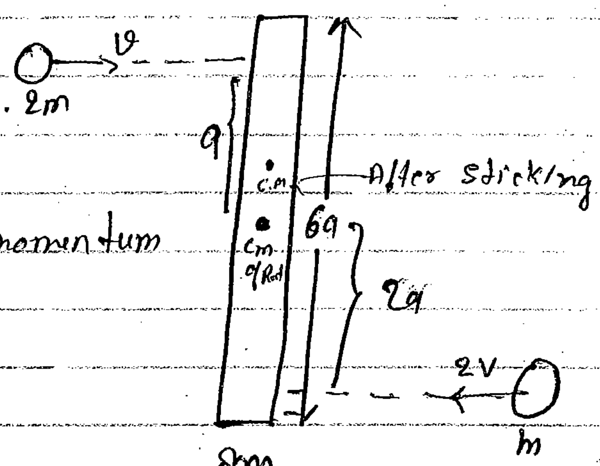
Denoting angular velocity (about the center of mass), total energy and centre of mass velocity by ω, E and v_c respectively, we have after collision.

(a) $v_c = 0$ (b) $\omega = \frac{3}{5}\left(\frac{v}{a}\right)$ (c) $\omega = \frac{v}{5a}$ (d) $E = \frac{3}{5}mr^2$

Solⁿ The whole system lying on the smooth table therefore $\vec{F}_{ext} = 0$

$$\because \vec{F}_{ext} = \frac{d\vec{p}}{dt}$$
$$\Downarrow$$
$$0 \Rightarrow \vec{p} = \text{Const.}$$



Now apply conservation of momentum

$$P_i = P_f$$

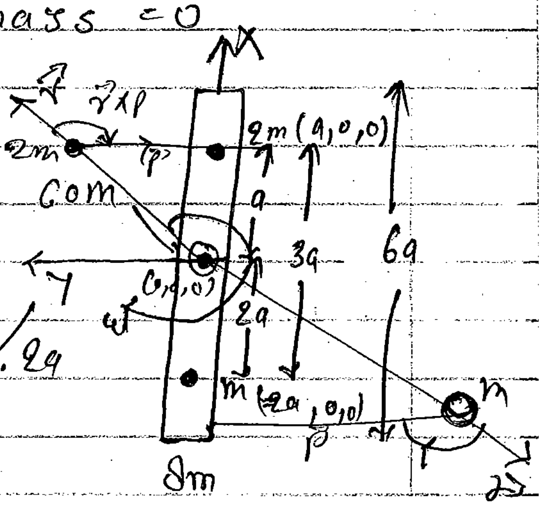
$$\cancel{2mv + m\,2v} = P_f$$

$$\boxed{P_f = 0} \Rightarrow \boxed{V_c = 0}$$

So Here initial and final momentum is $= 0$
So velocity of centre of mass $= 0$

$$X_{cm} = \frac{\sum m_i x_i}{\sum m_i}$$

$$= \frac{2m\cdot a + 8m\cdot 0 - m\cdot 2a}{11a}$$



$$= 0$$

So C.M. of system is mid of the rod.

Here $F_{ext} = 0$ therefore there is no external force So there is no external torque.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0$$

So $\vec{L}$ = Constant

So we apply Conservation of Angular momentum

$$L_i = L_f$$

$$V \times 2m \times a + m \times 2V \times 2a = I\omega$$

$$6maV = I\omega$$

$$6mVa = \Big(\underset{\substack{\uparrow \\ \text{Rod}}}{\frac{MR^2}{12}} + \underset{\substack{\uparrow \\ \text{Particle}}}{m_1 r_1^2} + \underset{\substack{\uparrow \\ \text{Particle}}}{m_2 r_2^2}\Big)\omega$$

$$= \left[\frac{2m(6a)^2}{12} + 2ma^2 + m(2a)^2\right]\omega$$

$$6mva = \left(\frac{2m \cdot \overset{3}{\cancel{36}}a^2}{\cancel{12}} + 2ma^2 + 4a^2m\right)\omega$$

$$\cancel{6m}v\cancel{a} = \frac{\cancel{36}}{5}\cancel{ma^2}\omega$$

$$\boxed{\omega = \frac{V}{5a}}$$

Total Energy : Pure Rotation:-

$$\text{Energy} = \frac{1}{2} I\omega^2$$

$$= \left(\frac{1}{\cancel{2}} \overset{\cancel{6}\,3}{\cancel{36}} m\cancel{a^2} \times \frac{V^2}{\underset{5}{\cancel{25}}\cancel{a^2}}\right)$$

$$\boxed{E = \frac{3mv^2}{5}}$$

Ans

11/Sep/2014

Special theory of Relativity

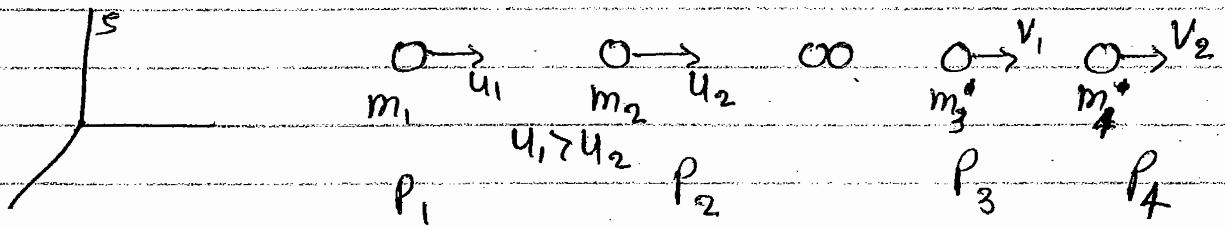
Relativistic Mechanics (It deals with dynamics at very high speed $v \sim c$) :-

It will reduce in to newtonian mechanics if $v << c$ or $c \longrightarrow \infty$.

- Special theory of Relativity deals with inertial frame of reference.

* Postulates of STR :-

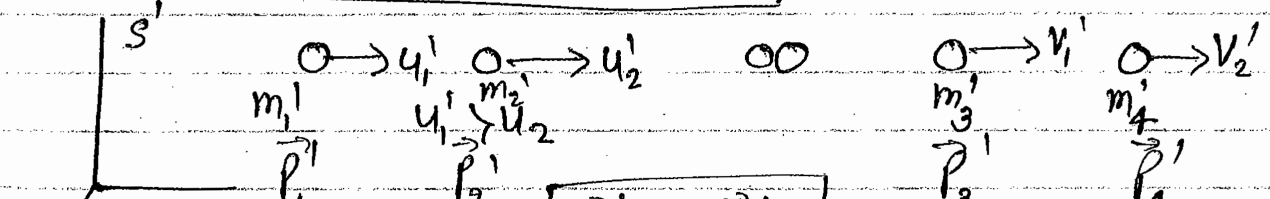
(i) Laws of physics are invariant in all inertial frame of reference. It means that mathematical mathematical form of the law remains unchanged in all inertial frame.



Law of Conservation of momentum -

$$\vec{P_i} = \vec{P_f}$$

$$\boxed{\vec{P_1} + \vec{P_2} = \vec{P_3} + \vec{P_4}}$$



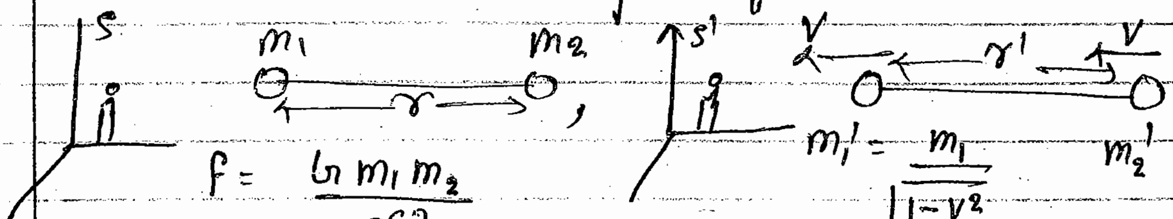
$$\boxed{\vec{P}_i' = \vec{P}_f'}$$

$$\vec{P}_1' + \vec{P_2} \neq \vec{P}_3' + \vec{P}_4'$$

In classical mechanics $c \to \infty$ where c is speed of information. In classical mechanics we tells that information is travelling in infinite speed.

---

* If mathematical form of the law doesn't remain invariant then that law should be modified.
for example :- Newton's Law of gravitation changes its form in different frames due to which Einstein modified it in his General theory of relativity.



$$f = \frac{G m_1 m_2}{r^2}$$

$$m_1' = \frac{m_1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad m_2' = \frac{m_2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$r' = r\sqrt{1-v^2/c^2}$$

$$\vec{f}' = \frac{G m_1' m_2'}{r'^2}$$

$$\boxed{\vec{f}' = \frac{G m_1 m_2}{r^2\left(1-\frac{v^2}{c^2}\right)^2}}$$

So newton's law of gravitation is not true.

$\Rightarrow$

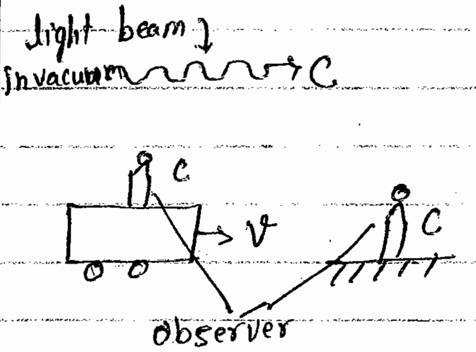
$$\nabla^2 E = \frac{1}{\mu_0 \epsilon_0} \frac{\partial^2 E}{dt^2} = \frac{1}{c^2} \frac{d^2 E}{dt^2}$$

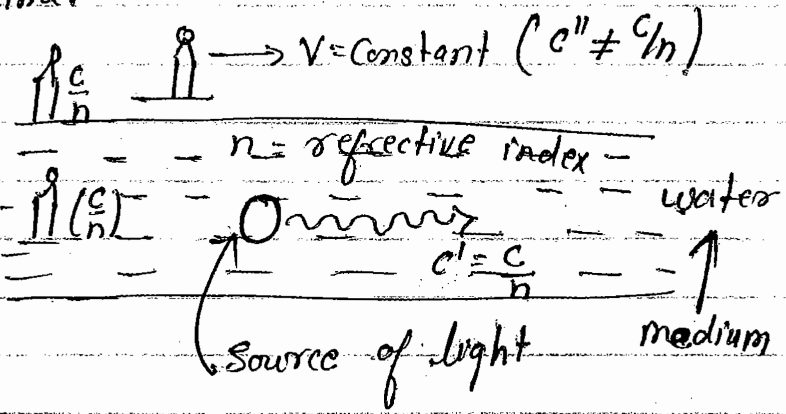
* **Second Postulate of Relativity :-** It states that speed of light ~~in vacuu~~ (magnitude of resultant velocity) in vacuum is same in an inertial frame and it is also ultimate speed.

* Direction may change.

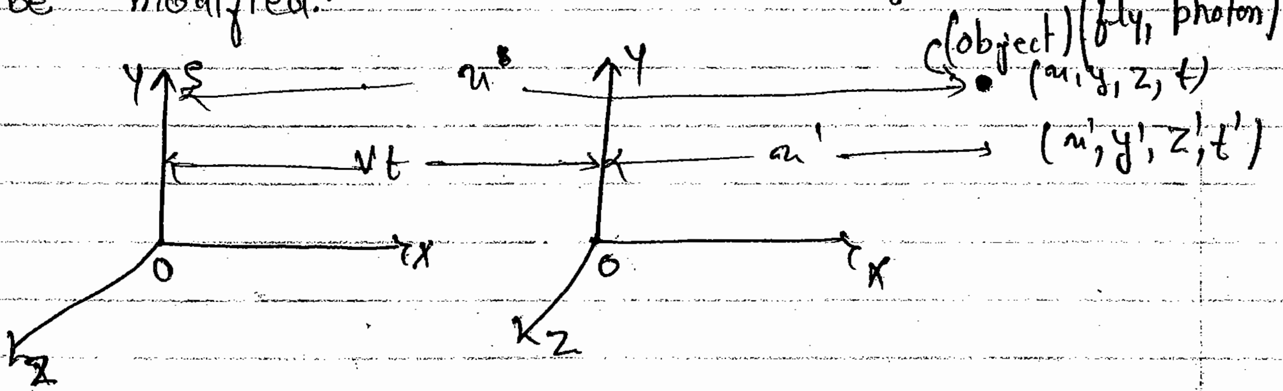
* Components may chance $\{ c = \sqrt{c_x^2 + c_y^2 + c_z^2} \}$.

* If there is a medium involved then speed of light will not be same in different ~~invert~~ inertial frame.





* **Lorentz's Transformation :-** Einstein said due to relative motion geometric relation must change therefore Galilean transformation must be modified.



Note :- Einstein using his two postulatee show that $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

See :- Aurthur Beiser's book.

Galilian Transformation :-

$$
\left.\begin{aligned}
x &= x' + vt \\
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}\right\}
$$

G. T. for coordinate
It's a Geometric relation or Euclidean Geometry

$$
\left.\begin{aligned}
v'_x &= v_x - v \\
v'_y &= v_y \\
v'_z &= v_z
\end{aligned}\right\}
$$

G-T for velocity

if light is viewed from s and s' its speed will be different in s and s'. So Galilian transformation is must be modified.

Here in $x' = x - vt$ for modification we can't change its linearty. becoz if we change their linearty then some contradiction will arises. Like if we modified x' in place of x by x^2 then we find two value. So at a same time we find two position. So modification is takes in that manner :-

L. T. :- {Direct Transformation}

$$
\left.\begin{aligned}
x' &= (x - vt)\gamma \\
y' &= y \\
z' &= z \\
t' &= \left(t - \frac{xv}{c^2}\right)\gamma
\end{aligned}\right\}
$$

It is valid if s' is moving in x direction.

