I}$$

$$\frac{2\pi}{T} = \sqrt{\frac{mgd}{I}}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgd}}}$$

Que. A thin rod of mass m and length L is suspended from 1 m. what is its time period.

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$



$$I = \frac{ML^2}{3}$$

$$d = \frac{L}{2}$$

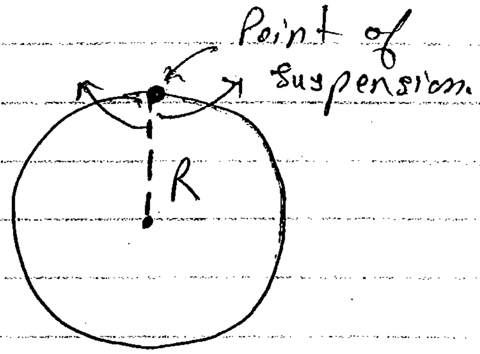
$$T = 2\pi\sqrt{\frac{ML^2/3}{MgL/2}}$$

$$\boxed{T = 2\pi\sqrt{\frac{2L}{3g}}}$$

Q. A disc of radius R is suspended from a point of its periferi. If T_1 and T_2 be time period of oscillation parallel to and perpendicular to the plane of disc. What is the value of T_1 and T_2?

Sol^n

$\because \ d = R$



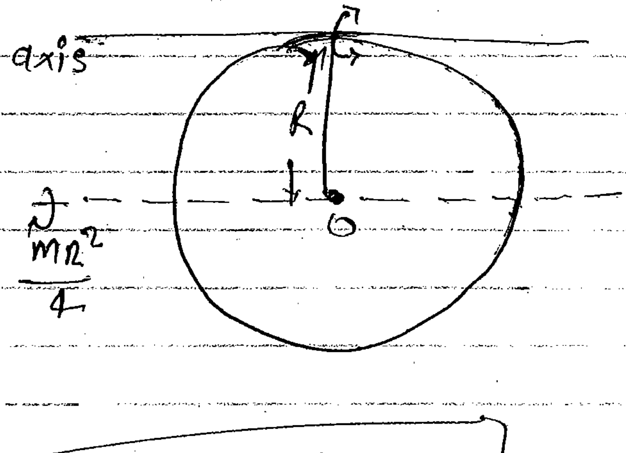
By parallel axis theorem –

$$I = \frac{MR^2}{2} + MR^2 = \frac{3}{2} MR^2$$

$$T_1 = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$= 2\pi\sqrt{\frac{\frac{3}{2}MR^2}{M.g.R}}$$

$$\boxed{T_1 = 2\pi\sqrt{\frac{3R}{2g}}}$$



$$I = \frac{MR^2}{4} + MR^2$$

$$I = \frac{5}{4} MR^2$$

$$T_2 = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$T_2 = 2\pi\sqrt{\frac{5MR^2/4}{Mg.R}} \qquad \boxed{T_2 = 2\pi\sqrt{\frac{5R}{4g}}}$$

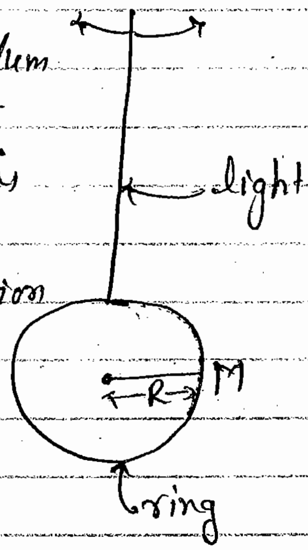
$$\frac{T_1}{T_2} = \sqrt{\frac{3}{2} \times \frac{4}{5}}$$

$$\boxed{\frac{T_1}{T_2} = \sqrt{\frac{6}{5}}}$$

Yet 2013

Q. System oscillates parallel ~~of~~ to plane of ring. What is time period.

It is not a simple pendulum becoz in simple pendulum ~~mass of bod is~~ boob is taken as a particle which not have radius and dimension but here Radius and dimension is
$I = MR^2 + M(R+L)^2$ there.



$d = R + l$

$$T = 2\pi\sqrt{\frac{MR^2 + M(R+L)^2}{Mg(R+l)}}$$ It is compound ~~ple~~ pendulum.

$$= 2\pi\sqrt{\frac{R^2 + R^2 + l^2 + 2Rl}{(R+l)}}$$

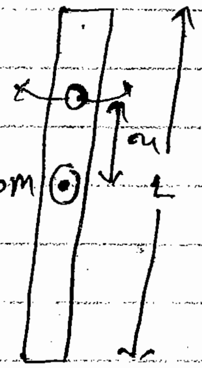
$$\boxed{T = 2\pi\sqrt{\frac{2R^2 + l^2 + 2Rl}{(R+l)}}}$$

Q. A thin rod of length L is suspended from some point on its length what should be distance of point of suspension from COM. So that time period should be minimum.

Sol^n

$d = x$

$$I = \frac{ML^2}{12} + Mx^2$$



$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$= 2\pi\sqrt{\frac{M\left(\frac{L^2}{12} + x^2\right)}{Mgx}}$$

$$T^2 = \frac{4\pi^2}{g}\left(\frac{L^2}{12x} + x\right)$$

If T is minimum then T^2 is also minimum.

$$\frac{dT^2}{dx} = 0$$

$$\frac{4\pi^2}{g}\left(\frac{-L^2}{12x^2} + 1\right) = 0$$

$$-\frac{L^2}{12x^2} + 1 = 0$$

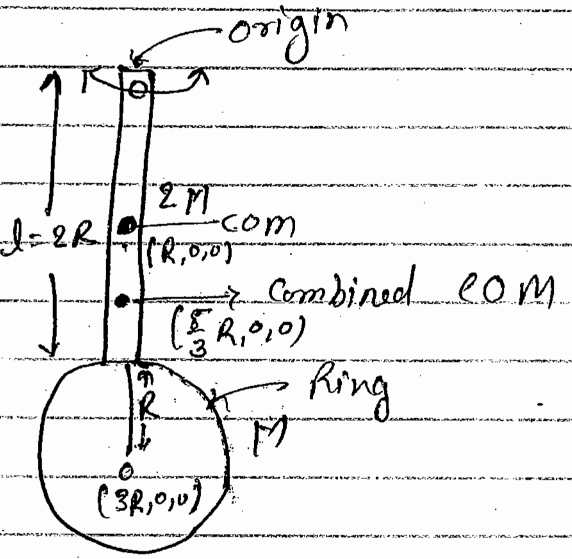
$$x^2 = \frac{L^2}{12}$$

$$\boxed{x = \frac{L}{\sqrt{12}}}$$ Ans

Q. Suppose there is a rod and there is a ring, mass of Rod is $2M$ and mass of ring is M. What is T for oscillation.

Sol^n

$$T = 2\pi\sqrt{\frac{I_{total}}{M_{total}\, g d}}$$



d = distance of combined COM from point of suspension

$$I = I_c + M(dis)^2$$

$$I_{total} = MR^2 + M(R+l)^2 + \frac{(2M)l^2}{3}$$

Co-ordinate of center of mass :-

$$x_{com} = \frac{\sum m_i x_i}{\sum m_i}$$ } discreate case or for combination of object.

$$x_{com} = \frac{\int x\,dm}{\int dm}$$ } for a single continuous object.

In this case -

$$x_{com} = \frac{2M \times R + M \times 3R}{3M}$$

$$x_{com} = \frac{5}{3}R = d$$

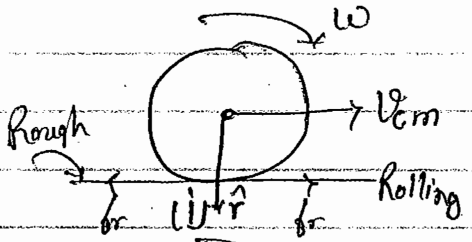
$$T = 2\pi\sqrt{\frac{\cancel{M}R^2 + \cancel{M}(R+l)^2 + (2\cancel{M})l^2/3}{3\cancel{M}\cdot g\cdot\frac{5}{\cancel{3}}R}}$$

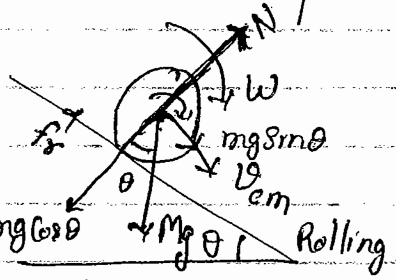
$$= 2\pi\sqrt{\frac{R^2+R^2+l^2+2Rl+2l^2/3}{5gR}}$$

$$= 2\pi\sqrt{\frac{6R^2+3l^2+2Rl+2l^2}{15gR}}$$

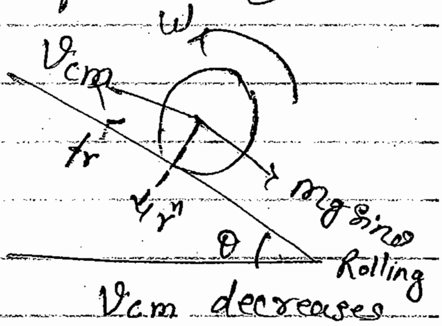
$$\boxed{T = 2\pi\sqrt{\frac{6R^2+5l^2+2Rl}{15gR}}}$$ Ans

* Rolling Motion on Stationary Surface (horizontal or inclined) :-

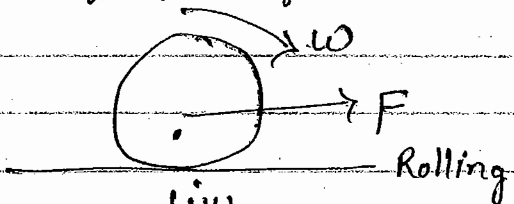




V_{cm} increases due to mgsinθ.
(ii)



V_{cm} decreases mgsinθ.
(iii)



(iv)

Condition for Rolling :-

$V_{cm} = \omega R$

$a_{cm} = \alpha R$

from fig(ii) :-

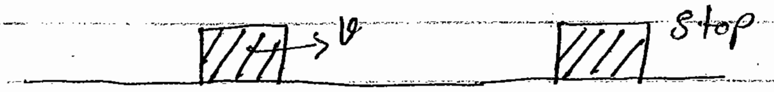
→ Here Vcm increases due to Mg sinθ
∴ $V_{cm} = \omega R$
So due to this relation ω should also be increases but, Here torque is zero

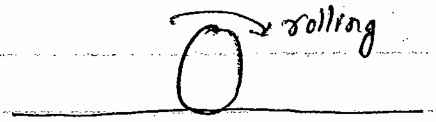
because Mg and N both are passing through center.
So Here ω is not increases so this Rolling is not possible.
So Rolling is not possible on smooth inclined plane.
For Rolling motion on incliened plane friction must be present.
When incliened plane is a rough surface then friction is works here.

(i) from fig. (i) :-
In this case when friction works opposite to the motion then v_{cm} decreases but $r \times F$ and ω are in same direcn then ω is increases so this can not satisfies the condition of Rolling.
In other hand when friction in the direcn of motion then v_{cm} increases but $r \times F$ and ω is in opposite direction so ω is decreases so it is also not satisfies the condition of rolling.
So here for Rolling motion surface might be smooth then $v_{cm} = $ Constt. and $\omega = $ Constt.