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Einstein using his two postulates showed that

$$\boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

* L.T tells us that measurement has been done in s frame and we are concluding what the results will be in s'.

* Inverse L-T :-

$$\left.\begin{aligned} x &= (x' + vt)\gamma \\ y &= y' \\ z &= z' \\ t &= \left(t' + \frac{x'v}{c^2}\right)\gamma \end{aligned}\right\}$$

Here $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$s' \longrightarrow s$$

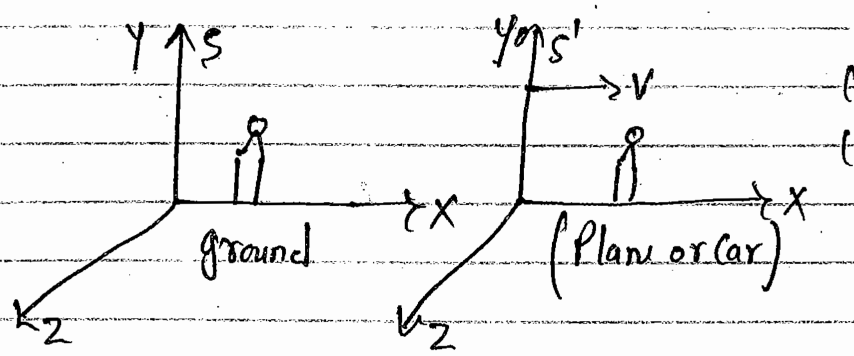
* Differential form of L-T :-

Direct
$$\begin{cases} \Delta x' = (\Delta x - v\Delta t)\gamma \\ \Delta t' = \left(\Delta t - \frac{v\Delta x}{c^2}\right)\gamma \end{cases}$$

$$\begin{cases} \Delta x = (\Delta x' + v\Delta t')\gamma \\ \Delta t = \left(\Delta t' + v\frac{\Delta x'}{c^2}\right. \end{cases}$$

These are used when time difference of coordinate are to be calculated.

* Velocity Transformation {Addition}:-



$$V_x = \frac{dx}{dt}$$

$$V_x' = \frac{dx'}{dt'} = \frac{(dx - v\,dt)\cancel{\gamma}}{\left(dt - \frac{v\,dx}{c^2}\right)\cancel{\gamma}}$$

$$= \frac{\left(\frac{dx}{dt} - v\right)}{\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{V_x - v}{1 - \frac{V_x v}{c^2}}$$

$$\boxed{V_x' = \frac{V_x - v}{1 - \frac{v}{c^2}V_x}} \qquad S \longrightarrow S'$$

$$V_y = \frac{dy}{dt}$$

$$V_y' = \frac{dy'}{dt'} =$$

So

$$V_x' = \frac{V_x - V}{\left(1 - \frac{V_x V}{c^2}\right)}$$

$$V_y' = \frac{V_y \sqrt{1 - V^2/c^2}}{\left(1 - \frac{V_x . V}{c^2}\right)}$$

$$V_z' = \frac{V_z \sqrt{1 - V^2/c^2}}{\left(1 - \frac{V_x V}{c^2}\right)}$$

mesurement in $S \longrightarrow S'$

Inverse :-

$$V_x = \frac{V_x' + V}{1 + \frac{V_x V}{c^2}}$$

$$V_y = \frac{V_y' \sqrt{1 - V^2/c^2}}{1 + V_x' V/c^2}$$

$$V_z = \frac{V_z' \sqrt{1 - V^2/c^2}}{1 + \frac{V_x' V}{c^2}}$$

mesurement is done in $S' \longrightarrow S$.

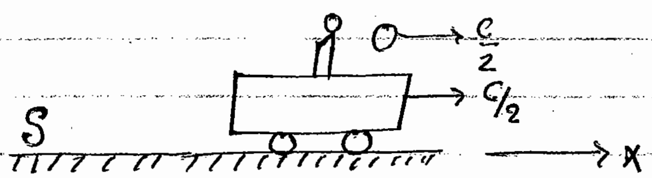
Q. A car is moving horizontally with speed $\frac{c}{2}$ a ball is thrown from the car with speed $c/2$ in the direction of movement of the car. What is velocity of ball as seeing from ground.?

Sol^n Given - $V = c/2$

$V_x' = c/2$

$V_y' = 0$

$V_z' = 0$



So $V_x = \dfrac{V_x' + V}{1 + \dfrac{V_x' V}{c^2}}$

$$= \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{1}{4}} = \frac{4c}{5}$$

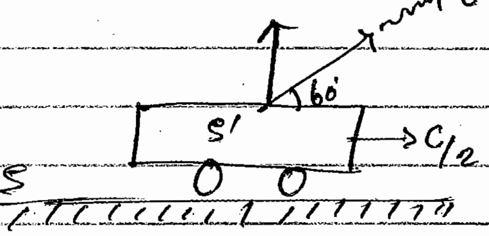
$$V_y = \frac{V_y' \sqrt{1 - V^2/c^2}}{1 + \frac{V_x' V}{c^2}} = 0$$

$$V_z = \frac{V_z' \sqrt{1 - V^2/c^2}}{1 + \frac{V_x' V}{c^2}} = 0$$

Q. A car is moving with speed $c/2$ a light beam is emitted from the car at angle $60°$ with the direction of velocity of car. What is component of velocity of light in the direc^n of velocity of car as seen from ground.

Sol^n

given $V = c/2$



$V_x' = c \cos 60 = \frac{c}{2}$

$V_y' = c \sin 60 = \frac{\sqrt{3} c}{2}$

$$V_x = \frac{V_x' + V}{1 + \frac{V_x' V}{c^2}} = \frac{\frac{c}{2} + \frac{c}{2}}{1 + 1/4}$$

$$= \frac{4c}{5} \quad \underline{\text{Ans}}$$

Extending :-

$$V_y = \frac{V_y' \sqrt{1 - v^2/c^2}}{1 + \frac{V_x' v}{c^2}}$$

$$= \frac{\frac{c\sqrt{3}}{2}\sqrt{1-\frac{1}{4}}}{1+\frac{1}{4}} = \frac{3c}{5}$$

Resultant :- in S :-

$$\sqrt{v_x^2+v_y^2} = \sqrt{\left(\frac{4}{5}c\right)^2+\left(\frac{3c}{5}\right)^2} = c$$ = velocity of light,

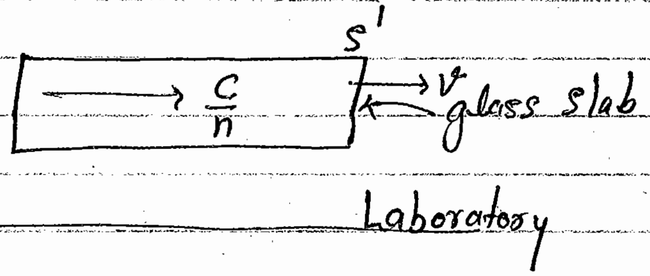
A-8

Q52 A beam of light moves in a slab of glass of refractive index n in the positive x-direction. The slab itself is also moving in the positive dirⁿ with a speed v in laboratory frame. What is the speed of the beam of light as measured in the laboratory frame?

Solⁿ

Velocity of light w.r.to glass

$= \frac{c}{n}$



$$v_x' = \frac{c}{n}$$

$$v_x = ? = \frac{v_x' + v}{1 + \frac{v_x' \cdot v}{c^2}} = \frac{\frac{c}{n} + v}{1 + \frac{\frac{c}{n}\cdot v}{c^2}}$$

=

* If there are two objects then the object w.r.to. which speed is to be calculate is taken as S' frame.

gate-2012

Q. An e^- and photon is moving in same direcn in lab frame. if speed of e^- is $0.85c$ what is speed of e^- w.r.to. photon?

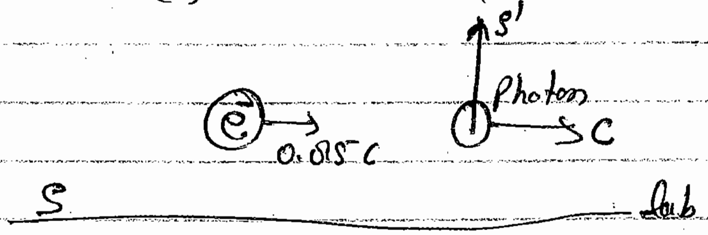
(a) c (b) $-c$ (c) $0.15c$ (d) $-0.15c$

Soln

$V_x = 0.85c$

$V_x' = ?$

$V = c$



$$V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}} = \frac{0.85c - c}{1 - \frac{0.85 \cancel{c} \cdot \cancel{c}}{\cancel{c^2}}}$$

$$= \frac{-0.15c}{0.15} = -c \quad \text{Ans}$$

{ when e^- is goin in any angle then ans. same.

* Velocity of any object w.r.to photon is $\pm c$ (means magnitude is always c.)
If photon is going in positive direction then it is $-c$ and when photon is going in negative direction is then it is $+c$.

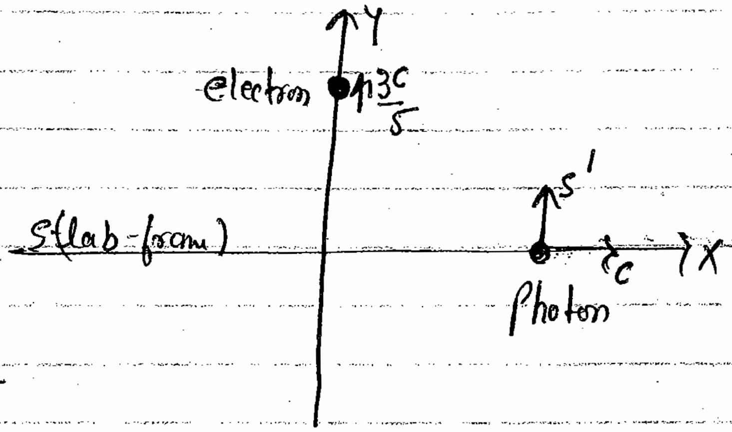
Q.13

Sol^n

$V = c$

$V_x = 0$

$V_y = \frac{3c}{5}$



$$V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$$

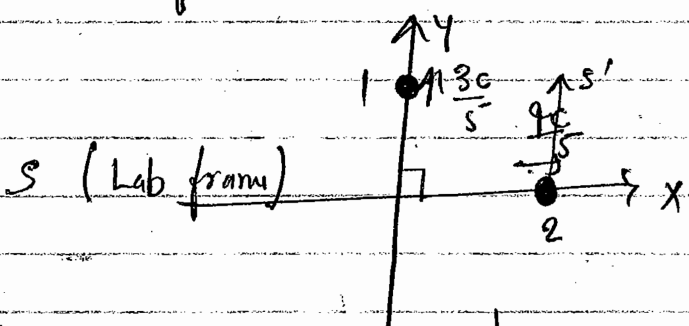
$$= \frac{0 - c}{1 - 0} = -c$$

$$V_y' = \frac{V_y \sqrt{1 - V^2/c^2}}{1 - \frac{V_x \cdot V}{c^2}} = \frac{\frac{3c}{5}\sqrt{1-1}}{1-0} = 0$$

So option (d) is correct.

Q.12 Two electrons are moving as shown in the figure. Velocity of 1 with respect to 2 is?

Sol^n



$V = \frac{4c}{5}$

$V_x = 0$

$V_y = \frac{3c}{5}$

So $V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$

$$\frac{0 - \frac{4c}{5}}{1 - 0} = \frac{-4c}{5}$$

$$V_y' = \frac{V_y \sqrt{1 - V^2/c^2}}{1 - \frac{V_x \cdot V}{c^2}}$$

$$= \frac{\frac{3c}{5}\sqrt{1-\frac{16}{25}}}{1-0} = \frac{9}{25}c$$

Resultant velocity of 1 w.r. to 2:-

$$= \sqrt{v_x'^2 + v_y'^2} = \sqrt{\left(\frac{-4c}{5}\right)^2 + \left(\frac{9c}{25}\right)^2}$$

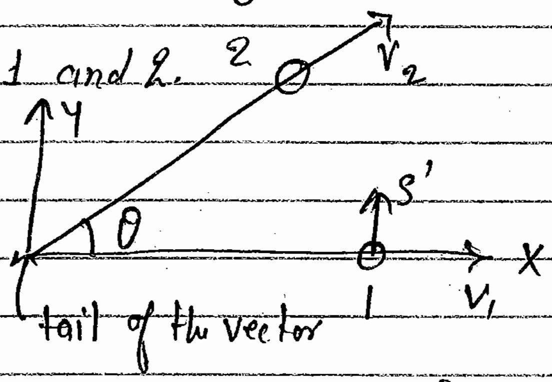
$$= \sqrt{\frac{16c^2}{25} + \frac{81}{625}c^2}$$

$$= \frac{\sqrt{481}}{25}c$$

Ans

* General Formula for relative velocity of two object moving at same angle:-

Relative velocity b/w 1 and 2.



$$|V_{12}| = |V_{21}|$$

$$\vec{V_{12}} = -\vec{V_{21}}$$

* Non-Relativistic Case :- $\{V_1, V_2 << c\}$

$$\boxed{|V_{12}| = |\vec{V_{21}}| = \sqrt{V_1^2 + V_2^2 - 2V_1V_2\cos\theta}}$$

* Relativistic Case :-

Let us calculate -

v_{21} (velocity of 2 w.r.t. 1).