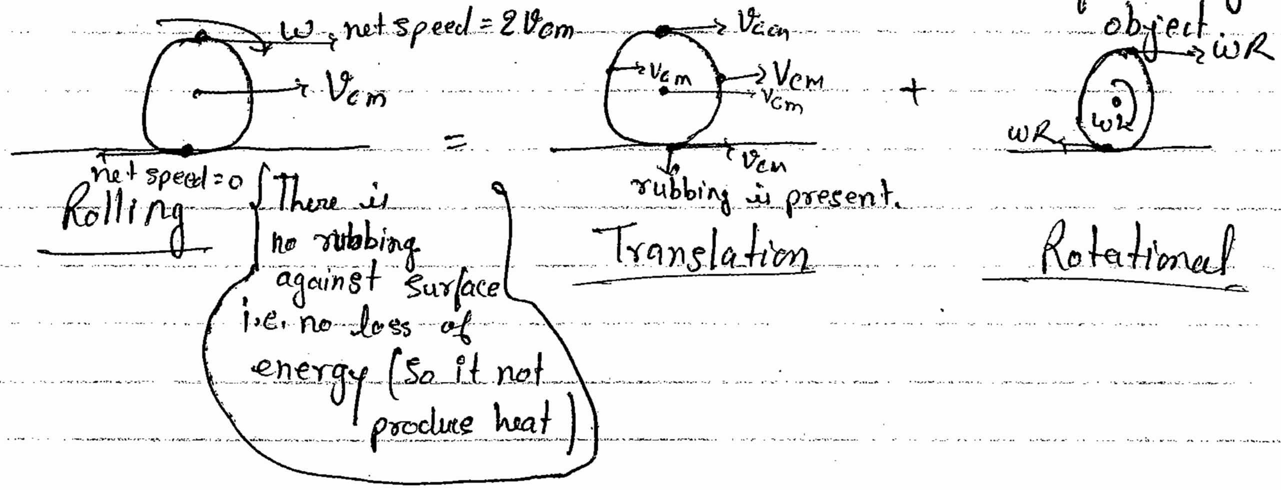
refectely Rigit that means we can't deform.
Iron are not ~~ref~~ perfectily rigid.

* In rolling motion although surface might be rough there is no loss of energy against ~~further~~ friction.





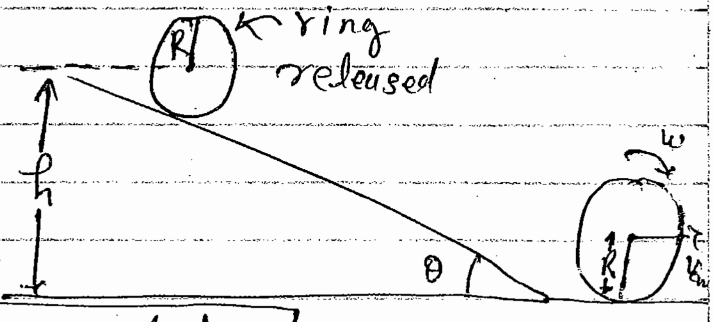
never stop { True for perfect rigid object



Net 2013 June.

Q. A ring is released from a inclined plane whose center is at h distance from ground. If ring rolls down the plane what is its angular speed when it reaches the bottom.

Sol^n



Here center decreases by (h-R) height.

$v_{cm} = \omega R$

Apply conservation of energy:-

$\boxed{\text{Loss of P.E.} = \text{Gain of K.E.}}$

$$Mg(h-R) = \frac{1}{2} I\omega^2 + \frac{1}{2} M v_{cm}^2$$

$$= \frac{1}{2} MR^2\omega^2 + \frac{1}{2} M\omega^2 R^2$$

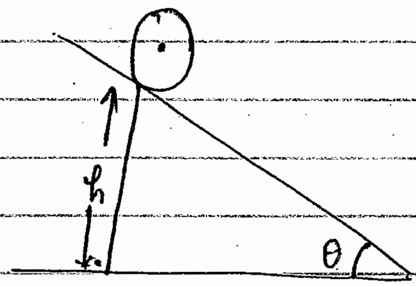
$$\cancel{M}g(h-R) = \cancel{M}R^2\omega^2$$

$$\boxed{\omega = \sqrt{\frac{g(h-R)}{R^2}}}$$

Q. A ~~hollow~~ solid sphere and a disc rolls down from an inclined plane from same point what is ratio of velocity of sphere and velocity of disc when they reach at the bottom.

Soln

For Disc:-



By Conservation of energy :-

Loss in P.E. = Gain in K.E.

$$mgh = \frac{1}{2} I\omega^2 + \frac{1}{2} m v_{cm}^2 \quad \{ v_{cm} = \omega r$$

$$= \frac{1}{2} \frac{m\cancel{r^2}}{2} \cdot \frac{v_{cm}^2}{\cancel{r^2}} + \frac{1}{2} m v_{cm}^2$$

$$\cancel{m}gh = \frac{3}{4} \cancel{m} v_{cm}^2$$

$$\boxed{(v_{cm})_{disc} = \sqrt{\frac{4}{3} gh}}$$

$s = ut + \frac{1}{2}at^2$ when $a = 0$ then $s = ut$

and $\boxed{t = \frac{s}{u}}$ only When u = Constant

time = $\frac{\text{distance}}{\text{Velocity}}$

for sphere :-

solid sphere

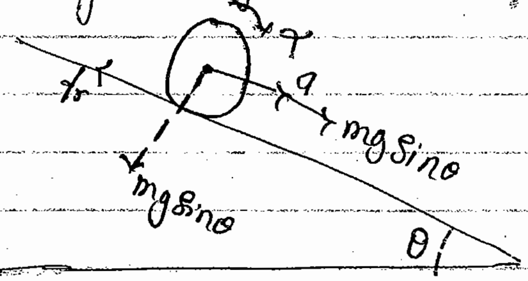
$$Mgh = \frac{1}{2}\left(\frac{2}{5}MR^2\right)\cdot\frac{V_{cm}^2}{R^2} + \frac{1}{2}MV_{cm}^2$$

$$\cancel{M}gh = \frac{7}{10}\cancel{M}V_{cm}^2$$

$$\boxed{(V_{cm})_{sphere} = \sqrt{\frac{10}{7}gh}}$$

$$\boxed{\frac{(V_{cm})_{sphere}}{(V_{cm})_{disc}} = \sqrt{\frac{15}{14}}}$$

* Acceleration of Rolling object on inclined plane :-



Equation of motion for Translation :-

$$mg\sin\theta - f_r = ma \quad \text{—} \ (1)$$

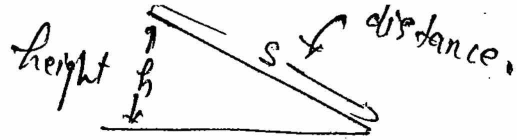
for Rotation :-

$$\tau = I\alpha$$

$$f_r R = I\alpha$$

for Rolling :-

$$a = \alpha R$$



$$\boxed{\alpha = \frac{a}{R}}$$

$$f_r R = \frac{Ia}{R}$$

$$\boxed{f_r = \frac{Ia}{R}} \leftarrow (ii)$$

(I) + (II) :-

$$mg\sin\theta = a\left[m + \frac{I}{R^2}\right]$$

$$g\sin\theta = a\left[1 + \frac{I}{mR^2}\right]$$

$$\boxed{a = \frac{g\sin\theta}{1+\frac{I}{mR^2}}}$$

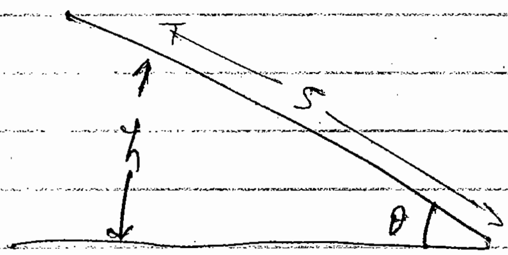
for friction put 'a' in eq^n (II).

A-7
Q.5 A solid cylinder starts Rolling down an inclined plane of inclination θ. from height h. Time taken to reach the bottom is ?.

(a) $\sqrt{\frac{2h}{g}}$ (b) $\sqrt{\frac{3h}{g}}$ (c) $\frac{1}{\sin\theta}\sqrt{\frac{3h}{2g}}$

✓(d) $\frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}$

Sol^n



$$s = \frac{h}{\sin\theta}$$

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 + \frac{1}{2}at^2$$

$$\boxed{t = \sqrt{\frac{2s}{a}}} \quad \text{--- (a)}$$

$$\left\{ \because a = \frac{g\sin\theta}{1+\frac{I}{mR^2}} \right\}$$

$$a = \frac{g\sin\theta}{1+\frac{1}{2}} = \frac{2g\sin\theta}{3}$$

$$\left\{ \because \text{M.I. of } \cancel{\text{Sphere}} \text{ cylinder} = \frac{1}{2}MR^2 \right.$$

$$t = \sqrt{\frac{\cancel{2}h}{\sin\theta} \cdot \frac{3}{\cancel{2}g\sin\theta}}$$

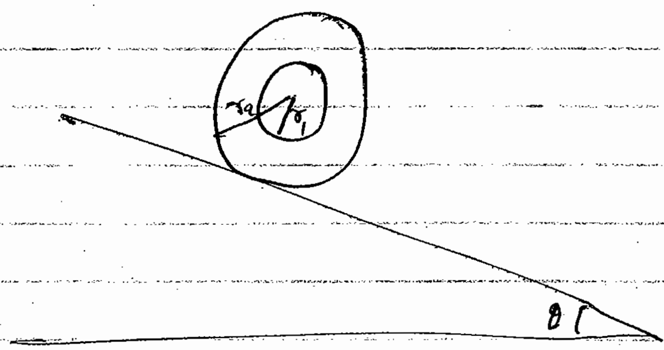
$$\boxed{t = \frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}}$$

A-7

Q.15 A circular loop of inner and outer radii r_1 and r_2 rolls down an inclined plane of inclination θ. Angular acceleration of loop on the inclined plane is -?

(a) $\frac{2r_2}{3r_2^2+r_1^2}g\sin\theta$ (b) $\frac{2r_1}{3r_2^2+r_2^2}g\sin\theta$ (c) $\frac{r_2}{r_2^2+r_1^2}g\sin\theta$

(d) $\frac{r_1}{r_2^2+r_1^2}g\sin\theta$.



Sol $a = \frac{g\sin\theta}{1+\frac{I}{mr_2^2}}$

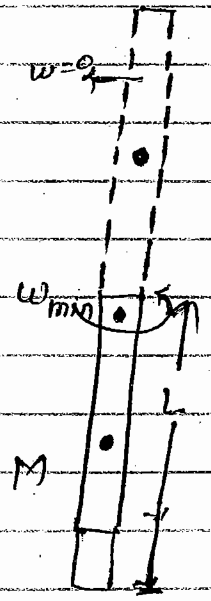
$$\boxed{a = \alpha r_2}$$

$$\boxed{a = \frac{g \sin\theta}{1 + \frac{\frac{1}{2} m(r_1^2 + r_2^2)}{m r_2^2}}}$$

$$\alpha = \frac{2 r_2^{\cancel{2}} g \sin\theta}{2 r_2^2 + (r_1^2 + r_2^2) \times \cancel{r_2}}$$

$$\boxed{\alpha = \frac{2 r_2 g \sin\theta}{r_1^2 + 3 r_2^2}}$$ Ans

A-7
Q.6



Since here one point is fixed so it is a case of pure rotation.

Loss in K.E. = Gain in P.E.

$$\frac{1}{2} I \omega_{min}^2 - 0 = mgL$$ { here L = height accended by center

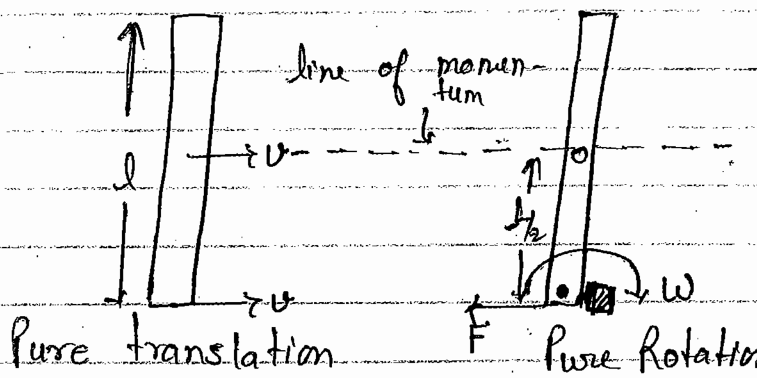
$$\Rightarrow \frac{1}{2} \frac{\cancel{M} L^{\cancel{2}}}{3} \omega_{min}^2 = \cancel{M} g \cancel{L}$$

$$\boxed{\omega_{min} = \sqrt{\frac{6g}{L}}}$$

A-7

Q.7 A rod of length 'L'

Sol^n If initially no point is fixed but of towards one point fixed then conservation of energy will not be applied. because it is like perfectly inelastic collesion.