So $\quad V = V_1$

$v_x = V_2 \cos\theta$

$v_y = V_2 \sin\theta$

So $v_x' = \dfrac{v_x - V}{1 - \dfrac{v_x . V}{c^2}} = \dfrac{v_2 \cos\theta - V_1}{1 - \dfrac{V_2 \cos\theta . V_1}{c^2}}$

$v_y' = \dfrac{v_y \sqrt{1 - V^2/c^2}}{1 - \dfrac{v_x . V}{c^2}} = \dfrac{V_2 \sin\theta \sqrt{1 - \dfrac{V_1^2}{c^2}}}{1 - \dfrac{V_1 V_2 \cos\theta}{c^2}}$

Resultant velocity (in s' frame):-

i.e. Resultant velocity of 2 w.r. to 1.

$|V_{21}| = \sqrt{V_x'^2 + V_y'^2}$

$= \dfrac{\sqrt{(V_2 \cos\theta - V_1)^2 + V_2^2 \sin^2\theta \left(1 - \dfrac{V_1^2}{c^2}\right)}}{1 - V_1 V_2 \cos\theta / c^2}$

$$\boxed{|V_{21}| = \dfrac{\sqrt{V_1^2 + V_2^2 + 2 V_1 V_2 \cos\theta - \dfrac{V_1^2 V_2^2 \sin^2\theta}{c^2}}}{1 - \dfrac{V_1 . V_2}{c^2} \cos\theta}}$$

If $V_1, V_2 << c$

$V_1^2 . V_2^2 \sim 0$

$\dfrac{V_1 V_2}{c^2} \sim 0$

Q.12

Sol^n $V_1 = \frac{3c}{5}$, $V_2 = \frac{4c}{5}$, $\theta = 90°$

$$|V_{21}| = |V_{12}| = \sqrt{\left(\frac{3c}{5}\right)^2 + \left(\frac{4c}{5}\right)^2 - \frac{\left(\frac{3c}{5}\right)^2\left(\frac{4c}{5}\right)^2}{c^2}}$$

$$= \sqrt{\frac{9c^2}{25} + \frac{16c^2}{25} - \frac{\frac{9c^2}{25} \times \frac{16c^2}{25}}{c^2}}$$

$$= \sqrt{\left(\frac{25}{25}\right)c^2 - \frac{144c^4}{625c^2}}$$

$$= \sqrt{c^2 - \frac{144c^2}{625}} = \sqrt{\left(1 - \frac{144}{625}\right)c^2}$$

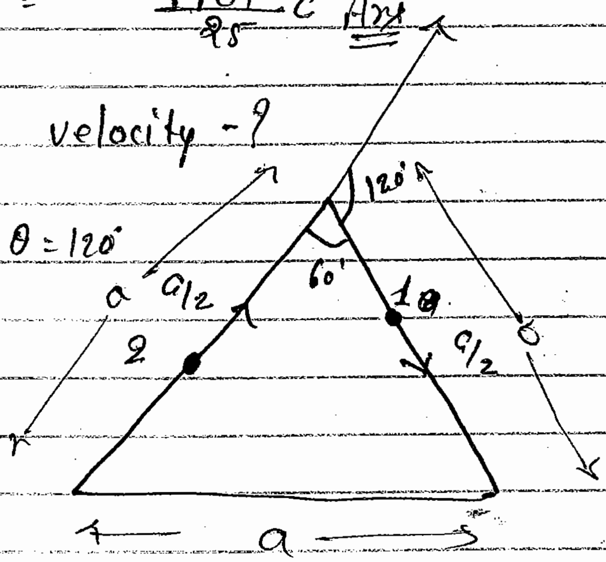
$$= c\,\cancel{\sqrt{\frac{89}{125}}} = c\sqrt{\frac{625-144}{625}}$$

$$= c\sqrt{481/625} = \frac{\sqrt{481}}{25}c \text{ Ans}$$

2012 TIFR

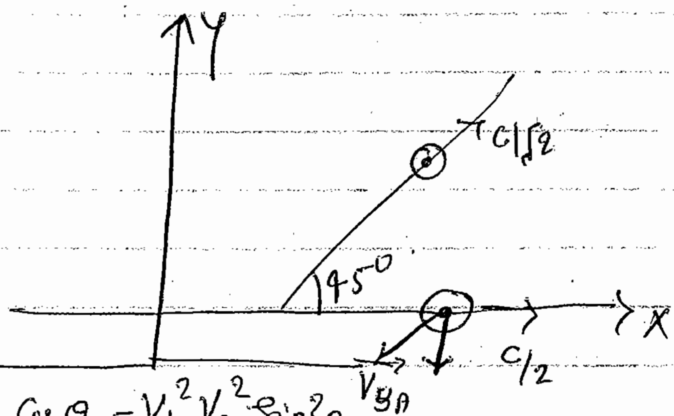
Q. What is their relative velocity -?

Sol^n $V_1 = c/2$, $V_2 = c/2$, $\theta = 120°$



Q.50

$V_{AB} = ?$
$\vec{V}_{BA} = -\vec{V}_{AB}$



$$|V_{12}| = \frac{\sqrt{v_1^2 + v_2^2 - 2V_1V_2\cos\theta - \frac{V_1^2V_2^2\sin^2\theta}{c^2}}}{1 - \frac{V_1V_2\cos\theta}{c^2}}$$

$$= \frac{\sqrt{\frac{c^2}{2} + \frac{c^2}{4} - 2\frac{c}{\sqrt{2}}\frac{c}{2}\frac{1}{\sqrt{2}} - \frac{\frac{c^2}{2}\frac{c^2}{4}\cdot\frac{1}{2}}{c^2}}}{1 - \frac{2\frac{c}{\sqrt{2}}\frac{c}{2}\cdot\frac{1}{\sqrt{2}}}{c^2}}$$

$$= \frac{\sqrt{\frac{c^2}{4} - \frac{c^2}{16}}}{1 - \frac{1}{4}} = \frac{c\sqrt{\frac{1}{4} - \frac{1}{16}}}{(1 - 1/4)}$$

$$= \frac{\frac{c}{2}\sqrt{1 - \frac{1}{4}}}{3/4} = \frac{c\sqrt{1 - \frac{1}{4}}}{3/2} \quad \frac{\sqrt{3}c}{2} \times \frac{2}{3}$$

$$\boxed{|V_{12}| = \frac{c}{\sqrt{3}}}$$ Ans

Second Method :-

Let us calculate -

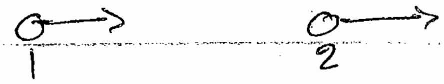
$$V_{u}' = \frac{V_u - V}{1 - \frac{V_u V}{c^2}} = \frac{\frac{c}{2} - \frac{c}{2}}{1 - \frac{c}{4}} = 0$$

$$V_y' = \frac{V_y \sqrt{1 - V^2/c^2}}{1 - \frac{V_x V}{c^2}} = \frac{\frac{c}{2}\sqrt{1-\frac{1}{4}}}{1-\frac{1}{4}} = \frac{c}{\sqrt{3}}$$

$$\boxed{V_y' = \frac{c}{\sqrt{3}}}$$

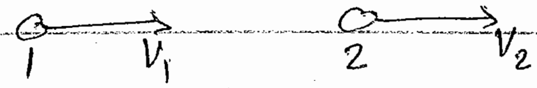
Non Relativistic Case :-

$$|V_{12}| = \sqrt{V_1^2 + V_2^2 - 2V_1V_2\cos\theta}$$



$$\boxed{V_{12} = V_1 - V_2}$$

Relativistic Case :-



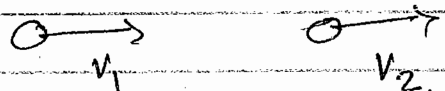
$$V_{12} = V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$$

$$\boxed{V_{12} = \frac{V_1 - V_2}{1 - \frac{V_1 V_2}{c^2}}}$$

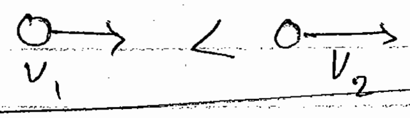
* Velocity of approach / Separation :-

$$= |\vec{V_1} - \vec{V_2}|$$

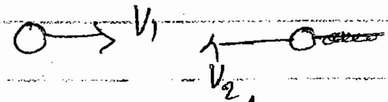
In both relativistic and non relativistic case.



Velocity of approach $= V_1 - V_2$



$$\boxed{\text{Velocity of separation} = v_2 - v_1}$$



$$\text{velocity of approach} = v_1 - (-v_2)$$
$$= v_1 + v_2$$

* When two moving objects are observed by a stationary observer then velocity of approach is used for time calculation.

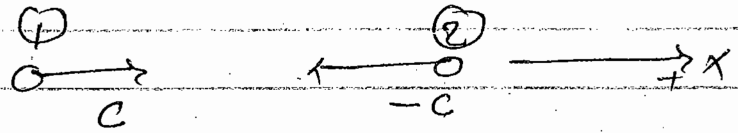
* If one moving object is observed by another moving object then relative velocity is used for time calculation.

* Velocity of approach may be less equal or greater than velocity of light.

* Relative velocity is never be greater than velocity of light.

A-8
Qu#7

Q.8 In the previous question velocity of approach of the two photons is -



Velocity of approach $= |V_1 - V_2|$

$= |V_1 - (-V_2)|$

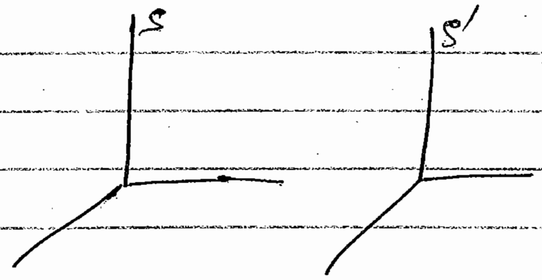
$= |c - (-c)|$

$= 2c$ Ans

Q.62 Velocity of a particle is $\vec{V} = \frac{c}{\sqrt{2}}(\hat{i}+\hat{j})$ in s frame and $\vec{V'} = \frac{c}{\sqrt{2}}(-\hat{i}+\hat{j})$ in s' frame which is moving along x direction w.r.to s velocity of s' w.r.to s frame is ?

Sol^n

$\vec{V} = \frac{c}{\sqrt{2}}(\hat{i}+\hat{j})$ in s



$\vec{V}' = \frac{c}{\sqrt{2}}(-\hat{i}+\hat{j})$ in s'

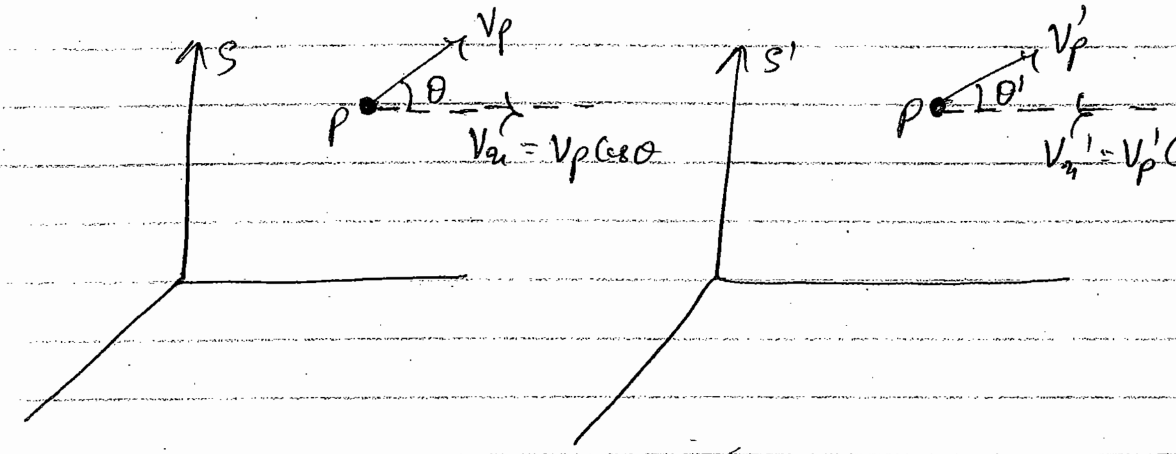
$$V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$$

$$\frac{-c}{\sqrt{2}} = \frac{\frac{c}{\sqrt{2}} - V}{1 - \frac{\frac{c}{\sqrt{2}} \cdot V}{c^2}}$$

$$-\frac{c}{\sqrt{2}} + \frac{V}{2} = \frac{c}{\sqrt{2}} - V$$

$$\frac{3V}{2} = \frac{\sqrt{2}\,2c}{\sqrt{2}} \Rightarrow \boxed{V = \frac{2\sqrt{2}c}{3}}$$

* Direction velocity vector in two frames :-



Let V_p and V_p' be speed of a particle as as seen from S and S' frames.
Let θ and θ' be its angle with x-dire^n

$$\cos\theta = \frac{V_x}{V_p} \qquad \cos\theta' = \frac{V_x'}{V_p'}$$

$$V_p = \sqrt{V_x^2+V_y^2+V_z^2}\ , \quad V_p' = \sqrt{V_x'^2+V_y'^2+V_z'^2}$$

{Calculate component of velocity by lorentz transformation then we get angle.}

or $\boxed{\tan\theta = \frac{V_y}{V_x}}$

$\boxed{\tan\theta' = \frac{V_y'}{V_x'}}$

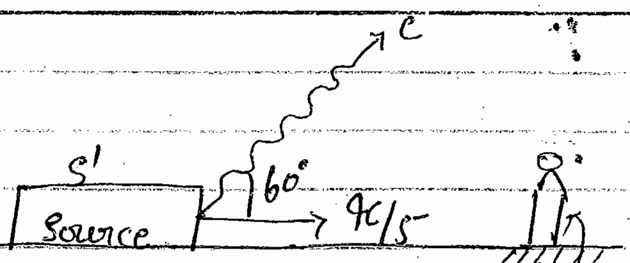
JEST 2013

Q. A monocromatic wave propagate in the direc^n making an angle 60° with x-axis in refrence frame of the source. The source moves ~~towards~~ at a speed $\frac{4c}{5}$ towards the observer the dire^n of wave as seen by observer is

(a) $\cos^{-1}\frac{13}{14}$ (b) $\cos^{-1}\frac{3}{14}$ (c) $\cos^{-1}\frac{\sqrt{3}}{6}$ (d) $\cos^{-1}\left(\frac{1}{2}\right)$.

Monocromatic → Electromagnetic wave.

Sol^n.



$V_x = V\cos 60 = \frac{c}{2}$

Let θ is the angle as seen by ~~obser ver~~ observer.

$$\cos\theta = \frac{V_x}{V_p} = \frac{V_x}{c}$$

$$V_x = \frac{V_x' + V}{1 + \frac{V_x \cdot V}{c^2}} = \frac{\frac{c}{2} + \frac{9c}{5}}{1 + \frac{\frac{c}{2}\cdot\frac{9c}{5}}{c^2}}$$

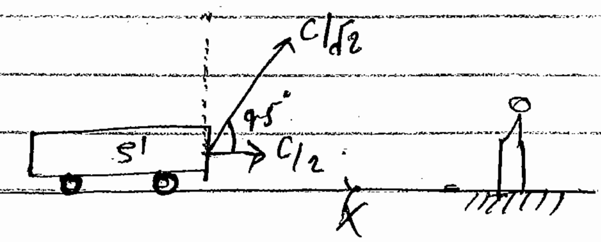
$$\boxed{V_x = \frac{13c}{14}}$$

$$\cos\theta = \frac{V_x}{c} = \frac{13\cancel{c}}{14\cancel{c}} = \frac{13}{14}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{13}{14}\right)}$$ Ans

2. A car is moving with speed $c/2$ a stone is ~~throug~~ thrown from the car with speed $c/\sqrt{2}$ at $45°$ with the direc^n of velocity of the car. what is the angle as seen from the ground.

Sol^n



$V_x' = \frac{c}{\sqrt{2}}\cos 45 = \frac{c}{2}$

$V_y' = \frac{c}{\sqrt{2}}\cos 45 = \frac{c}{2}$

$$V_x = \frac{V_x' + V}{1 + \frac{V_x' \cdot V}{c^2}} = \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{1}{4}} = \frac{4c}{5}$$

$$v_y' = \frac{v_y'\sqrt{1-v^2/c^2}}{1+\frac{v_x v}{c^2}} = \frac{\frac{c}{2}\sqrt{1-\frac{1}{4}}}{1+\frac{1}{4}} = \frac{\sqrt{3}c}{5}$$

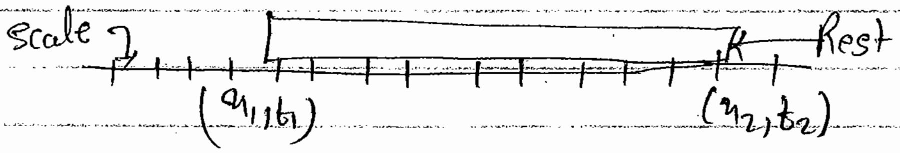
$$v_p = \sqrt{v_x^2+v_y^2} = \frac{\sqrt{19}\,c}{5}$$

$$\boxed{\cos\theta = \frac{v_x}{v_p} = \frac{4}{\sqrt{19}}}$$

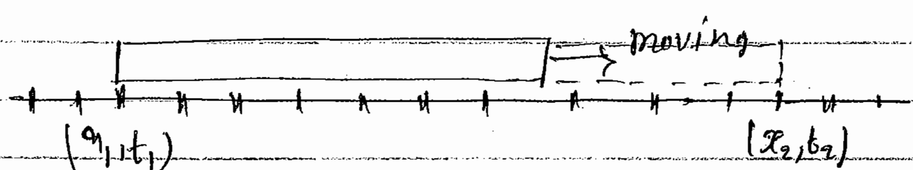
* Relativistic of Geometry :-

* Length Contraction :-

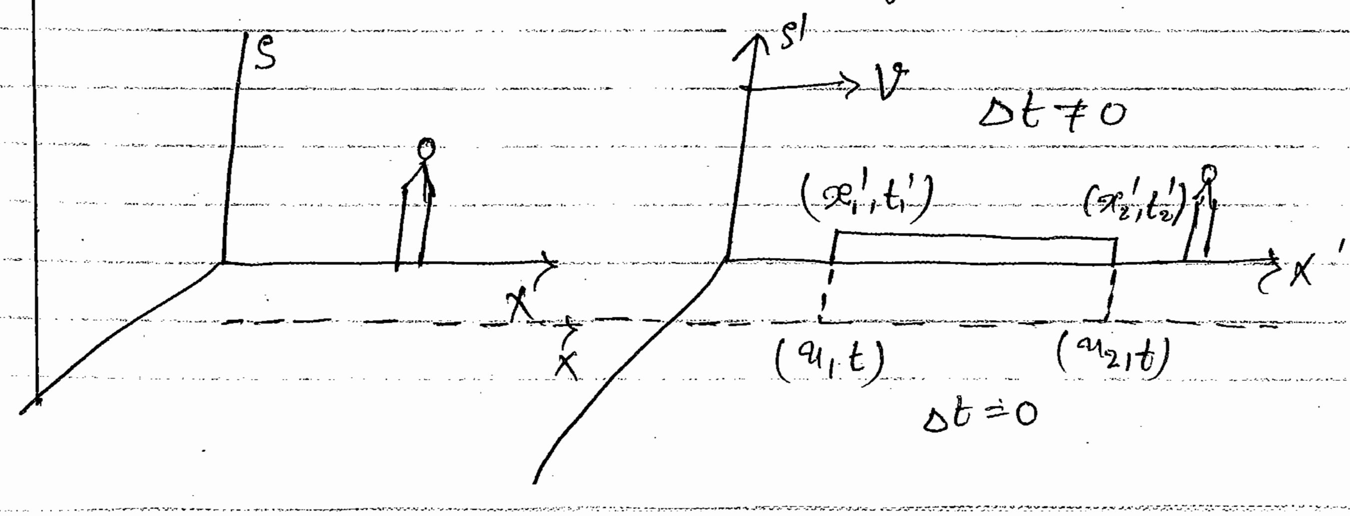
If length of a moving object is to be measured its end Cordinates must be noted simultaneously. Then cordinate difference will be equal to length.



$x_2 - x_1 = \text{length}$



$x_2 - x_1 \neq \text{length}.$



$\Delta t \neq 0$

$\Delta t = 0$

V is the velocity parallel to the rod.

Rod is lying in frame S'.

$\Delta x'$ = rest length $= L_0$ = Proper length
$\Delta x'$ = length in motion or apparent length $= L$

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\Delta x = \frac{\Delta x' + V\Delta t}{\sqrt{1-V^2/c^2}}$$

$$L_0 = \frac{L - 0}{\sqrt{1-V^2/c^2}}$$

$$\boxed{L = L_0\sqrt{1-\frac{V^2}{c^2}}}$$

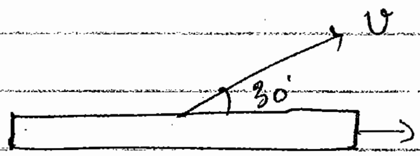
$$\boxed{L = L_0\sqrt{1-\frac{V_{\parallel}^2}{c^2}}}$$

$V_{\parallel}$ = Relative velocity $\parallel^{lel}$ to the length of the rod.

A-8
Q.1

Sol^n

$$L = \frac{L_0}{2}$$



$$L = L_0\sqrt{1 - V_{\parallel}^2/c^2}$$

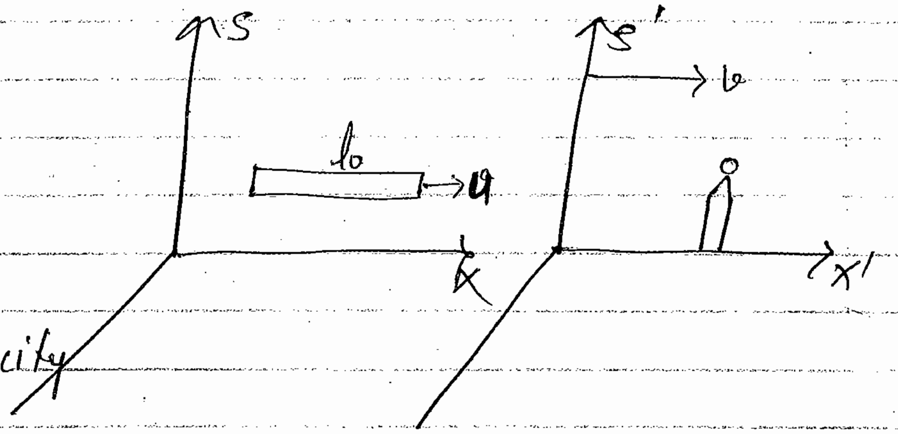
$$\frac{\cancel{L_0}}{2} = \cancel{L_0}\sqrt{1 - V^2\frac{3}{4c^2}}$$

$$\frac{1}{4} = 1 - \frac{3}{4}\frac{V^2}{c^2}$$

$$\boxed{V = c}$$

Q.19 A rod of proper length L_0 oriented parallel to the x-axis

Soln



$$L = L_0\sqrt{1 - \frac{V_{11}^2}{c^2}}$$

V_{11} = relative velocity

Relative Velocity $V_u' = \dfrac{V_u - V}{1 - \dfrac{V_u V}{c^2}} = \dfrac{u - v}{1 - \dfrac{uv}{c^2}} = V_{11}$

$$L = L_0\sqrt{1 - \frac{1}{c^2}\left(\frac{(u-v)^2}{\left(1 - \frac{uv}{c^2}\right)^2}\right)}$$

$$= \frac{l_0}{c\left(1 - \frac{uv}{c^2}\right)}\sqrt{c^2\left(1 - \frac{uv}{c^2}\right)^2 - (u-v)^2}$$

$$= \frac{l_0}{c\left(1 - \frac{uv}{c^2}\right)}\sqrt{c^2\left(1 + \frac{u^2v^2}{c^4} - \cancel{\frac{2uv}{c^2}}\right) - \left(u^2 + v^2 - \cancel{2uv}\right)}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{c^2 + \frac{u^2v^2}{c^2} - u^2 - v^2}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{(c^2-v^2) + u^2\left(\frac{v^2}{c^2}-1\right)}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{c^2\left(1-\frac{v^2}{c^2}\right) - u^2\left(1-\frac{v^2}{c^2}\right)}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{\left(1-\frac{v^2}{c^2}\right)(c^2-u^2)}$$

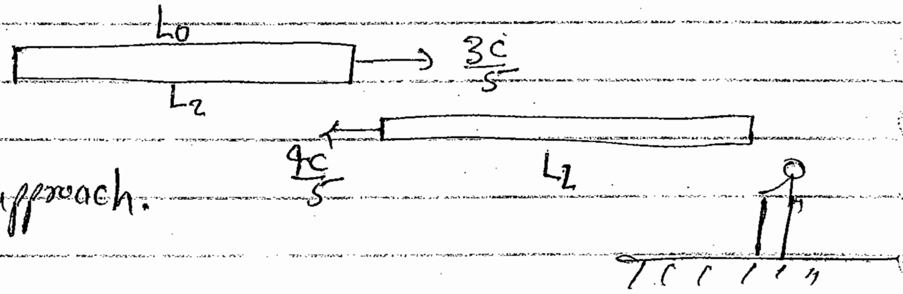
$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{\left(1-\frac{v^2}{c^2}\right) c^2\left(1-\frac{u^2}{c^2}\right)}$$

$$\boxed{l = \frac{l_0}{\left(1-\frac{uv}{c^2}\right)} \sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}}$$

Ans

Q.14 Two rods of rest length L_0 are moving towards each other with speeds $\frac{3c}{5}$ and $\frac{4c}{5}$. What is time taken by the two rods to cross each other as seen by a person on the ground.

Soln



Time = $\frac{\text{distance}}{\text{Velocity of approach.}}$

$$L_1 = L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-\frac{\left(\frac{3c}{5}\right)^2}{c^2}}$$

$$= L_0\sqrt{1-\frac{9\cancel{c^2}}{25\cancel{c^2}}} = L_0\sqrt{\frac{25-9}{25}}$$

$$\boxed{L_1 = \frac{4L_0}{5}}$$

$$L_2 = L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-\frac{16\cancel{c^2}}{25\cancel{c^2}}}$$

$$\boxed{L_2 = \frac{3L_0}{5}}$$

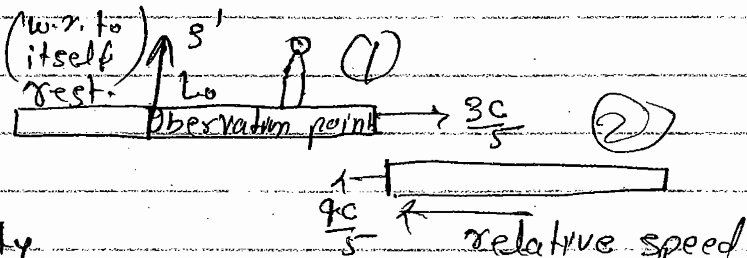
$$t = \frac{\text{distance}}{\text{Velocity of approach}} = \frac{L_1+L_2}{(V_1-[-V_2])}$$

$$= \frac{\frac{4L_0}{5}+\frac{3L_0}{5}}{\frac{3c}{5}+\frac{4c}{5}} = \frac{7L_0/5}{7c/5} = \frac{L_0}{c}$$

$$\boxed{t = \frac{L_0}{c}}$$

Q.15 In previous ques what is time taken by the two rods to cross each other as seen by the rod of speed $\frac{3c}{5}$.