$\tau = \frac{dL}{dt}$

If $\tau = 0$

$\frac{dL}{dt} = 0$

$L = $ constant

Apply conservation of angular momentum:- {about the point which is fixed}.

$L_i = L_f$

$p r_\perp = I\omega$

$\cancel{m} v \cancel{l}/2 = \frac{\cancel{m} l^{\cancel{2}}}{3}\omega$

$$\boxed{\omega = \frac{3v}{2l}}$$

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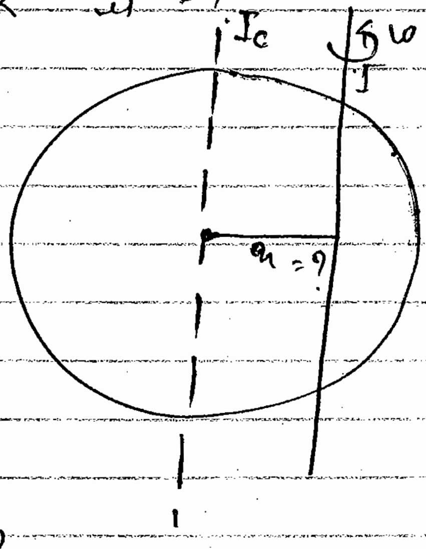
A-7

Q.3 A circular disc of mass M and Radius R rotates about an axis lying in its plane with constant angular velocity ω. If angular momentum of the disc about this axis is $\frac{3}{4}MR^2\omega$, distance of axis from centre of disk is -?

Sol



Given $L_{axis} = \frac{3}{4}MR^2\omega$

$$I_{axis}\,\omega = \frac{3}{4}MR^2\omega$$

$$\left(I_C + M\cancel{R}^2\right)\omega = \frac{3}{4}MR^2\omega$$

$$\therefore I_C = \frac{MR^2}{4}$$

$$\left(\frac{MR^2}{4} + Mx^2\right)\cancel{\omega} = \frac{3}{4}MR^2\cancel{\omega}$$

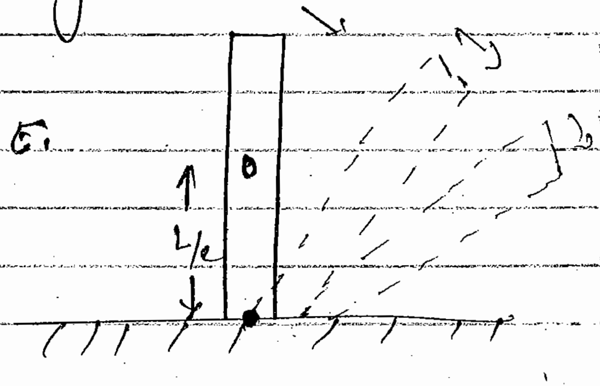
$$Mx^2 = \frac{1}{2}MR^2$$

$$\boxed{x = \frac{R}{\sqrt{2}}}$$ Ans

A-7

Q.8 A Rod of length L is held vertical with its one end fixed on ground. The rod can rotate about fixed end. If the rod is released from this position, what is its speed when it is about to hit the ground.

Sol Conservation of energy



Loss in P.E. = Gain in K.E.

$$Mg\frac{L}{2} = \frac{1}{2}I\omega^2$$

$$\cancel{M}g\frac{\cancel{L}}{\cancel{2}} = \frac{1}{\cancel{2}}\frac{\cancel{M}L^{\cancel{2}}}{3}\omega^2$$

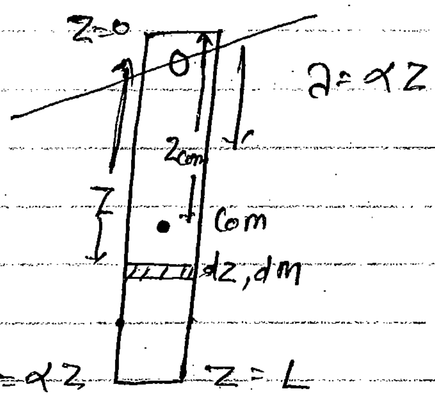
$$\omega^2 = \frac{3g}{L}$$

$$\boxed{\omega = \sqrt{\frac{3g}{L}}}$$ Angular speed.

A-7
Q.10 A thin rod of length L is suspended from one end as shown in figure. The linear mass density of rod varies with distance from point of suspension as $\lambda = \alpha s$. Time period of oscillation of the rod is?

Sol^n

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$



I = MI about point of susp.
d = dist of (P.S.) from C.O.M.

$$I = \int dm\, r_\perp^2 \qquad \lambda = \frac{dm}{dz} = \alpha z$$

$$dm = \alpha z\, dz$$

$$= \int_0^L \alpha z\, dz\, z^2$$

$$\boxed{M.I. = \frac{\alpha L^4}{4}}$$

Position of center of mass -

$$z_{com} = \frac{\int z\, dm}{\int dm} = \frac{\int z\, \alpha z\, dz}{\int \alpha z\, dz} = \frac{\int z^2 \alpha\, dz}{M}$$

$$= \frac{\alpha L^3}{3M} = d$$

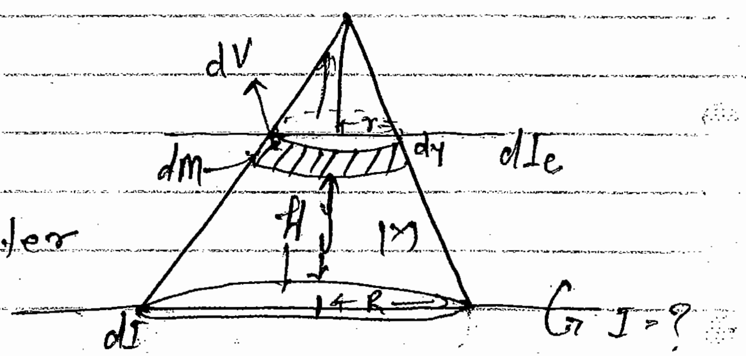
So $$T = 2\pi\sqrt{\frac{\frac{\alpha L^4}{4}}{\cancel{M}g\left(\frac{L^3\alpha}{3\cancel{M}}\right)}}$$ $$\boxed{T = 2\pi\sqrt{\frac{3L}{4g}}}$$ Ans

A-7

Q.19 A right circular cone has mass M, Radius R and height h. Moment of inertia of cone about a diameter of its base is?

Sol^n

$$I = \int dI$$



M.I of disc about the given axis (base diameter

$$dI = dI_c + dM y^2$$

$$= \frac{dm\, r^2}{4} + dM y^2$$

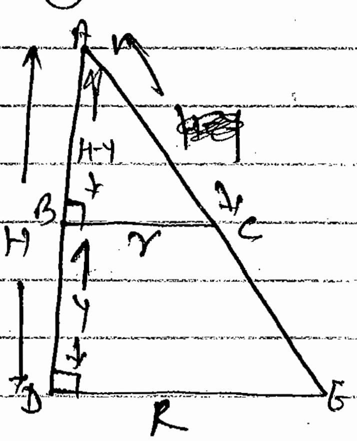
$$\boxed{dI = dM\left(\frac{r^2}{4} + y^2\right)} \quad \text{— (I)}$$

$\because$ $\rho = \dfrac{M}{\frac{1}{3}\pi R^2 H} = \dfrac{dM}{dV}$

$$dm = \frac{M}{\frac{1}{3}\pi R^2 H} dV$$

$$\boxed{dM = \frac{3M}{\pi R^2 H} \pi r^2 dy} \quad \text{— (II)}$$

$\because$ Here ΔABC and ΔADG is similar so –



$$\frac{r}{R} = \frac{H-y}{H}$$

$$\boxed{r = R\left(1 - \frac{y}{H}\right)} \quad \text{— (III)}$$

$$\boxed{y = H\left(1 - \frac{r}{R}\right)}$$

$$dr = -\frac{R}{H}\,dy$$

So $\boxed{dy = -\frac{H}{R}\,dr}$

So from (I)

$$dI = \frac{3M}{\cancel{\pi} R^2 \cancel{H}}\,\cancel{\pi} r^2 \left(-\frac{\cancel{H}}{R}dr\right)\left(\frac{r^2}{4} + H^2\left(1-\frac{r}{R}\right)^2\right)$$

$$\cancel{dI = \frac{3M}{R^3}\left[\frac{r^4}{4}dr + H r^2 dr - \frac{H}{R} r^3 dr\right]}$$

$$= \frac{3M}{R^3} r^2 dr \left[\frac{r^2}{4} + H^2\left(1 + \frac{r^2}{R^2} - 2\frac{r}{R}\right)\right]$$

$$\int dI = \frac{3M}{R^3}\left[\int_0^R \frac{r^4}{4}dr + H^2\left(\int_0^R r^2 dr + \int_0^R \frac{r^4}{R^2}dr - \frac{2}{R}\int_0^R r^3 dr\right)\right]$$

$$= \frac{3M}{R^3}\left[\frac{1}{4}\frac{R^5}{5} + H^2\left(\frac{R^3}{3} + \frac{1}{\cancel{R^2}}\left(\frac{R^{\cancel{5}}}{5}\right) - \frac{\cancel{2}}{\cancel{R}}\left(\frac{R^{\cancel{4}}}{\cancel{4}_2}\right)\right)\right]$$

$$= \frac{3M}{R^3}\left[\frac{R^5}{20} + H^2R^3\left[\frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right]\right]$$

$$= \frac{3MR^2}{20} + 3MH^2\left(\frac{10+6-15}{30}\right)$$

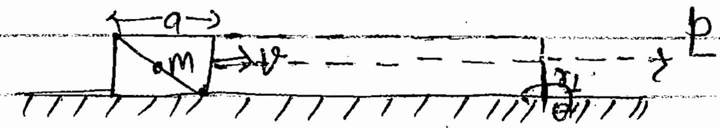
$$= \frac{3MR^2}{20} + \frac{\cancel{3}MH^2}{\cancel{30}\,10}$$

$$\boxed{I = \frac{3MR^2}{20} + \frac{MH^2}{10}}$$

Ans

A-7

Q.22 A cubical block of side 'a' is moving with velocity 'v' on a horizontal smooth plane as shown. It hits a ridge at point O. The angular speed of the block after it hits O is?

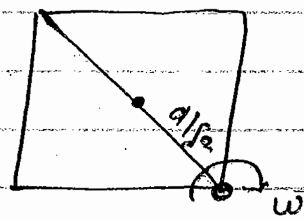


Soln So applying Conservation of Angular momentum -

$L_i = L_f$

$p r_\perp = I\omega$

$MV \cdot \frac{a}{2} = I\omega$



$\frac{Ma^2}{2}$

So $MV \cdot \frac{a}{2} = \left[\frac{Ma^2}{6} + \frac{Ma^2}{2}\right]\omega$

$\Rightarrow \cancel{M}V \cdot \frac{\cancel{a}}{2} = \cancel{M}a^{\cancel{2}}\omega\left[\frac{1}{6}+\frac{1}{2}\right]$

$\Rightarrow \frac{V}{2} = \omega a\left[\frac{1+3}{6}\right]$

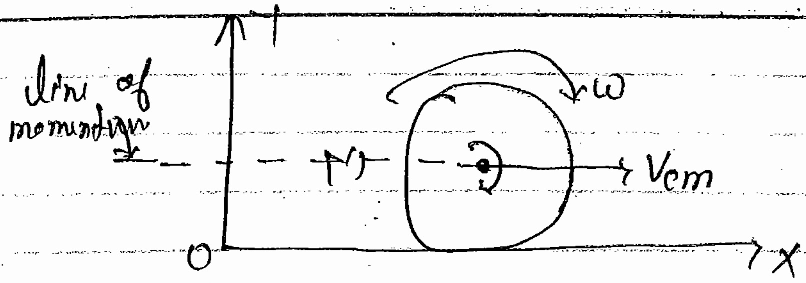
$\Rightarrow \frac{V}{\cancel{2}} = \frac{4\omega a}{\cancel{6}\,3}$

$\Rightarrow \boxed{\omega = \frac{3V}{4a}}$ So ans (a) is correct.

A-7

Q.23 A disc of mass M and Radius R is rolling with angular speed ω on a horizontal plane as shown in figure. The magnitude of angular momentum of the disc about the origin O is?

Soln



$$L = I\omega + M V_{cm} R$$

$$= \frac{MR^2}{2}\omega + M\omega R.R$$

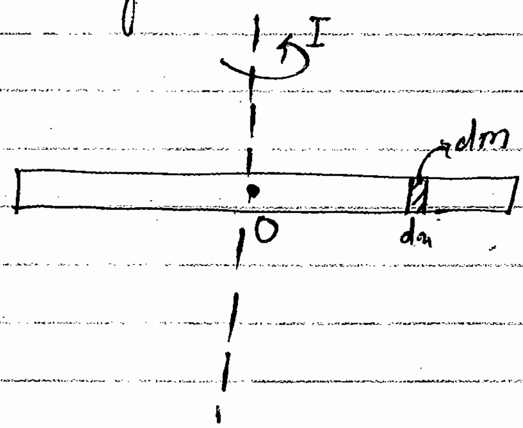
$$= \frac{MR^2\omega}{2} + M\omega R^2$$

$$\boxed{L = \frac{3}{2} M\omega R^2}$$

A-6 Q.9

Q. modified: Calculate moment of Inertia about a perpendicular axis through its center of mass.

$\lambda = \lambda_0 |x|$ $\quad x$ = dist. from center



$$\lambda = \frac{dm}{dx}$$

$$\lambda_0 |x| = \frac{dm}{dx}$$

$$dm = \lambda_0 |x| dx$$

$$I = \int dm\, x^2$$

$$= \int_{-l}^{+l} \lambda_0 |x| dx . x^2$$

$$= \int_{-l}^{0} \lambda_0 (-x)\, x^2 dx + \int_{0}^{l} \lambda_0 (x)\, x^2 dx$$

$$I = -\lambda_0\left[\frac{x^4}{4}\right]_{-l}^{0} + \lambda_0\left[\frac{x^4}{4}\right]_0^{l}$$

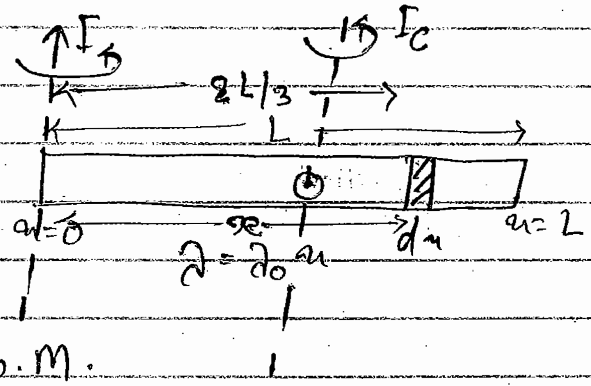
$$= +\lambda_0\frac{l^4}{4} + \lambda_0\frac{l^4}{4} = \frac{2\lambda_0 l^4}{4}$$

$$\boxed{I = \frac{\lambda_0 l^4}{2}}$$

$$x_{com} = \frac{\int x\,dm}{\int dm} = \frac{\int_{-L}^{+L} x\lambda_0|x|\,dx}{\int_{-L}^{+L}\lambda_0|x|\,dx} = 0$$

$$\boxed{x_{com} = 0}$$

Q. When x is measured from one end. What is the M.I of Rod about a $\perp^{r}$ axis through its c.o.m.



Sol^n

$$\lambda = \frac{dm}{dx} = \lambda_0 x$$

$$dm = \lambda_0 x\,dx$$

$$I = \int dm\, r_\perp^2$$

$$\because r_\perp = x$$

$$\therefore I = \int_0^L \lambda_0 x\,dx \cdot x^2$$

$$\boxed{I = \frac{\lambda_0 L^4}{4}}$$

$$X_{com} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^l x \cdot \lambda_0 x\,dx}{\int_0^l \lambda_0 x\,dx}$$

$$= \frac{A_0 \frac{L^3}{3}}{\underbrace{\lambda_0 \frac{L^2}{2}}_{M}}$$

$$\boxed{X_{com} = \frac{2L}{3}}$$

So $\quad I = I_c + M\left(\frac{2L}{3}\right)^2$

$$\frac{\lambda_0 L^4}{4} = I_c + \lambda_0 \frac{L^2}{\cancel{2}} \cdot \frac{\cancel{4}^{2} L^2}{9}$$

$$\frac{\lambda_0 L^4}{4} = I_c + \frac{2\lambda_0 L^4}{9}$$

$$I_c = \lambda_0 L^4\left(\frac{1}{4} - \frac{2}{9}\right) = \lambda_0 L^4\left(\frac{9-8}{36}\right)$$

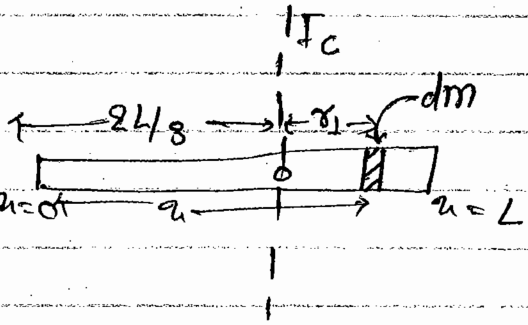
$$\boxed{I_c = \frac{\lambda_0 L^4}{36}}$$ Ans

Method-II :- Giving I_c directely :-

first calculate ~~c.o.~~ position of com.

$$I = \int dm\, r_1^2$$

$$= \int_0^L \lambda_0 x\,dx \left(x - \frac{2L}{3}\right)^2$$



$$r_1 = x - \frac{2L}{3}$$

$$I = \lambda_0 \int_0^L x\left(x^2 + \frac{4}{9}L^2 - \frac{4}{3}Lx\right)dx$$

$$= \lambda_0\left[\int_0^L x^3\,dx + \frac{4L^2}{9}\int_0^L x\,dx - \frac{4L}{3}\int_0^L x^2\,dx\right]$$

$$= \lambda_0\left[\frac{L^4}{4} + \frac{\cancel{4}^{2}L^2}{9}\cdot\frac{L^2}{\cancel{2}} - \frac{4}{3}L\,\frac{L^3}{3}\right]$$

$$= \lambda_0\left[\frac{L^4}{4} + \frac{2L^4}{9} - \frac{4L^4}{9}\right]$$

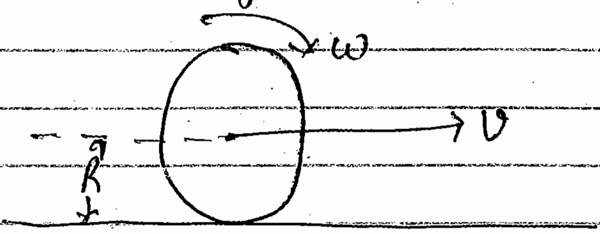
$$= \lambda_0 L^4\left[\frac{1}{4} - \frac{2}{9}\right] = \lambda_0 L^4\left[\frac{9-8}{36}\right]$$

$$I = \frac{1}{36}\lambda_0 L^4$$

So $\boxed{I = \frac{\lambda_0 L^4}{36}}$ Ans

Q. A circular disc of mass M and radius R is moving on a horizontal surface with speed v if angular momentum of the disc about the point on its path is $2MVR$. what is K.E. of the disc?

Sol



$$L = I\omega + mvR$$

$$2mvR = I\omega + mvR$$

$$mvR = I\omega$$

$$\cancel{m}v\cancel{R} = \frac{\cancel{m}R^{\cancel{2}}}{2}\omega \Rightarrow \boxed{\omega = \frac{2v}{R}}$$

$$K.E. = \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2$$

$$= \frac{1}{\cancel{2}} \frac{m\cancel{R^2}}{\cancel{2}} \cdot \frac{\cancel{4} v^2}{\cancel{R^2}} + \frac{1}{2} m v^2$$

$$\boxed{K.E. = \frac{3}{2} m v^2}$$

* Pure Rotatimal Motion of Rigid Body :- At least one point of rigid body is fixed.

It has two types -

(I) Rotation about a fixed axis (Here point is also fixed)

(II) Rotation about a fixed point (axis is not fixed.).

Case I :-

$L_{axis} = I\omega$ → This may or may not be total angular momentum.

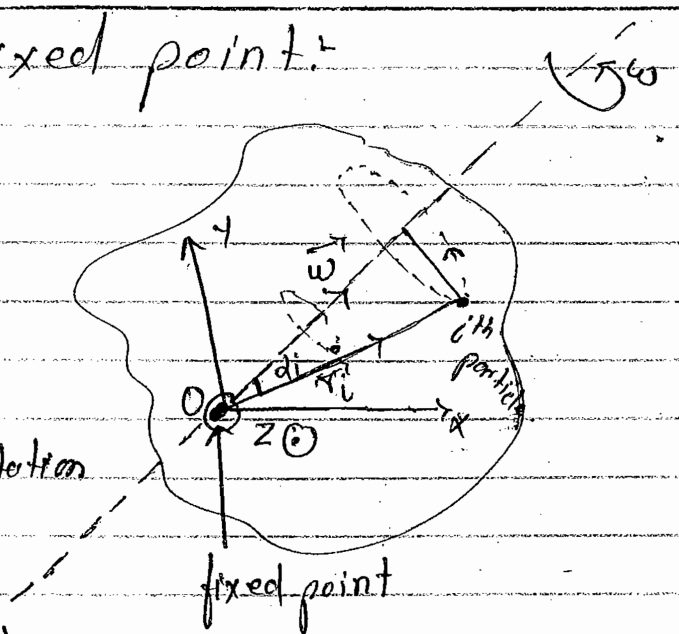
$$\boxed{K.E. = \frac{1}{2} I \omega^2}$$

Case II :-

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

$$\boxed{K.E. = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}}$$

* Rotation about a fixed point:-



Radius of circul in which i^{th} particle moves is :- $r_i \sin\alpha_i$

tangential velocity of i^{th} particle due to rotation

$v_i = \omega r \rightarrow$ Radius

So speed of i^{th} particle

$v_i = \omega r_i \sin\alpha_i$

$$\boxed{\vec{v_i} = \vec{\omega} \times \vec{r_i}}$$

$\searrow$ Axis about which the rigid body is ~~osci~~ rotating at semi instant of time.

{ We can't write $\vec{r_i} \times \vec{\omega}$ becoz it can't give correct sence of velocity.

Angular momentum of i^{th} particle about point 'O'

$$\vec{L}_{i^{th}} = \vec{r_i} \times \vec{p_i}$$

$$= \vec{r_i} \times m_i \vec{v_i}$$

$$= m_i (\vec{r_i} \times \vec{v_i}) = m_i [\vec{r_i} \times (\vec{\omega} \times \vec{r_i})]$$

Angular momentum of rigid body about point O.

$$\vec{L} = \sum_{i=1}^{N} \vec{L_i}$$

$$\boxed{\vec{L} = \sum_{i=1}^{N} m_i [\vec{r_i} \times (\vec{\omega} \times \vec{r_i})]}$$

$$\vec{r_i} = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$$

$$\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$$

$$\vec{L} = L_x\hat{x} + L_y\hat{y} + L_z\hat{z}$$

Put $\vec{r_i}$, $\vec{\omega_i}$ and $\vec{L}$ in above expression to get [equate Coefficients of $\hat{x}$, $\hat{y}$, and $\hat{z}$)

$$L_x = \underbrace{\sum m_i(y_i^2 + z_i^2)}_{I_{xx}}\omega_x - \underbrace{\sum m_i x_i y_i}_{I_{xy}}\omega_y - \underbrace{\sum m_i x_i z_i}_{I_{xz}}\omega_z$$

$$L_y = -\underbrace{\sum m_i y_i x_i}_{I_{yx}}\omega_x + \underbrace{\sum m_i(x_i^2 + z_i^2)}_{I_{yy}}\omega_y - \underbrace{\sum m_i y_i z_i}_{I_{yz}}\omega_z$$

$$L_z = -\underbrace{\sum m_i z_i x_i}_{I_{zx}}\omega_x - \underbrace{\sum m_i z_i y_i}_{I_{zy}}\omega_y - \underbrace{\sum m_i(x_i^2 + y_i^2)}_{I_{zz}}\omega_z$$

So. $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}}_{\text{Co-ordinate dependent}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

values of elements will change if axis chosen are changed.