Sol^n



$$t = \frac{\text{distance}}{\text{relative Velocity}}$$

Relative speed $\quad V_x' = \frac{V_x - V}{1-\frac{V_x \cdot V}{c^2}} = \frac{-\frac{4c}{5}-\frac{3c}{5}}{1+\frac{\frac{4\cancel{c}}{5}\cdot\frac{3\cancel{c}}{5}}{\cancel{c^2}}}$$

$$\boxed{V_{41} = -\frac{35}{37}c}$$

Apparant length of rod 2 :-

$$L_2 = L_0\sqrt{1 - \frac{V_{11}^2}{c^2}} = L_0\sqrt{1 - \frac{\left(\frac{-35}{37}\right)^2 \cancel{c^2}}{\cancel{c^2}}}$$

$$= L_0 \frac{\sqrt{(37)^2 - (35)^2}}{37}$$

$$= L_0 \frac{\sqrt{(37+35)(37-35)}}{37}$$

$$= L_0 \frac{\sqrt{72 \times 2}}{37} = \frac{L_0\sqrt{144}}{37}$$

$$\boxed{L_2 = \frac{12L_0}{37}}$$

So $\quad t = \frac{L_1 + L_2}{\frac{35}{37}c} = \frac{L_0 + \frac{12L_0}{37}}{\frac{35}{37}c}$

$$= \frac{49L_0/\cancel{37}}{35c/\cancel{37}} = \frac{7L_0}{5c}$$

$$\boxed{t = \frac{7L_0}{5c}}$$ Ans

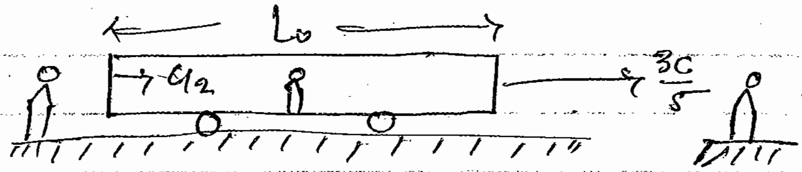
rear end = back end.

Q. A car of rest length L_0 is moving with speed $V = \frac{3c}{5}$. A stone is thrown from rear end of the car with speed $c/2$ relative to the car. What is the time after which the stone will hit front end of the car as measured in car frame and also ~~at~~ as measured in ground frame.

Sol^n

In Car frame :-

time = $\frac{\text{dist.}}{\text{relative velo. (here speed of stone)}}$



$= \frac{L_0}{c/2}$

$$\boxed{t' = \frac{2L_0}{c}}$$

In ground frame :- $t = \frac{\text{dis.}}{\text{Velocity of approach}}$

$L = L_0\sqrt{1 - V_{11}^2/c^2}$ $= L_0\sqrt{1 - \frac{9c^2}{25c^2}}$

$= L_0 \frac{\sqrt{25-9}}{5}$ $= L_0 \frac{\sqrt{16}}{5}$

$L = \frac{L_0 4}{5} = \frac{4L_0}{5}$

$\Rightarrow \boxed{L = \frac{4L_0}{5}}$

Velocity of stone w.r. to ground :-

$= |\vec{V_1} - \vec{V_2}|$

$V_a = \frac{V_a' + V}{1 + \frac{V_a V}{c^2}} = \frac{\frac{c}{2} + \frac{3c}{5}}{1 + \frac{3c^2}{10c^2}}$

$$\boxed{V_x = \frac{11c}{13}}$$

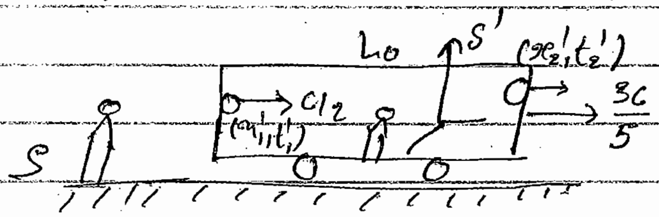
$$time = \frac{L_0\,4/5}{\left|\frac{3c}{5} - \frac{11c}{13}\right|} = \frac{4L_0/5}{(39-55)\frac{c}{5\times13}}$$

$$= \frac{4L_0 \times 13}{16_4 c} = \frac{13L_0}{4c}$$

$$\boxed{t = \frac{13L_0}{4c}}$$ Ans

Second Method :- By Lorentz's Transformation

$\Delta x' = x_2' - x_1' = L_0$



$\Delta t' = t_2' - t_1' = \frac{2L_0}{c}$

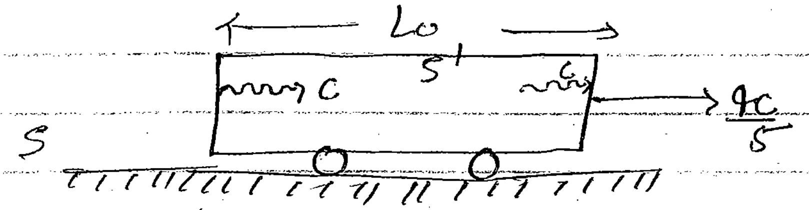
$\Delta t = ?$

$$\Delta t = \frac{\Delta t' + \Delta x' \frac{v}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{\frac{2L_0}{c} + L_0 \frac{3c}{5}/c^2}{\sqrt{1 - \frac{9}{25}}}$$

$$= \frac{\frac{2L_0}{c} + \frac{3L_0}{5c}}{4/5} = \frac{13L_0}{4c}$$ Ans

Q. A car of rest length L_o is moving with speed ~~is~~ $\frac{4c}{5}$ a light beam is emited from rear end of the car and it is absorbed at the front end. What is distance traveled by the car during this process.

Sol^n



In S' frame :-

$$\Delta t' = \frac{L_o}{c}$$

$$\Delta x' = L_o$$

Time duration of process in ground frame :-

$$\Delta t = \frac{\Delta t' + \frac{\Delta x' V}{c^2}}{\sqrt{1 - V^2/c^2}} = \frac{\frac{L_o}{c} + \frac{L_o \frac{4}{5} c}{c^2}}{3/5}$$

$$\boxed{\Delta t = \frac{3L_o}{c}}$$

Distance travelled by car = Speed × time in ground frame.

$$= \frac{4\cancel{c}}{5} \times \frac{3L_o}{\cancel{c}}$$

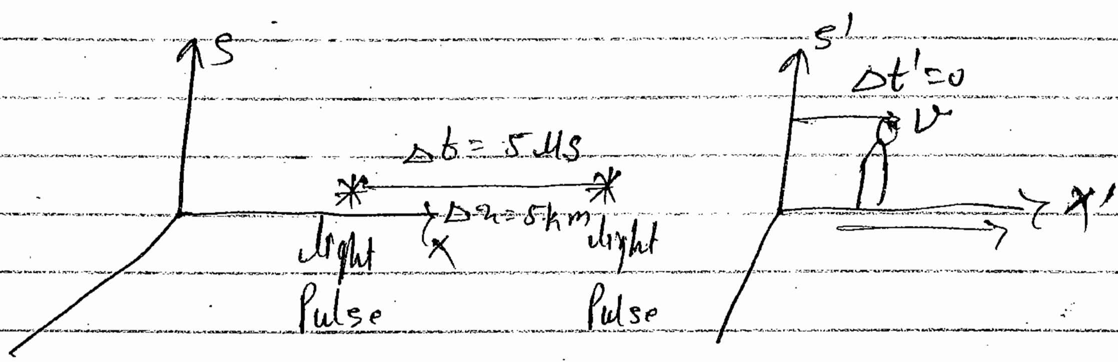
$$= \frac{12 L_o}{5}$$

Ans

JEST 2014

2. In a certain inertial frame two light pulses are emited, a distance ~~distance~~ 5 km apart and separated by 5 μs. An observer who is travelling parallel to the line ~~going~~ joning the two points where pulses are emited at a velocity v w.r.t. this frame note that the pulses are simultaneous. There fore v is?

Sol^n



$$\Delta t' = \frac{\Delta t - \frac{\Delta x v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$0 = 5 \times 10^{-6} - \frac{5 \times 10^{3} v}{c^2}$$

$$\cancel{5} \times 10^{-6} = \frac{\cancel{5} \times 10^{3} v}{c^2}$$

$$v = c^2 \times 10^{-9}$$

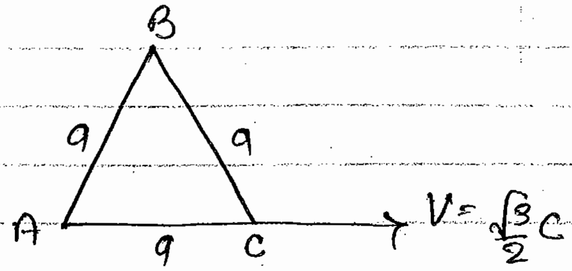
$$= c \times 3 \times 10^{8} \times 10^{-9}$$

$$\boxed{v = 0.3c}$$ Ans

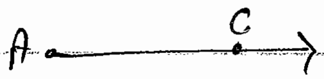
A-9

Q.2

Proper ~~length~~ / Rest perimeter $= 3a$.



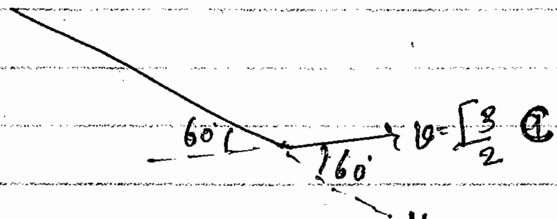
AC :-



$V = \frac{\sqrt{3}}{2}\, c$

$$AC = a\sqrt{1-\frac{3}{4}} = \frac{a}{2}$$

BC :-

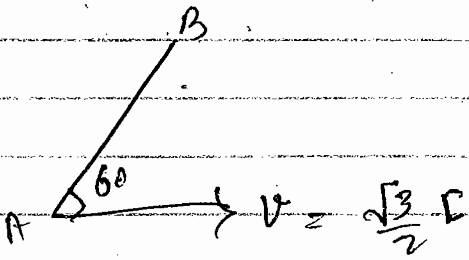


$$BC = a\sqrt{1-\frac{V_{11}^2}{c^2}}$$

$V_{11} = V\cos 60 = \frac{\sqrt{3}}{4}c$

$$= a\sqrt{1-\frac{3}{16}} = \frac{a\sqrt{13}}{4}$$

AB :-



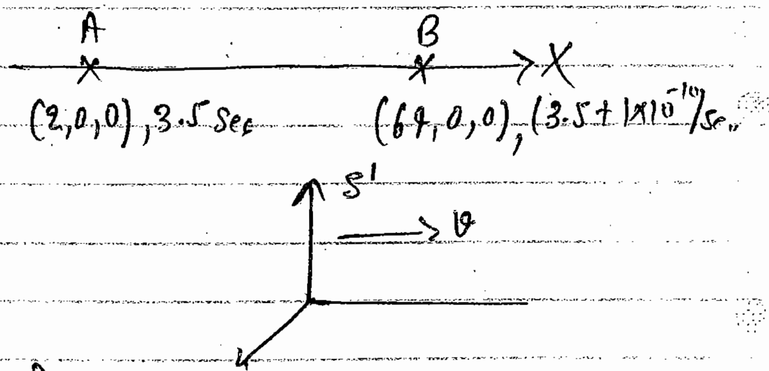
$$V_{11} = V\cos 60^\circ = \frac{\sqrt{3}}{4}\, c$$

$$AB = \frac{a\sqrt{13}}{4}$$

So Perimeter $= AB + BC + CA$

$$= \frac{a\sqrt{13}}{4} + \frac{a\sqrt{13}}{4} + \frac{a}{2} = \frac{2a\sqrt{13}}{4} + \frac{a}{2}$$

Q.16



$$\Delta t' = \frac{\Delta t - \Delta x \cdot v/c^2}{\sqrt{1 - v^2/c^2}}$$

$$0 = \frac{\left(1\times10^{-10} - \frac{62\times v}{c^2}\right)}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \frac{9\times10^{16}\times10^{-10}}{62} = v$$

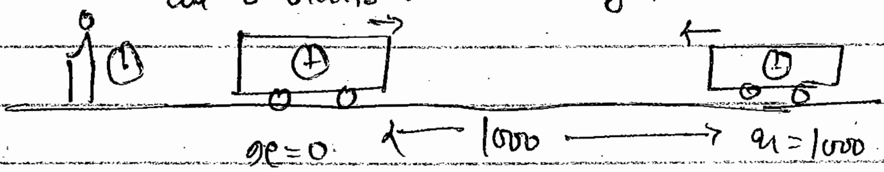
$$\Rightarrow v = \frac{9\times10^{6}}{62}$$

$$\Rightarrow \boxed{v = 1.5\times10^{5}\ \text{m/sec (A to B)}}$$

Q.17

all 3-clocks are schcronized.

$$t' = \frac{t - x v/c^2}{\sqrt{1 - v^2/c^2}}$$



$$t_1' = \frac{0 - 0\times v/c^2}{\sqrt{1 - v^2/c^2}} = 0$$

$$t_2' = \frac{0 - 1000 \times \frac{(-\frac{c}{2})}{c^2}}{\sqrt{1 - \frac{1}{4}}}$$

$$t_2' = \frac{500/c}{\sqrt{3}/2} = \frac{1000}{\sqrt{3}\,c}$$

Time difference b/w two clocks is $t_2' - t_1'$

$S' \longrightarrow S$ $\quad t = \dfrac{t' + x'v/c^2}{\sqrt{1 - v^2/c^2}}$ Observer is in S' (first measurement in S')

$S \longrightarrow S'$ $\quad t' = \dfrac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$ Observer is in S first measured in S

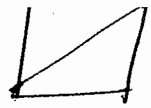
* Relativity of Geometrical Structure :-

(i) Length Contraction $\Rightarrow$ $L = L_0\sqrt{1 - v_{\parallel}^2/c^2}$ | Result of measurement

(ii) Area Contraction :-

$$A = A_0\sqrt{1 - v_{\parallel}^2/c^2}$$

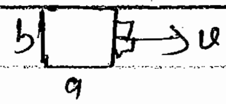
$v_{\parallel}$: Velocity $\parallel^{lel}$ to surface.



some formula used.