Here the axes is choosen in that manner that off diagonal elements will be zero.

- Here matrix (3×3) is II^{nd} rank Tensor.
- It is symmetric and real.
- So it is a Hermitian matrix.
- its eigen values are real.
- This matrix is known as Inertia Tensor

So Symbolic representation:-

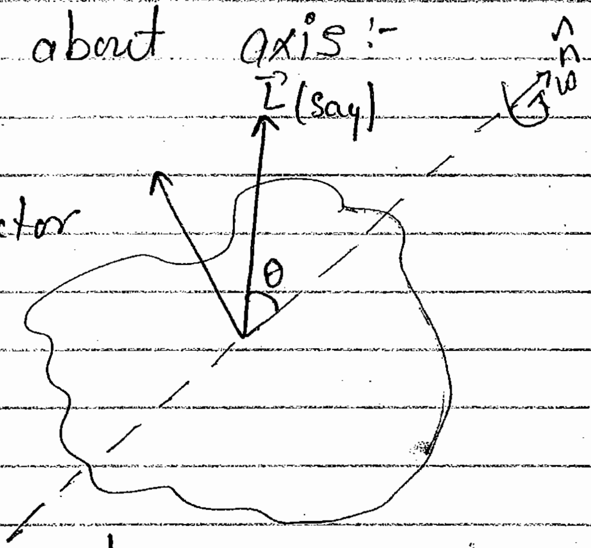
$$\boxed{\vec{L} = \overleftrightarrow{II} \cdot \vec{\omega}}$$

Compact form of matrix relation.

Symbol for Inertia Tensor.

It is the general formula for Angular momentum.

* Angular Momentum about axis:-



Let $\hat{n}$ is unit vector along the axis

$$\vec{\omega} = \omega \hat{n}$$

$$L_{axis} = \vec{L} \cdot \hat{n}$$

$$= \hat{n} \cdot \vec{L}$$

$$= \hat{n} \cdot \overleftrightarrow{II} \cdot \vec{\omega}$$

$$= (\hat{n} \cdot \overleftrightarrow{II} \cdot \hat{n})\, \omega$$

$L_{axis} = L\cos\theta$

we know that -

$$L_{axis} = I\omega$$

Row Matrix

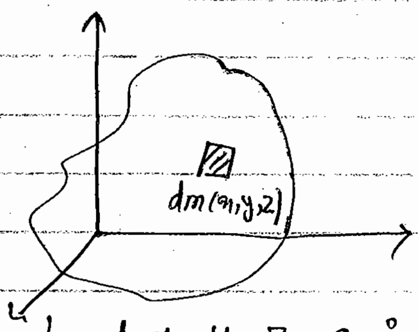
3x3 Matrix

Coloumn matrix.

So $\boxed{I = \hat{n} \cdot \overleftrightarrow{II} \cdot \hat{n}}$ (Memorize)

* Inertia Tensor :-

$$\mathbb{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$



$I_{xx}, I_{yy}, I_{zz} \longrightarrow$ M.I. about x, y, z axis

$I_{ij}, \; i \neq j \longrightarrow$ Product of inertia.

$$I_{xx} = \sum_{i=1}^{N} m_i (y_i^2 + z_i^2) \quad , \quad I_{xx} = \int dm\,(y^2 + z^2)$$

(x, y, z Co-ordinates of dm)

discreate   Continuous

$$I_{xy} = -\sum_{i=1}^{N} m_i x_i y_i \quad , \quad I_{xy} = -\int dm\, xy$$

* Principal Axes :-

The set of three axes for which product of inertia are zero, are called principal Axes [Mathematical Defination].

If x, y, z are principal axes -

$I_{ij} = 0 \quad i \neq j \quad , \quad I_{xy} = 0 \, , \; I_{yx} = 0$ etc.

$$\mathbb{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$\longrightarrow$ Principal Moment of Inertia.

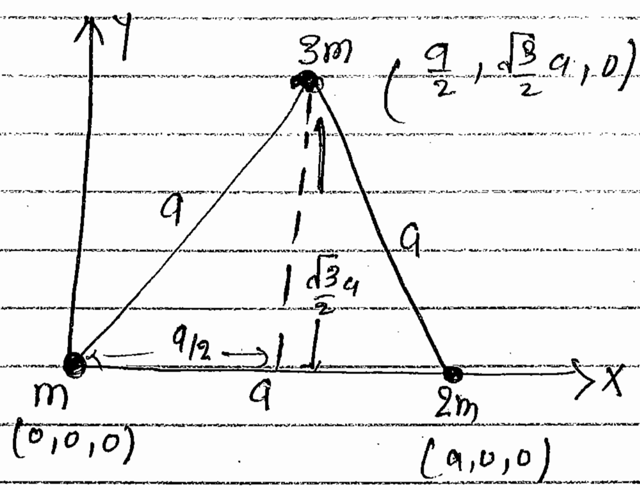
$$\boxed{\text{Principal M.I = Egen Value of Inertia Tensor}}$$

⇒ Direction of Principal Axes :-

Direction of principal Axes are given by eigen vectors of inertia tensor.

A-6
Q.18 In the figure shown value of I_{xy} is ?

$$I_{xy} = -\sum_{i=1}^{3} m_i x_i y_i$$



$$= -\left(m \times 0 + 2m \times 0 + 3m \times \frac{a}{2} \times \frac{\sqrt{3}a}{2}\right)$$

$$\boxed{I_{xy} = -\frac{3\sqrt{3}ma^2}{4}}$$

∵ $I_{xy} \neq 0$ (Product of Inertia $\neq 0$)

So X, Y, Z will ~~be~~ not be principal axes

* Draw the principal Axes about (0,0,0) :-

If We want to find orientation of principal axis then we need to write inertia tensor and find its eigen vectors.

$\therefore \; I_{yx} = I_{xy} = \frac{-3\sqrt{3}a}{4}$

$I_{yz} = I_{zy} = -\sum m_i y_i z_i = 0$

$I_{zx} = I_{xz} = 0$

$I_{xx} = \sum m_i (y_i^2 + z_i^2) = m \times 0 + 2m \times 0 + 3m\left(\frac{3a^2}{4} + 0^2\right)$

$I_{xx} = \frac{9a^2}{4}$

$I_{yy} = \sum m_i (x_i^2 + z_i^2) = m \times 0 + 2m(a^2 + 0) + 3m\left(\frac{a^2}{4} + 0\right)$

$= 2ma^2 + \frac{3ma^2}{4} = \frac{11ma^2}{4}$

$I_{zz} = \sum m_i (x_i^2 + y_i^2) = m \times 0 + 2m(a^2 + 0) + 3m\left(\frac{a^2}{4} + \frac{3a^2}{4}\right)$

$I_{zz} = 2ma^2 + 3ma^2 = 5ma^2$

$$\mathbb{I} = ma^2 \begin{pmatrix} \frac{9}{4} & -\frac{3\sqrt{3}}{4} & 0 \\ -\frac{3\sqrt{3}}{4} & \frac{11}{4} & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

To find the direction of principal axis we will calculates eigen vector.

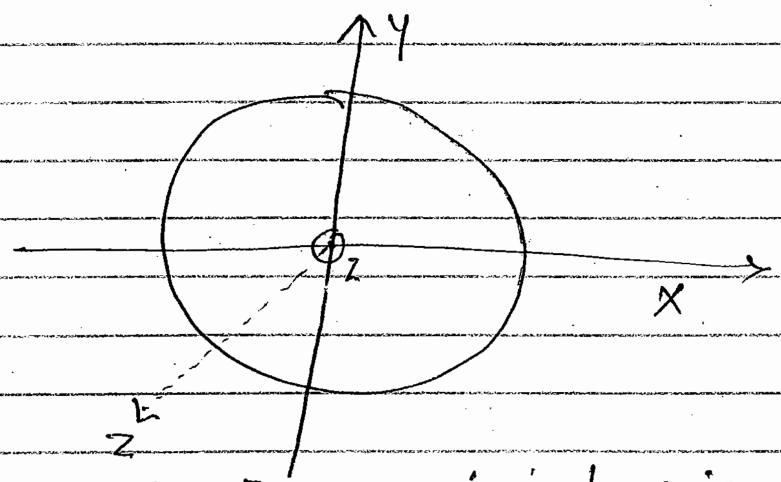
* **Principal Axes** :- {Defined through a point of rigid body}:-

Physical Defination :-

A symmetric axis is always a principal axis if the object has axial symmetry.

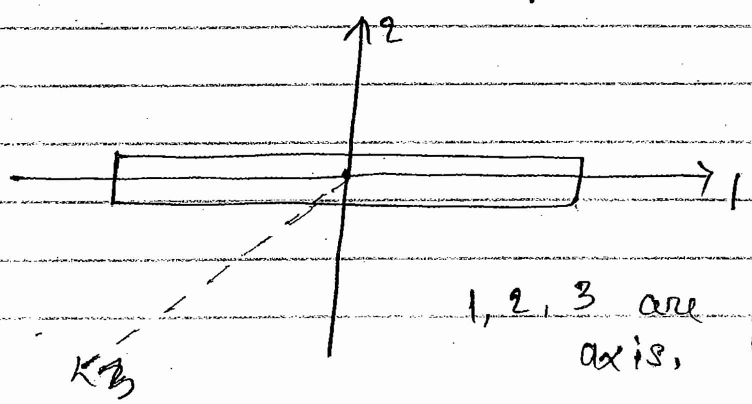
Ex -

(1) Sphere.



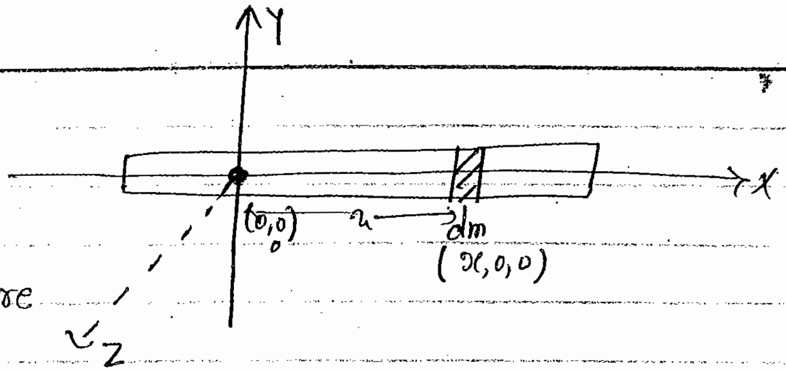
x, y, z are principal axis.

(2) Rod



1, 2, 3 are principal axis,

$\therefore I_{ij} = 0 \quad i \neq j$



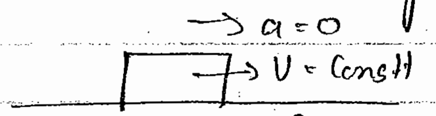
Product of inertia are zero $\therefore$ x, y, z are principal axes.

* Principal axes may or may not be symmetric axes.
Every symmetric axes is a principal axes but every principal axes will not be symmetric.

* Actual Physical Defination of Principal Axes :-

"The set of ~~axe~~ axes about which uniform rotational motion can be maintained without application of a torque are called Principal Axes."

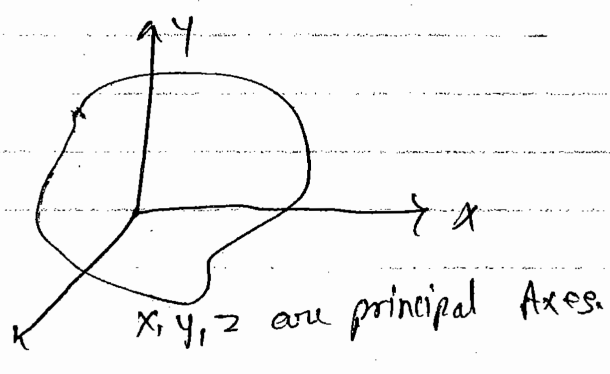
* The axes for which $\vec{L} \parallel \vec{\omega}$ are principal axes.



* Property of principal moment of inertia :-

X, Y, Z are principal axes
Then,

$$I_{xx} = \sum m_i (y_i^2 + z_i^2)$$
$$I_{yy} = \sum m_i (x_i^2 + z_i^2)$$
$$I_{zz} = \sum m_i (x_i^2 + y_i^2)$$



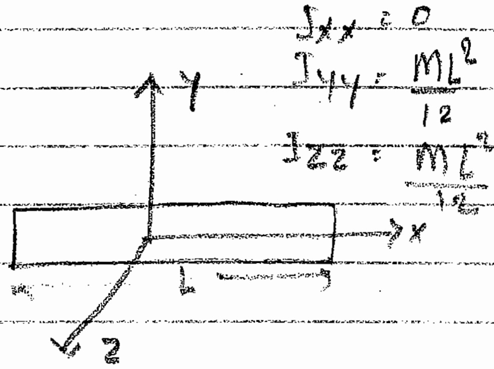
$$I_{xx} + I_{yy} = \sum m_i (x_i^2 + y_i^2 + 2z_i^2)$$

$$I_{xx} + I_{yy} = I_{zz} + 2\sum m_i z_i^2$$

$$\boxed{I_{xx} + I_{yy} - I_{zz} = 2\sum m_i z_i^2 > 0}$$

$$I_{xx} + I_{yy} - I_{zz} \geq 0$$

$$\boxed{I_{xx} + I_{yy} \geq I_{zz}}$$



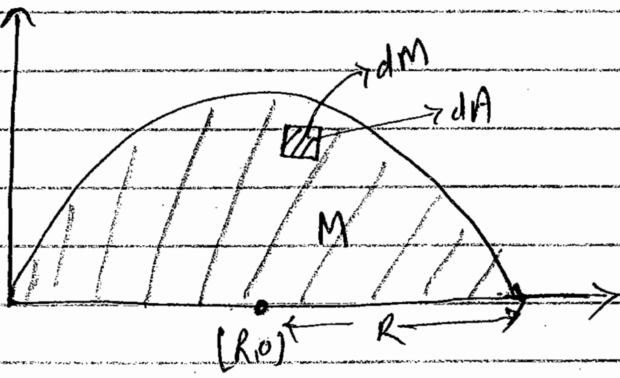
Similarly -

$$\boxed{I_{yy} + I_{zz} \geq I_{xx}} \text{ and } \boxed{I_{xx} + I_{zz} \geq I_{yy}}$$

A-6

Q.11 A semicircular disc of Radius R and mass M is kept in x-y plane as shown in figure. The product of inertia I_{xy} is?

Sol^n



$$I_{xy} = -\int dm\, x \cdot y$$

$$dm = \frac{M}{A} dA = \frac{M}{\frac{\pi R^2}{2}}$$

$$\frac{dm}{dA} = \frac{2M}{\pi R^2}$$

$$dm = \frac{2M}{\pi R^2} dx\,dy$$

$$I_{xy} = -\frac{2M}{\pi R^2} \int\int x\,y\,dx\,dy$$

∵ Here if we moving along the boundary then x and y both change. So limits are dependents.

So first we write equation of boundary –

$$(x-R)^2 + (y-0)^2 = R^2$$

$$\boxed{y = \sqrt{R^2-(x-R)^2}}$$

So
$$I_{xy} = -\frac{2M}{\pi R^2}\int_0^{2R} x\,dx \int_0^{\sqrt{R^2-(x-R)^2}} y\,dy$$

$$\cancel{= -\frac{2M}{\pi R^2}\left[\frac{x^2}{2}\right]_0^{2R}\left[\frac{y^2}{2}\right]_0^{\sqrt{R^2-(x-R)^2}}}$$

$$\cancel{= -\frac{2M}{\pi R^2}\left[\frac{4R^2}{2}\right]\left[\frac{R^2-(x-R)^2}{2}\right]}$$

$$\cancel{= -\frac{2M}{4\pi}\left[4(R^2-x^2-R^2+2xR)\right]}$$

$$= -\frac{2M}{\pi R^2}\int_0^{2R}\left[\frac{y^2}{2}\right]_0^{\sqrt{R^2-(x-R)^2}} x\,dx$$

$$= -\frac{2M}{\pi R^2}\int_0^{2R}\left\{\frac{R^2-(x-R)^2}{2}\right\}\cdot x\,dx$$

$$= -\frac{2M}{\pi R^2}\int_0^{2R}\left[\frac{R^2-x^2-R^2+2xR}{2}\right]x\,dx$$

$$= -\frac{M}{\pi R^2}\int_0^{2R}(-x^3+2x^2R)\,dx$$

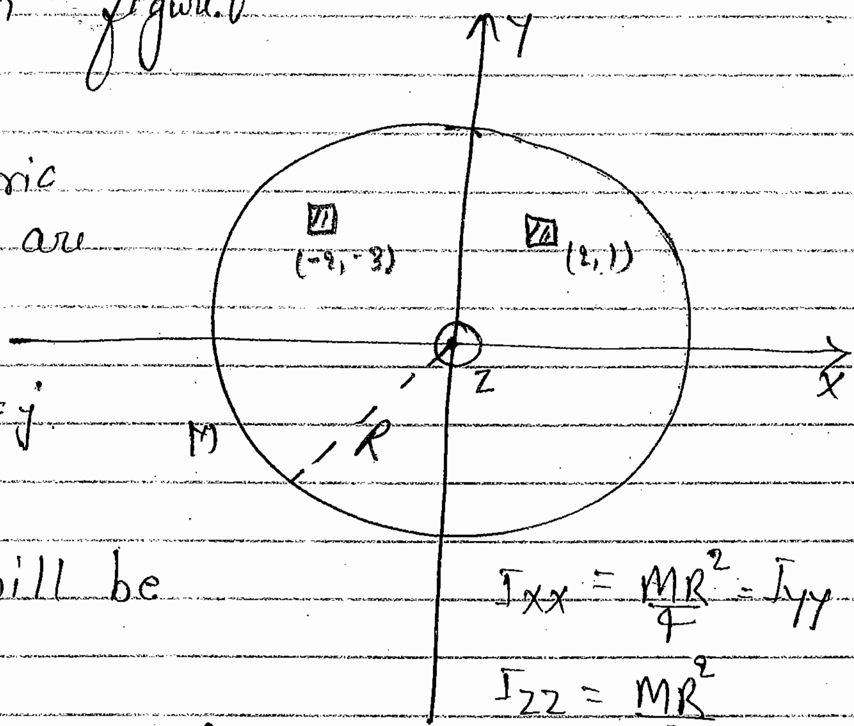
$$= \frac{-M}{\pi R^2}\left[-\left(\frac{x^4}{4}\right)_0^{2R} + 2R\left(\frac{x^3}{3}\right)_0^{2R}\right]$$

$$= \frac{-M}{\pi R^2}\left[\frac{-16R^4}{4} + \frac{2R \times 8R^3}{3}\right] = \frac{-M}{\pi R^2}\left[\frac{-16R^4}{4} + \frac{16R^4}{3}\right]$$

$$= +\frac{16R^4 M}{\pi \cancel{R^2}}\left[\frac{1}{4} - \frac{1}{3}\right] = \frac{\cancel{16}^{4} MR^2}{\pi}\left[\frac{3-4}{\cancel{12}_{3}}\right]$$

$$I = -\frac{4MR^2}{3\pi} \quad \underline{\text{Ans}}$$

Q. Write Inertia tensor of the disc about the axes shown in figure.



X, Y, Z are symmetric axes so these are principal axes.

$\therefore \quad I_{ij} = 0$ for $i \neq j$.

$$I_{xx} = \frac{MR^2}{4} = I_{yy}$$

$$I_{zz} = \frac{MR^2}{2}$$

Inertia tensor will be written -

$$\mathbb{I} = \frac{MR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

* Calculation of I from $\mathbb{I}$ matrix :-

B.A.
Q.10

Sol^n

$$\mathbb{I} = \begin{pmatrix} 8 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 8 \end{pmatrix}$$

Magnitude of M.I about an axis.

$$\hat{n} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$$

$$I = \underset{\substack{\text{row matrix} \\ 1\times 3}}{\hat{n}} \cdot \underset{\substack{3\times 3 \\ \text{Matrix}}}{\overleftrightarrow{\mathbb{I}}} \cdot \underset{\substack{\text{Coloumn matrix} \\ 3\times 1}}{\hat{n}}$$

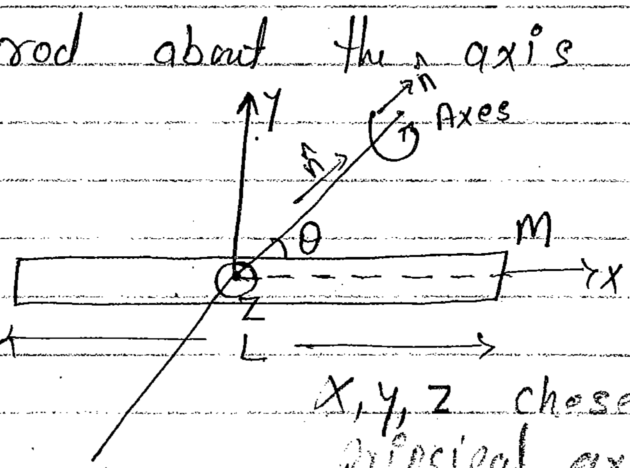
$$I = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \begin{pmatrix} 8 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 8 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}$$

$$I = \left(4, 2\sqrt{3}, -2\right) \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}$$

$$= 2 + 3 = 5$$

$\boxed{I = 5}$ Ans

Q. Find M.I. of thin rod about the axis shown in fig.



$$\mathbb{I} = \frac{ML^2}{12} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

X, Y, Z chosen as principal axis.

$$\hat{n} = (\cos\theta, \sin\theta, 0)$$

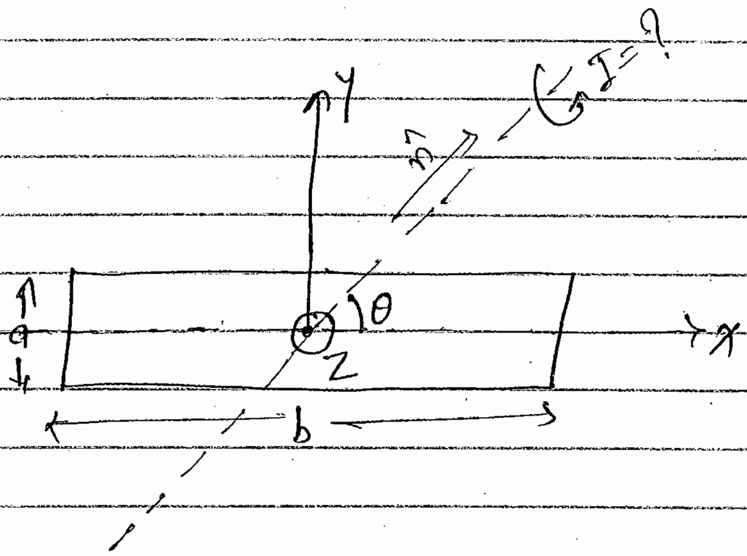
$$I = \hat{n} \cdot \overleftrightarrow{\mathbb{I}} \cdot \hat{n}$$

$$I = (\cos\theta, \sin\theta, 0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \frac{ML^2}{12}$$

$$= (0, \sin\theta, 0) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \frac{ML^2}{12}$$

$$\boxed{I = \frac{ML^2}{12} \sin^2\theta}$$ Ans

A-6
Q.21



$$I_{xx} = \frac{Ma^2}{12}$$

$$I_{yy} = \frac{Mb^2}{12}$$

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = \frac{M}{12}(a^2+b^2)$$

$$\overleftrightarrow{\mathbb{I}} = \frac{M}{12} \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & (a^2+b^2) \end{pmatrix}$$

$$\hat{n} = (\cos\theta, \sin\theta, 0)$$

So $\quad I = \hat{n} \cdot \overleftrightarrow{I} \cdot \hat{n}$

$$= \left(\cos\theta, \sin\theta, 0\right) \frac{m}{12} \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & (a^2+b^2) \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$= \left(a^2\cos\theta, b^2\sin\theta, 0\right) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \frac{m}{12}$$

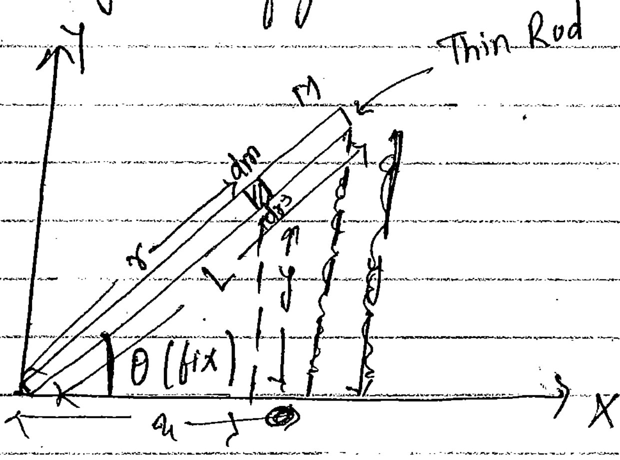
$$= \left(a^2\cos^2\theta + b^2\sin^2\theta\right) \frac{m}{12}$$

$$\boxed{I = \frac{ma^2}{12}\cos^2\theta + \frac{mb^2}{12}\sin^2\theta}$$

So option (d) is correct.