(iii) Volume Contraction :-

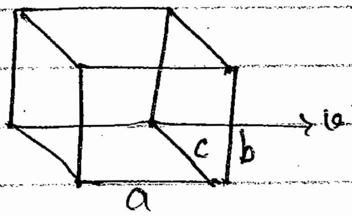
$$\boxed{V = V_0\sqrt{1 - v^2/c^2}}$$

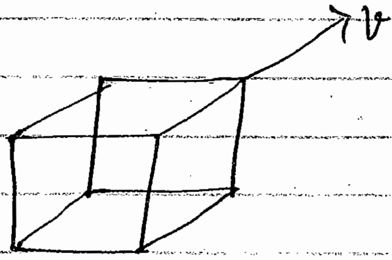


$A_0 = ab$

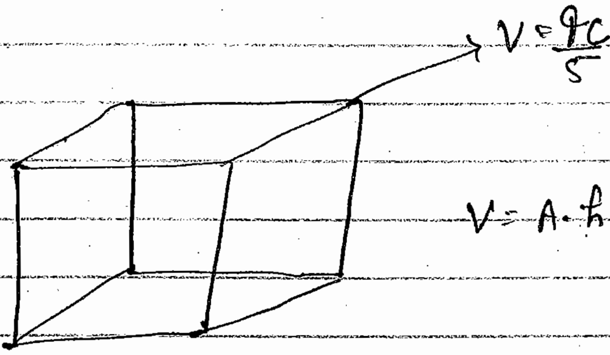
$A = (a\sqrt{1-v^2/c^2})\,b$

$A = A_0\sqrt{1-v^2/c^2}$





Q.5



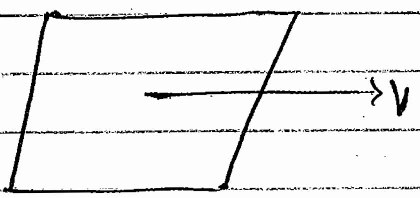
$V = V_0\sqrt{1-v^2/c^2}$

$= a^3\sqrt{1-\frac{16}{25}}$

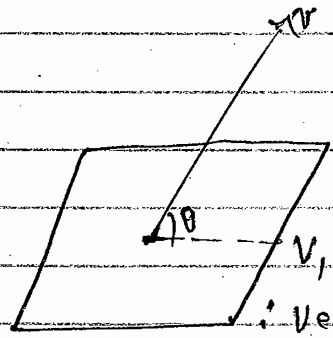
$$\boxed{V = \frac{3a^3}{5}}$$

$V = A \cdot h$

→



$v_{\parallel} = v$



$v_{\parallel} = v_{\parallel}\cos\theta$

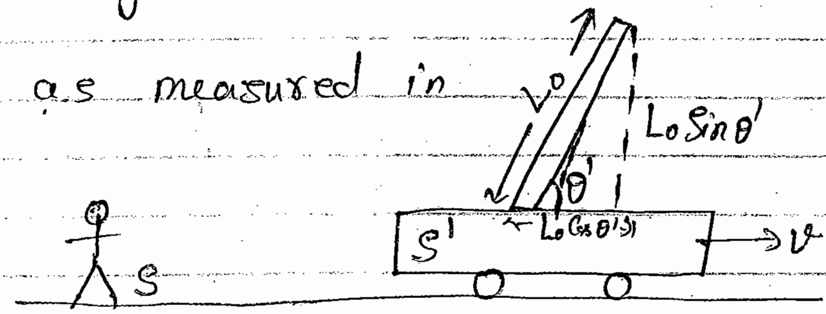
: velocity ∥ to surface.

→ But in case of volume velocity is directly put whatever direcⁿ it will move.

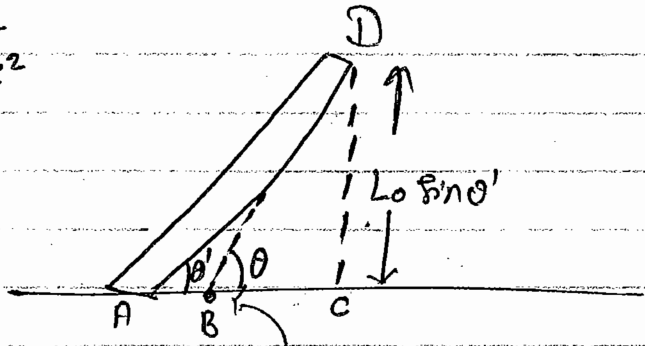
* Change of Geometrical Angle of an object:-

Let θ' = angle as measured in S'

θ = Angle as measured in S



$$\tan\theta = \frac{\cancel{L_0}\sin\theta'}{\cancel{L_0}\cos\theta'\sqrt{1-v^2/c^2}}$$



Angle measured from ground

$$\boxed{\tan\theta = \frac{\tan\theta'}{\sqrt{1-v^2/c^2}}}$$

$\theta > \theta'$ $\qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

as $\gamma < 1$

- In relativity their is no concept of rigidity.

Here θ is the angle has been measured with direction of velocity.

$$\boxed{\theta = \tan^{-1}\gamma\tan\theta'}$$

Gate - 2011

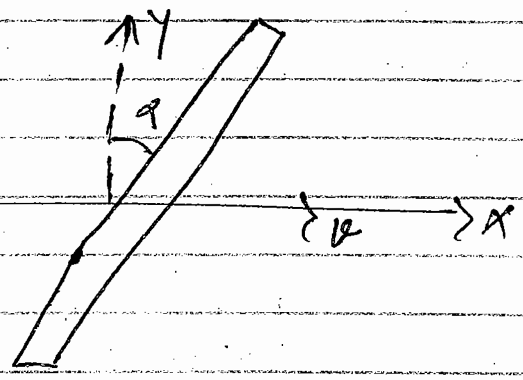
Q. A rod is moving along x-direction with speed V in the rest frame of the rod angle measured with y-axis is α. what is angle measured with y-axis in ground frame.

(a) $\tan^{-1}\gamma\tan\alpha$ ✓(b) $\tan^{-1}\dfrac{\tan\alpha}{\gamma}$

(c) $\tan^{-1}\gamma\cot\alpha$ (d) $\tan^{-1}\dfrac{1}{\gamma}\cot\alpha$.

<u>Soln</u>

Let β be the angle with y axis as seen from ground.



$$\theta' = 90 - \alpha$$

$$\theta = 90 - \beta$$

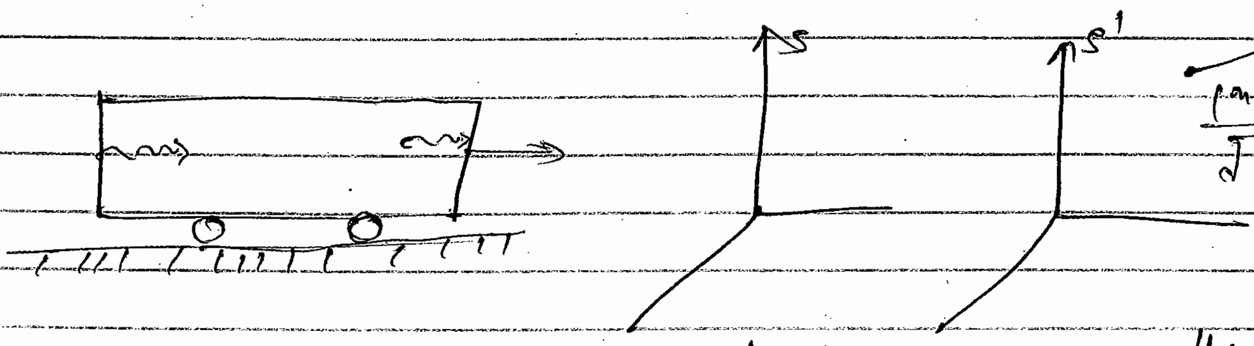
$$\therefore \boxed{\tan\theta = \gamma\tan\theta'}$$

So $\cot\beta = \gamma\cot\alpha$

$$\tan\beta = \frac{1}{\gamma}\tan\alpha$$

$$\boxed{\beta = \tan^{-1}\frac{1}{\gamma}\tan\alpha}$$

Note:-



$$\frac{(ct + xv/c^2)}{\sqrt{1 - v^2/c^2}}$$

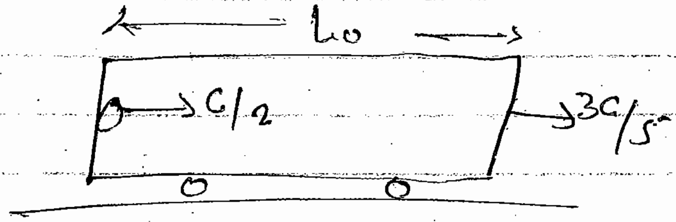
So we can not find length travelled by sun.

this is the co-ordinate of object not fram.

Q. {Related to previous Question.} Stone is ~~throwing~~ thrown from rear end of car to front end of car. find the distance travelled by stone.

Sol^n

Distance travelled by stone



$= \Delta x$

or $=$ Velocity of stone w.r. to ground $\times$ time measured in ground.

Relativistic Dynamics

* Newtons Second Law:-

$$F = \frac{dp}{dt} \quad \text{applicable when } m \text{ is constant}$$

$$= \frac{d}{dt}(mv)$$

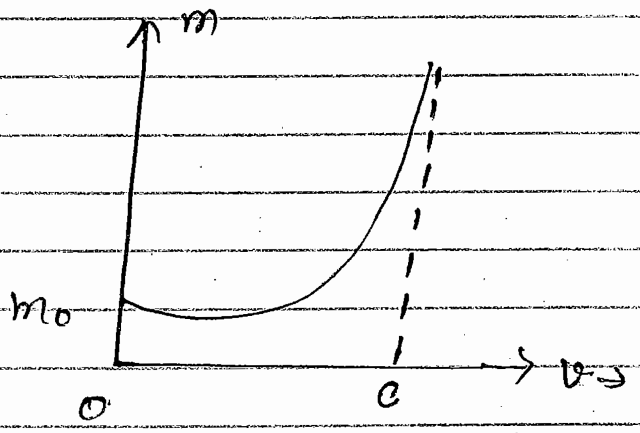
$\times$ $f = ma$ Not applicable when mass is variable.

* Variation of mass with velocity:-

Rest mass.

$$\boxed{m = \frac{m_0}{\sqrt{1 - v^2/c^2}}} \quad , m_0 \neq 0$$

Dynamic mass.



$$m\sqrt{1 - v^2/c^2} = m_0$$

If dynamic mass is not zero but rest mass is 0 then

$$m\sqrt{1 - v^2/c^2} = 0$$

$$1 - v^2/c^2 = 0$$

$$\boxed{v = c}$$

* If rest mass of a particle is zero then

it must move with speed of light.
e.g. Newtrino.

$$\nu \longrightarrow V = c$$

$$m_0 = 0$$

* Momentum :-

$$p = mv$$

$$\boxed{p = \frac{m_0 V}{\sqrt{1 - V^2/c^2}}} \qquad m_0 \neq 0$$

* Kinetic Energy :-

$$K = \text{Total Energy} - \text{Rest Energy}$$

$$K = E - E_0$$

$$\boxed{K = mc^2 - m_0 c^2}$$

In limit '$V \ll c$' then — $\underline{K = \frac{1}{2} m_0 V^2}$

$$K = \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}} - m_0 c^2$$

$$\boxed{E = mc^2} \longrightarrow \underset{(K.E. + R.E.)}{\text{Total Energy.}} = \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}}$$

* Kinetic Energy and Momentum Relation :-

$$pc = \sqrt{K(K + 2m_0 c^2)}$$

* Total Energy and Momentum Relation :-

$$\boxed{E = \sqrt{p^2c^2 + m_0^2c^4}}$$

* for $m_0 = 0$

$$\boxed{p = \frac{E}{c}}$$

* for Non - Relativistic Case :-

$$\boxed{E = K = \frac{1}{2} m_0 v^2}$$

or

$$\boxed{E = K = \frac{p^2}{2m}}$$

Q. Kinetic Energy of a particle is $2m_0c^2$ what is it's momentum.

Sol^n

$$p = \sqrt{K(K + 2m_0c^2)}$$

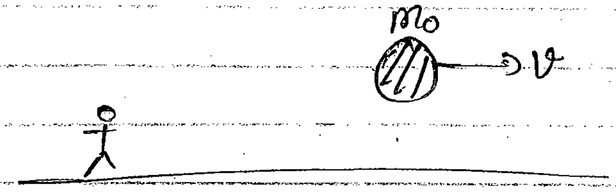
$$= \sqrt{2m_0c^2(2m_0c^2 + 2m_0c^2)}$$

$$= \sqrt{8m_0^2c^4}$$

$$\boxed{p = 2\sqrt{2}\, m_0 c}$$ Ans

* Important Note :-

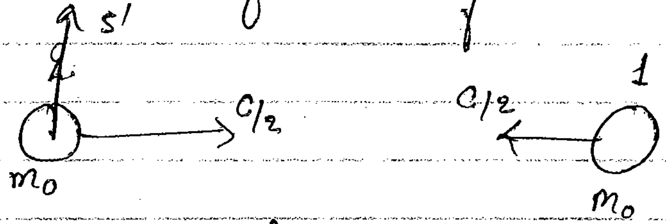
(1) $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$



V = Relative velocity in a frame in which mass is being measured.