Q. Calculate $I_{xy} = ?$ of the given figure.

Sol^n



$$I = -\int dm\, xy$$

$$dm = \frac{M}{L} dr$$

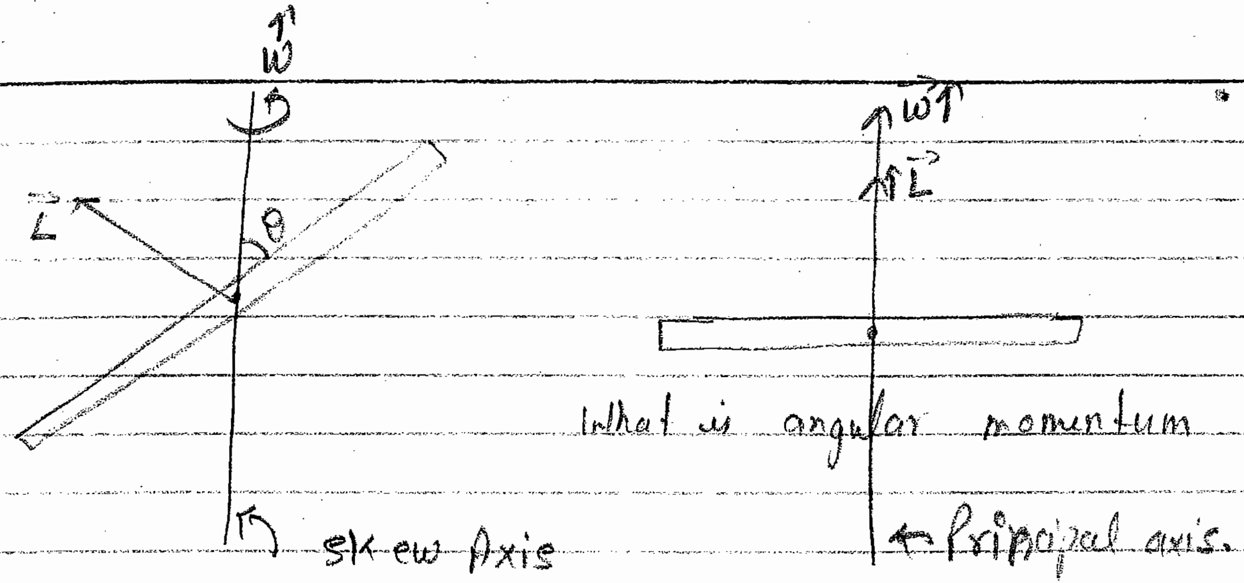
$$x = r\cos\theta$$
$$y = r\sin\theta$$

So $\quad I = -\int \frac{M}{L} dr \cdot r\cos\theta \cdot r\sin\theta$

$$= -\frac{M}{L}\cos\theta\sin\theta \int_0^L r^2 dr = -\frac{M}{L}\cos\theta\sin\theta \left[\frac{r^3}{3}\right]_0^L$$

$$\boxed{I = -\frac{ML^2}{3}\sin\theta\cos\theta}$$ Ans

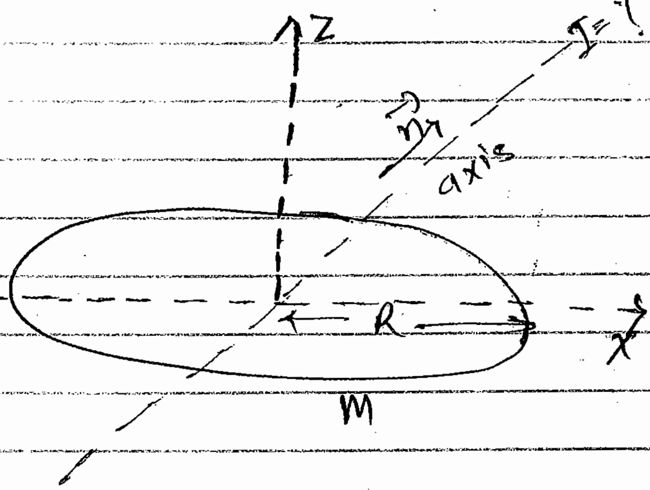
Q.



What is angular momentum

Q.

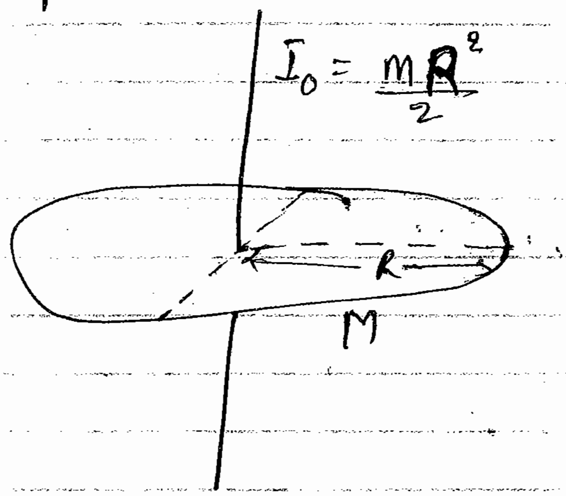
$$\mathbb{I} = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}$$

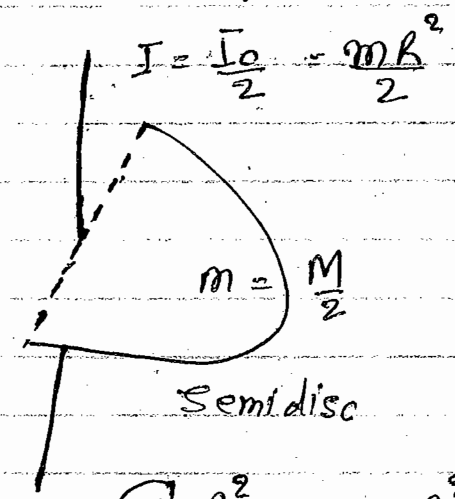


$$\hat{n} = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

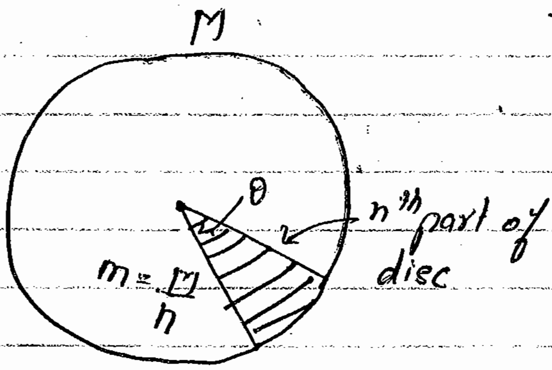
Note: Two important points related to M.I.

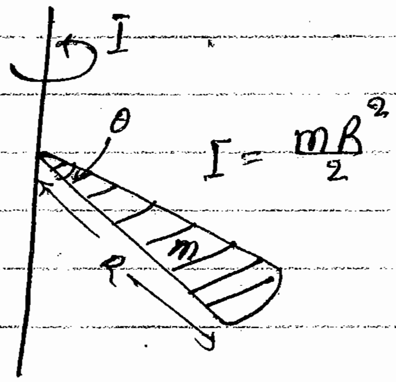
(i) If an object is cut symmetrically then form of M.I. of divided part is same as form of M.I of whole object when expressed in terms of mass of divided part.



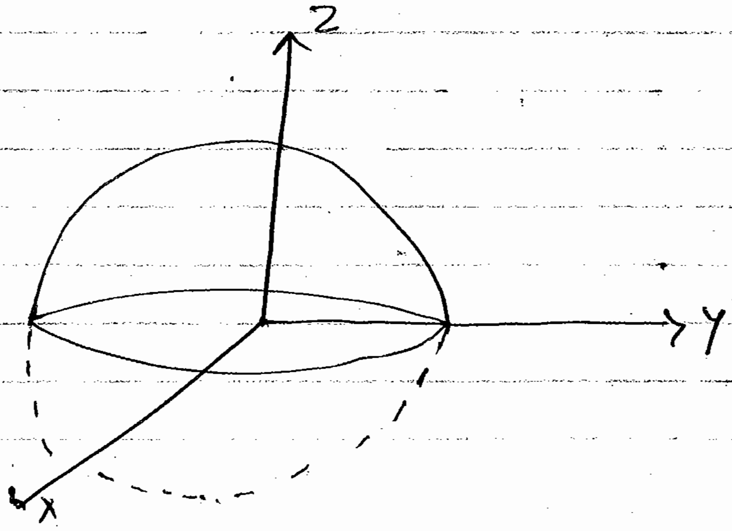


$$I = \frac{I_0}{2} = \frac{M}{2\times 2}R^2 = \frac{mR^2}{2}$$





A-6
Q.10

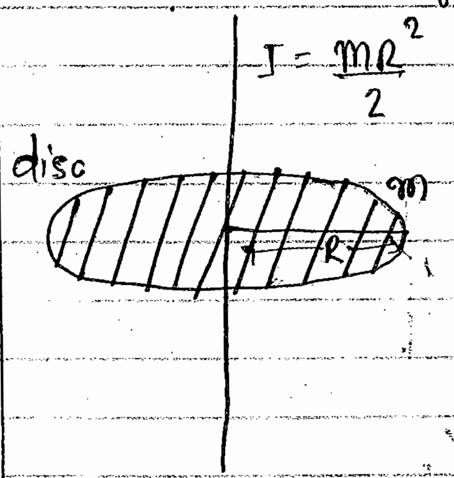


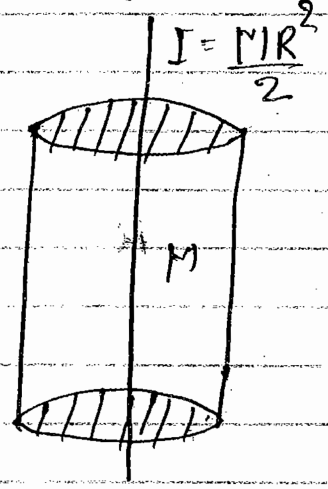
Sol^n I about $x = \frac{2}{3}MR^2$

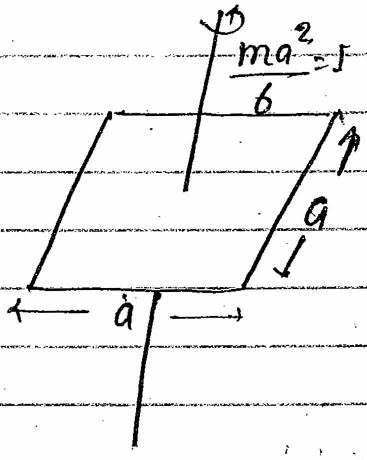
I about $z = \frac{2}{3}MR^2$

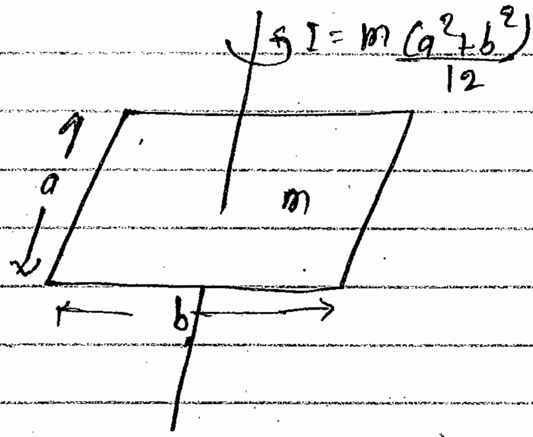
Assembling

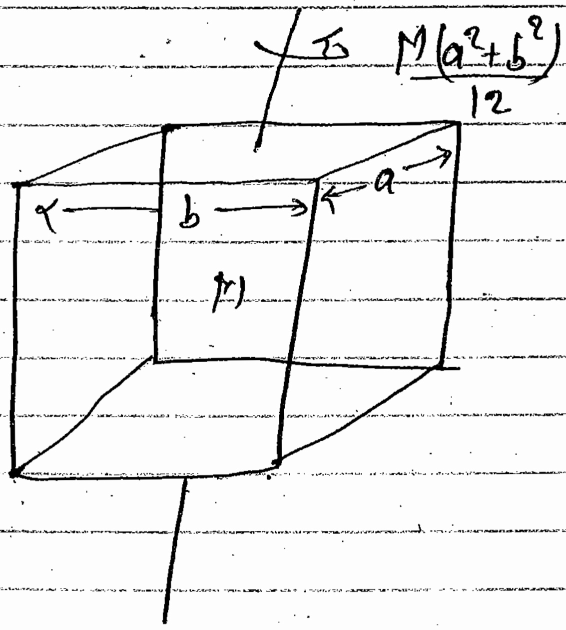
(ii) If stacking of small ~~obejts~~ ~~is~~ objects is done to make a big object then form of M.I. of big objects is same as of M.I. of small objects.

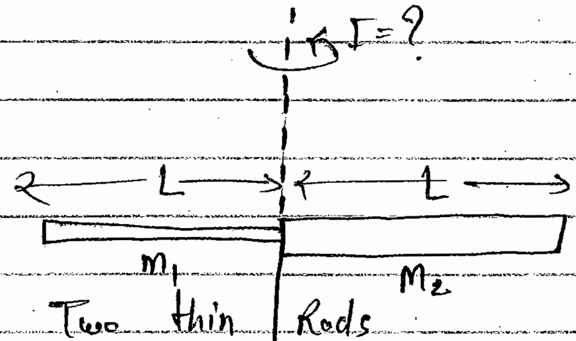












$$I = \frac{m_1L^2}{3} + \frac{M_2L^2}{3}$$

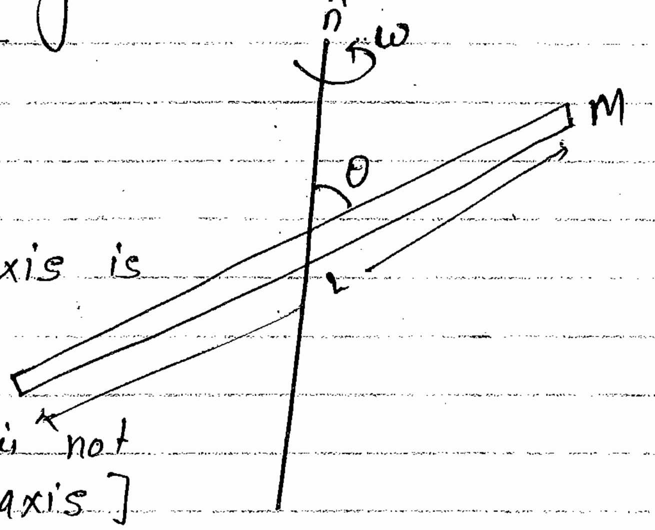
$$I = \frac{(m_1+m_2)L^2}{3}$$

* Angular Momentum about skew axis :-

Here two questions can be asked.

* What is the angular momentum about axis.
* What is total angular momentum.

(i) $\boxed{\vec{L}_{axis} = I_{axis}\,\omega}$

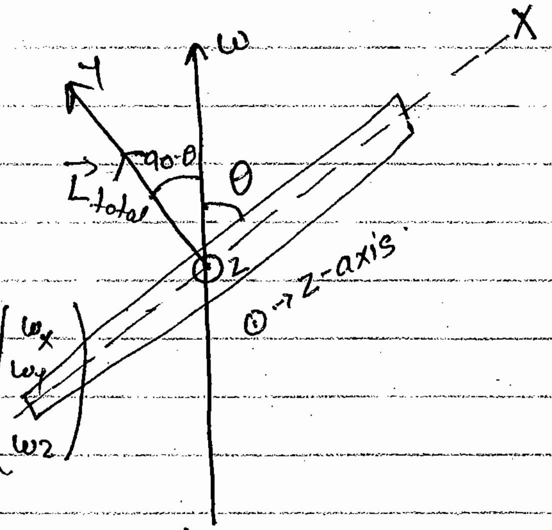


(ii) $\vec{L}_{total} = \vec{L}_{axis}$ (if axis is principal axis)

$\vec{L}_{total} = \overleftrightarrow{\mathbb{I}}\cdot\vec{\omega}$ [if axis is not a principal axis]

(i) $\vec{L}_{axis} = I_{axis}\cdot\omega$

$$\boxed{\vec{L}_{axis} = \frac{ML^2}{12}\sin^2\theta\cdot\omega}$$



(ii) $\vec{L}_{total} = \overleftrightarrow{\mathbb{I}}\cdot\vec{\omega}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{ML^2}{12} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{ML^2}{12} \end{pmatrix}\begin{pmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

Comparing Cofficient:-

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{mL^2}{12}\omega \sin\theta \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} L_x = 0 \\ L_y = \frac{mL^2}{12}\omega \sin\theta \\ L_z = 0 \end{matrix}$$

$$\vec{L}_{total} = \sqrt{L_x^2 + L_y^2 + L_z^2}$$

$$= \sqrt{0 + \left(\frac{mL^2}{12}\sin\theta\right)^2 + 0}$$

$$\boxed{\vec{L}_{total} = \frac{mL^2}{12}\omega \sin\theta}$$

Angular momentum about axis is just a component of total angular momentum.

$$L_{axis} = L_{y_{total}} \cos(90-\theta)$$

$$= L_{y_{total}} \sin\theta = \frac{mL^2}{12}\omega \sin\theta \cdot \sin\theta$$

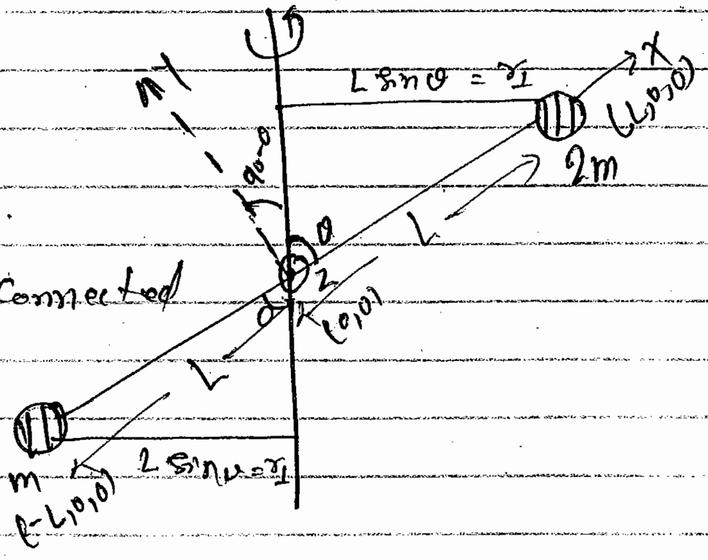
$$\boxed{L_{axis} = L_{total} \frac{mL^2}{12}\omega \sin^2\theta}$$

Ans

Q. find $L_{axis} = ?$

$L_{total} = ?$

Two small particle connected by a small rod.



$$L_{axis} = I_{axis} \cdot \omega$$

$$= \sum_{i=1}^{2} \left(m_i r_{\perp i}^2 \right) \omega$$

$$= \left(m \times L^2 \sin^2\theta + 2m \times L^2 \sin^2\theta \right) \omega$$

$$\boxed{\vec{L}_{axis} = 3mL^2 \sin^2\theta \cdot \omega}$$

Total Angular momentum :-

$$\text{Inertia Tensor } \mathbb{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3mL^2 & 0 \\ 0 & 0 & 3mL^2 \end{pmatrix}$$

$I_{yy} = \sum m_i (x_i^2 + z_i^2)$
$= m(l^2 + 0) + 2m \times l^2$
$= 3ml^2$

$I_{zz} = \sum m_i (x_i^2 + y_i^2)$

$$\vec{\omega} = \begin{pmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$$\vec{L}_{total} = \mathbb{I} \cdot \omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3mL^2 & 0 \\ 0 & 0 & 3mL^2 \end{pmatrix} \begin{pmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3mL^2 \sin\theta\,\omega \\ 0 \end{pmatrix}$$

$$\boxed{\vec{L}_{total} = L_{2m} + L_m = 3mL^2 \omega \sin\theta.}$$

$$K.E. = \frac{1}{2} I_{axis} \omega^2 = \frac{1}{2} 3mL^2 \omega \sin^2\theta \cdot \omega^2$$

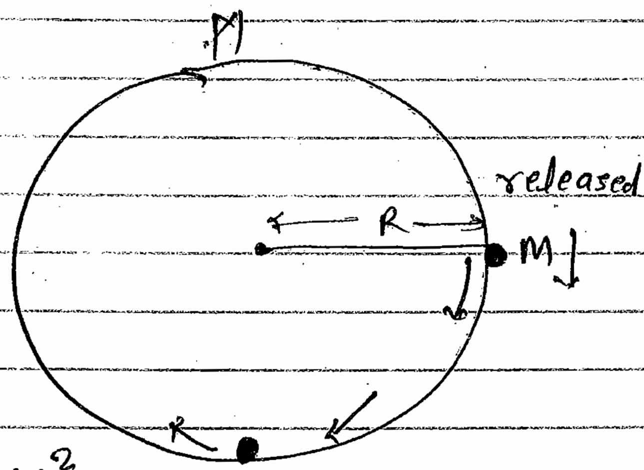
$$\boxed{K.E. = \frac{3}{2} mL^2 \omega^3 \sin\theta}$$

A-7

Q.20 A disc of mass M and Radius R is pivoted about a horizontal axis through its centre and a small body of the same mass M is attached to the rim of the disc. If the disc is released from rest with the small body at the end of a horizontal radius, the angular speed when the small body is at the bottom is?

(a) $\sqrt{\left(\frac{g}{4R}\right)}$ (b) $\sqrt{\left(\frac{g}{2R}\right)}$ (c) $\sqrt{\left(\frac{3g}{4R}\right)}$ ✓(d) $\sqrt{\left(\frac{4g}{3R}\right)}$

Sol^n Now apply Conservation of energy -



Loss in K.E. = Gain in K.E.

$$MgR = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2}\left(\frac{MR^2}{2} + MR^2\right)\omega^2$$

$$\cancel{M}g\cancel{R} = \frac{3\cancel{M}R^{\cancel{2}}}{4}\cdot\omega^2$$

$$\boxed{\omega = \sqrt{\frac{4g}{3R}}}$$

A-7

Q.21 A uniform bar of length 6a and mass 8m lies on a smooth horizontal table. Two point masses m and 2m moving in the same horizontal plane with speed 2v and v respectively strike the bar (see fig) and stick to the bar after collision.