Q. 2 e^{-s} of rest mass m_0 are moving towards each other with speed $c/2$ as seen from lab frame. What is dynamic mass of one e^- in the rest frame of another e^-.

Sol^n



Let us calculate mass of 1 in rest frame of 2 (w.r.to. 2.)

$$m_1 = \frac{m_0}{\sqrt{1 - \frac{v_{12}^2}{c^2}}}$$

$$v_{12} = v_x' = \frac{v_x - v}{1 - v_x v/c^2}$$

$$= \frac{-\frac{c}{2} - \frac{c}{2}}{1 + \frac{1}{4}} = \frac{-4c}{5}$$

So $$m_1 = \frac{m_0}{\sqrt{1 - \frac{16}{25}}} = \frac{5m_0}{3}$$

$$\boxed{m_1 = \frac{5m_0}{3}}$$

* Relativistic Collision / Breaking :-

* In all relativistic Case :-

Identification :- Words used
→ $v = \alpha c$
→ massless particle
→ rest mass then collision is relativistic.

(i) Momentum of the system is conserved.

$$\boxed{\vec{P_i} = \vec{P_f}}$$

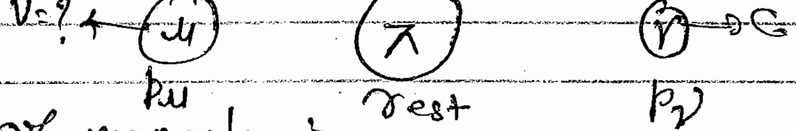
(ii) Total energy is conserved.

$$\boxed{E_{initial} = E_{final}}$$

Q.58 A pion of mass m_π (Rest mass) at rest decays into a muon of mass m_μ (rest mass) and a neutrino of zero mass. Speed of muon is ?

Sol^n

$$\pi \longrightarrow \mu + \nu$$



Conservation of momentum :-

$\therefore P_{initial} = 0$

So $\quad p_\mu = p_\nu$ (opposite in direc^n.)

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

Let v = speed of muon.

$$\frac{m_\mu v}{\sqrt{1 - \frac{v^2}{c^2}}} = p_\nu$$

$$\frac{m_\mu v}{\sqrt{1 - v^2/c^2}} = \frac{E_\nu}{c} \quad \{\because \text{rest mass is zero}\}$$

So $$E_\nu = \frac{m_\mu v c}{\sqrt{1 - v^2/c^2}}$$

Apply Energy Conservation:-

$$E_{initial} = E_{final}$$

Rest Energy $\longrightarrow m_\pi c^2 = E_\mu + E_\nu$

$$m_\pi c^2 = \frac{m_\mu c^2}{\sqrt{1 - v^2/c^2}} + \frac{m_\mu v c}{\sqrt{1 - v^2/c^2}}$$

$$m_\pi \cancel{c^2} = \frac{m_\mu \cancel{c^2}}{\sqrt{1 - v^2/c^2}} \left[1 + \frac{v}{c}\right]$$

$$\frac{m_\pi}{m_\mu} = \frac{(1 + v/c)}{\sqrt{1 - \frac{v}{c}}\sqrt{1 + \frac{v}{c}}}$$

$$\frac{m_\pi}{m_\mu} = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

$$\frac{m_\pi^2}{m_\mu^2} = \frac{1 + v/c}{1 - v/c}$$

Componento-dividento

$$\frac{m_\pi^2 - m_\mu^2}{m_\mu^2}$$

AE T-2010-11
Dated :- 17/Sep/2014

(61)

Decay Process :-

$$\tau^- \longrightarrow \pi^- + \nu$$

rest

$$m_0 = M_\tau \qquad M_\pi \qquad 0$$

$$p_\pi = p_\nu \quad \text{——— (1)}$$

Let E_π is energy of pion.
from conservation of Energy.

$$E_\tau = E_\pi + E_\nu$$

$$M_\tau c^2 = E_\pi + p_\nu c$$

$$\Rightarrow M_\tau c^2 = E_\pi + \sqrt{E_\pi^2 - m_\pi^2 c^4}$$

$$\Rightarrow (M_\tau c^2 - E_\pi)^2 = E_\pi^2 - m_\pi^2 c^4$$

$$\Rightarrow M_\tau^2 c^4 + \cancel{E_\pi^2} - 2M_\tau c^2 E_\pi = \cancel{E_\pi^2} - m_\pi^2 c^4$$

$$\Rightarrow c^4 (M_\tau^2 + M_\pi^2) = 2M_\tau \cancel{c^2} E_\pi$$

$$\Rightarrow \boxed{E_\pi = \frac{(M_\tau^2 + M_\pi^2) c^2}{2M_\tau}}$$

Ans

$$E_\nu = p_\nu c$$

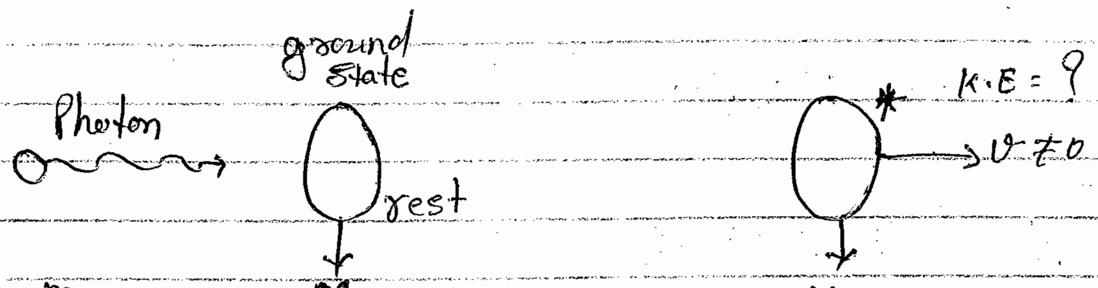
$$E_\pi = \sqrt{p_\pi^2 c^2 + M_\pi^2 c^4}$$

$$E_\pi = \sqrt{p_\nu^2 c^2 + M_\pi^2 c^4}$$

$$\sqrt{\frac{E_\pi^2 - m_\pi^2 c^4}{c^2}} = p_\nu$$

Total Energy $\rightarrow E = K + m_0c^2$

Q.60



$m_0 = M$ — initial, Before absorption

$M + \Delta$ — After Absorption, final.

Conservation of momentum :-

$$p_{initial} = p_{final}$$

$$\Rightarrow p_{photon} + p_{atom} = p^*_{atom}$$

$$\Rightarrow \frac{E_p}{c} + 0 = p^*_{atom}$$

$$\frac{E_p}{c} = p_a \quad \text{—— (1)}$$

Conservation of energy :-

$$E_{photon} + E_{atom} = E^*_{atom}$$

$$E_p + Mc^2 = E^*_{atom} \quad \text{—— (2)}$$

$$\Rightarrow p_a c + Mc^2 = E^*_{atom} = K + m_0c^2$$

$$\Rightarrow p_a c + Mc^2 = K + (M+\Delta)c^2$$

$$\therefore p_c = \sqrt{K(K + 2m_0c^2)}$$

So ~~$p_a c$~~

$$\sqrt{K(K+2m_0c^2)} + Mc^2 = K + (M+\Delta)c^2$$

$$\Rightarrow \sqrt{K[K+2(M+\Delta)c^2]} = K + \cancel{Mc^2} + \Delta c^2 - \cancel{Mc^2}$$

$$\Rightarrow \sqrt{K[K+2(M+\Delta)c^2]} = K+\Delta c^2$$

Squiring –

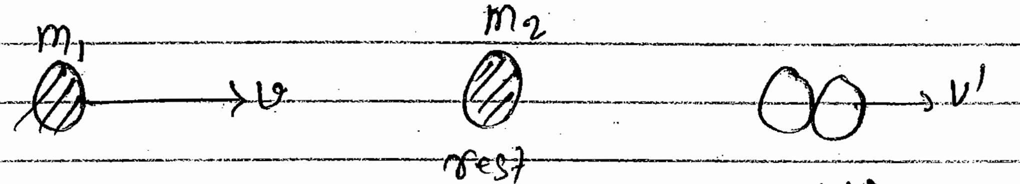
$$\cancel{K^2} + 2K(M+\Delta)c^2 = \cancel{K^2} + \Delta^2 c^4 + 2K\Delta c^2$$

$$2Mc^2K + \cancel{2\Delta c^2 K} = \Delta^2 c^4 + \cancel{2K\Delta c^2}$$

$$\boxed{K = \frac{\Delta^2 c^2}{2M}}$$

Q. A particle of rest mass m_1 moving with speed v collides with another particle of mass m_2 which is initially at rest and sticks to it. What is final velocity of the new system and what is new rest mass?

Sol



$m_0 = M \geq m_1 + m_2$

Let v' = final speed of combined masses

M – final rest mass of " "

$$\Rightarrow P_{initial} = P_{final}$$

$$\frac{m_1 v}{\sqrt{1 - v^2/c^2}} + 0 = \frac{M v'}{\sqrt{1 - v'^2/c^2}} \quad \text{—(1)}$$

$$\frac{m_1 v}{\sqrt{1-v^2/c^2}} = \frac{M V'}{\sqrt{1-V'^2/c^2}} \longleftarrow (1)$$

Conservation of energy -

$$E_{initial} = E_{final}$$

$$\frac{m_1 c^2}{\sqrt{1-v^2/c^2}} + m_2 c^2 = \frac{M c^2}{\sqrt{1-V'^2/c^2}} \longleftarrow (2)$$

Divide (I)/(II) -

$$\left[\frac{\frac{m_1 c^2}{\sqrt{1-v^2/c^2}} + m_2 c^2}{\frac{m_1 v}{\sqrt{1-v^2/c^2}}}\right]^{-1} = \frac{V'}{c^2}$$

$$\frac{V'}{c^2} = \left[\frac{c^2}{v} + \frac{m_2 c^2}{m_1 v} \times \sqrt{1-v^2/c^2}\right]^{-1}$$

$$\frac{V'}{\cancel{c^2}} = \frac{1}{\frac{\cancel{c^2}}{v}\left[1 + m_2/m_1 \times \sqrt{1-v^2/c^2}\right]}$$

$$\boxed{V' = \frac{v}{1 + \frac{m_2}{m_1}\sqrt{1-v^2/c^2}}}$$

For non Relativistic :- $v << c \Rightarrow \sqrt{1-v^2/c^2} = 1$

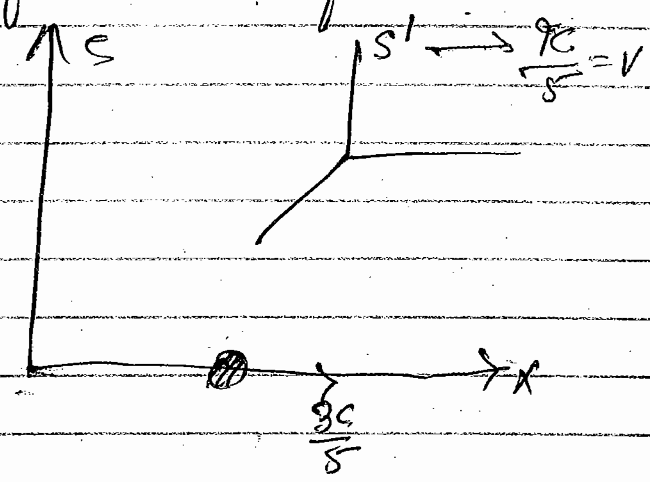
$$\boxed{V' = \frac{m_1 v}{m_1 + m_2}}$$

* Non Relativistic Case :-

m_1 O→ v O m_2 m_1+m_2 OO → v'

$$\boxed{m_1 v = (m_1+m_2)v'}$$

Q. An electron of rest mass m_0 is moving in x-direcn with velocity $\frac{3c}{5}$ in S frame. What is the energy of $\bar{e}^5$ in S' frame.



$$E = mc^2$$

$$= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$$

Energy in S' frame.

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{V_x'^2}{c^2}}}$$

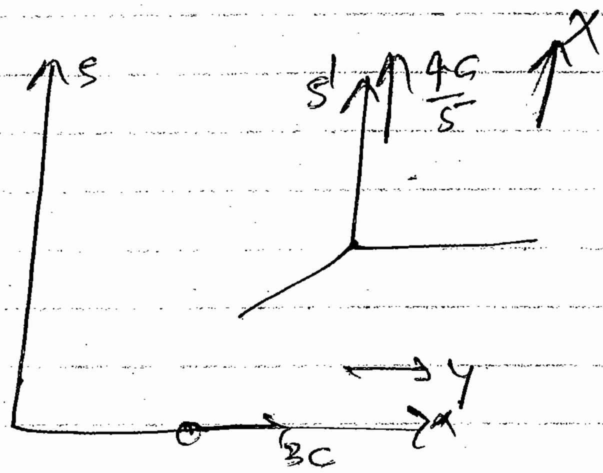
$$V_x' = \frac{V_x - V}{1 - \frac{V_x V}{c^2}}$$

$$= \frac{\frac{4c}{5} - \frac{3c}{5}}{1 - \frac{\frac{3c \cdot 4c}{5 \cdot 5}}{c^2}}$$

$$= \cancel{48} = \frac{5}{13}c$$

Q. In previous question what is energy of e^- in S' frame where S' frame is moving in y direcⁿ

Solⁿ ∵ Direcⁿ of movement of S' frame is considered as x-direcⁿ.
So particle is moving in y direcⁿ.



$V_y = \frac{3c}{5}$

$V_x = 0$

$$E' = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$V_x' = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{V_x - V}{1 - \frac{V_x V}{c^2}}$$

$$= \frac{0 - \frac{4c}{5}}{1-0} = -\frac{4c}{5}$$

$$\boxed{V_x' = -\frac{4c}{5}}$$

$$V_y' = \frac{V_y\sqrt{1-v^2/c^2}}{1-\frac{V_x \cdot V}{c^2}} = \frac{3c}{5} \times \frac{3}{5} = \frac{9c}{25}$$

$$v' = \sqrt{V_x'^2 + V_y'^2} = \sqrt{\frac{16}{25}c^2 + \frac{81}{625}c^2}$$

$$\boxed{v' = \sqrt{\frac{481}{625}c^2}}$$

Then $E' = \frac{m_0 c^2}{\sqrt{1-\frac{481}{625}}}$

$$\boxed{E' = \frac{25}{12} m_0 c^2}$$

Q. A photon of wavelength λ strikes an electron of rest mass m_0 which is initially at rest. If the photon is scattered by e^- in opposite direcn. what is new wavelength of photon and what is K.E. of e^-.

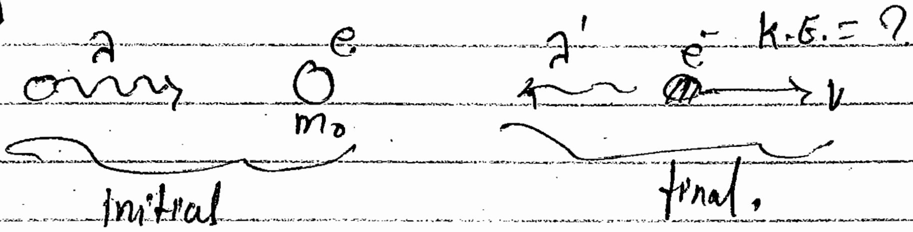
Sol^n

$$\lambda' = \lambda + \frac{h}{m_0 c}(1-\cos\theta)$$

$$\lambda' = \lambda + \frac{2h}{m_0 c}$$

$$\boxed{\lambda' - \lambda = \frac{2h}{m_0 c}}$$

K.E. of electron:-



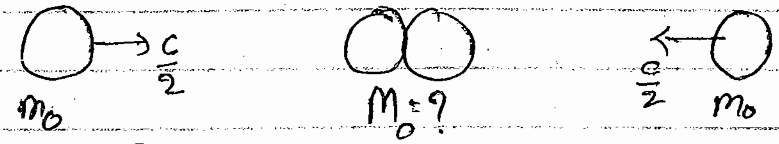
$$T.E_{initial} = T.E._{final}$$

$$\frac{hc}{\lambda} + \cancel{m_0 c^2} = \frac{hc}{\lambda'} + k + \cancel{m_0 c^2}$$

$$\boxed{k = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda'}\right]}$$

Q. Two particles of equal rest mass m_0 are moving in opposite direc^n with speed $c/2$. After collision they strick each other what is new rest mass?

Sol^n



(a) $\frac{4m_0}{\sqrt{3}}$ (b) $\frac{2m_0}{\sqrt{3}}$ (c) $2m_0$ (d) $\frac{5m_0}{\sqrt{3}}$

$$P_{initial} = P_{final}$$

$$\cancel{\frac{m_0 \cdot \frac{c}{2}}{\sqrt{1-\frac{1}{4}}}} - \cancel{\frac{m_0 \cdot c/2}{\sqrt{1-\frac{1}{4}}}} = P_{final}$$

$$\boxed{P_{final} = 0}$$

So final mass is at rest.

$$T.E_i = T.E_f$$

$$\frac{m_0}{\sqrt{1-\frac{1}{4}}}c^2 + \frac{m_0 c^2}{\sqrt{1-\frac{1}{4}}} = M_0 c^2$$

$$\frac{4m_0\cancel{c^2}}{\sqrt{3}} = M_0 \cancel{c^2}$$

$$\boxed{M_0 = \frac{4}{\sqrt{3}} m_0}$$

$$\boxed{M_0 > 2m_0}$$

Q. A particle of rest mass m_0 ~~is~~ has momentum $2m_0c$ what is total energy of the particle and what is its speed.

Sol

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

$$= \sqrt{5\, m_0^2 c^4}$$

$$\boxed{E = \sqrt{5}\, m_0 c^2}$$

$$\because \quad p = mv$$

and $E = mc^2$

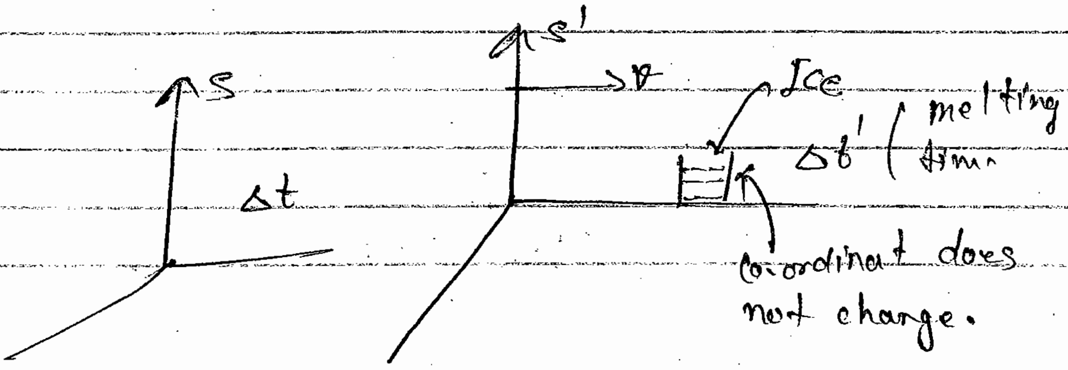
so $$\frac{p}{E} = \frac{v}{c^2}$$

so $$\boxed{v = \frac{p}{E} c^2}$$

$$\boxed{v = \frac{2}{\sqrt{5}} c}$$

* Time Dilation :-

All process (may be Physical, Chemical and biological) slow down while in motion. Due to which completion of process takes more time.



$\Delta x' = 0$

but $\Delta x \neq 0$

So $$\Delta t = \frac{\Delta t' + V\Delta x'/c^2}{\sqrt{1 - V^2/c^2}}$$

$$\boxed{\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}} \quad \{\because \Delta x' = 0$$

Relation b/w time duration.

$\Delta t'$ = time measured in the frame in which {must sample is at rest (proper time) $\{\Delta x' = 0\}$

Δt = is time measure in the frame in which it is moving.

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}$$

Proper life time of particle

Life time measured in Lab frame.

$$\boxed{\Delta t > \Delta t'}$$

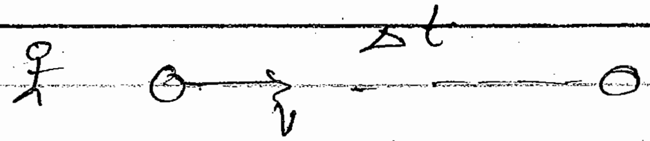
Q. A particle of rest mass m_0 has proper time 2×10^{-6} sec $\{2\,\mu s.\}$ if total energy of the particle is $2m_0c^2$ what distance it will travell during its lifetime,

Sol^n

$\Delta t' = 2 \times 10^{-6}$ sec

m_0

$E = 2m_0c^2$



Distance = Velocity × time (Δt)

$$= v \cdot \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$$

$$2\cancel{m_0 c^2} = \frac{\cancel{m_0 c^2}}{\sqrt{1 - v^2/c^2}}$$

$$1 - \frac{v^2}{c^2} = \frac{1}{4}$$

$$v = \frac{\sqrt{3}}{2} c$$

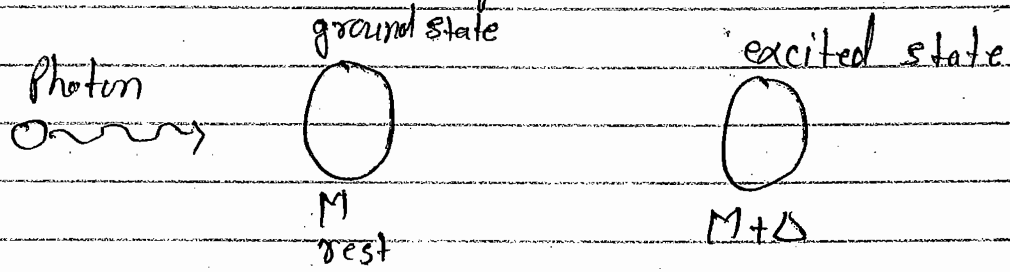
So Distance $= \frac{\sqrt{3}}{2} c \cdot \frac{\Delta t'}{\sqrt{1 - \frac{3}{4}}}$

$$= \frac{\sqrt{3}}{\cancel{2}} \, 3 \times 10^8 \times 2 \times 10^{-6} \cancel{\times 2}$$

$$= 6\sqrt{3} \times 10^2 \text{ meter.}$$

2011/12 V.E.T

Q. What is frequency of photon?



Sol^n Applying conservation of momentum –

let energy of photon is E_p. {momentum $= \frac{E_p}{c}$}

$$P_{initial} = P_{final}$$

$$\frac{E_p}{c} + 0 = P_{final} = P_{atom}$$

$$E_p = c P_{atom} = c P_a \quad \text{(1)}$$

momentum of atom in excited state

Applying conservation of Energy -

$$T.E_i = TE_f$$

$$E_p + Mc^2 = \sqrt{P_a^2 c^2 + (M+\Delta)^2 c^4}$$

$$= \sqrt{\left(\frac{E_p}{\cancel{c}}\right)^2 \cancel{c^2} + (M+\Delta)^2 c^4}$$

$$(E_p + Mc^2) = \sqrt{E_p^2 + (M+\Delta)^2 c^4}$$

Squiring on both side -

$$(E_p + Mc^2)^2 = E_p^2 + (M+\Delta)^2 c^4$$

$$\cancel{E_p^2} + \cancel{M^2 c^4} + 2E_p M c^2 = \cancel{E_p^2} + \cancel{M^2 c^4} + \Delta^2 c^4 + 2M\Delta c^4$$

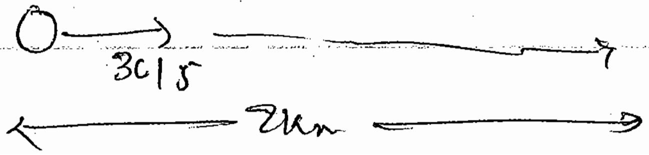
$$E_p = \frac{c^4(\Delta^2 + 2M\Delta)}{2Mc^2}$$

$$E_p = \frac{(\Delta^2 + 2M\Delta)c^2}{2M}$$

$$\frac{E_p}{h} =$$

Q. A particle (unstable) moving with speed $\frac{3c}{5}$ can travell the maximum distance of 2km. as measure in lab from what is proper life-time of the particle.

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}$$



Distance = Velocity × time.

$$2\times10^3 = \frac{3}{5}c \times \Delta t$$

$$2\times10^3 = \frac{3}{5}c \times \frac{\Delta t'}{\sqrt{1-v^2/c^2}}$$

$$= \frac{3}{5}c \times \frac{\Delta t'}{\frac{4}{5}}$$

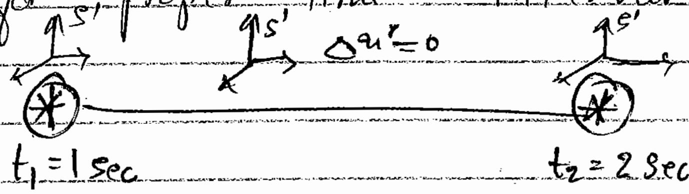
$$\text{so } \Delta t' = \frac{8\times10^3}{3c} = \frac{8\times10^3}{3\times3\times10^8}$$

$$= \frac{8}{9}\times10^{-5}\ \text{sec}$$

$$\boxed{\Delta t' = 8.8\ \mu\text{sec}}$$ Ans

2. The two events take place in lab frame at points (2,0,0), (22,0,0) at times $t_1 = 1$ sec and $t_2 = 2$ sec respectively. Is the time interval measured for these two events is proper time interval. What is velocity of a frame moving along the line joining the two events in which time interval is proper.

Solⁿ No, it is not a proper time interval becoz for proper time interval $\Delta x = 0$



$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1 - V^2/c^2}}$$

$$0 = \frac{20 - V.1}{\sqrt{1 - V^2/c^2}}$$

$$\boxed{V = 20 \text{ m/sec}}$$

Proper time interval $= \Delta t'$

$$\Delta t' = \frac{\Delta t - \Delta x . V/c^2}{\sqrt{1 - V^2/c^2}}$$

$$= \frac{1 - 20.20/c^2}{\sqrt{1 - 400/c^2}}$$

$$\Delta t' = \frac{1 - 400/c^2}{\sqrt{1 - \frac{400}{c^2}}} = \sqrt{1 - 400/c^2}$$

$$\boxed{\Delta t' = \sqrt{1 - 400/c^2}}$$

from time dilation formula :-

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}$$

$$1 = \frac{\Delta t'}{\sqrt{1 - \frac{400}{c^2}}}$$

$$\boxed{\Delta t' = \sqrt{1 - \frac{400}{c^2}}}$$

By L.T. :-

$$\Delta x' = (\Delta x - v\Delta t)\gamma \quad \text{—①}$$

$$\Delta t' = \left(\Delta t - \frac{\Delta x . v}{c^2}\right)\gamma \quad \text{—②}$$

$$\Delta y' = \Delta y \quad \text{—③}$$

$$\Delta z' = \Delta z \quad \text{—④}$$

$① ^2 + ③ ^2 + ④ ^2 - c② ^2$:-

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$

for movement along x-direcⁿ :-

$$\boxed{\Delta x'^2 - c^2 \Delta t'^2 = \Delta x^2 - c^2 \Delta t^2}$$

By putting the value of Δx and Δt we get proper time interval.

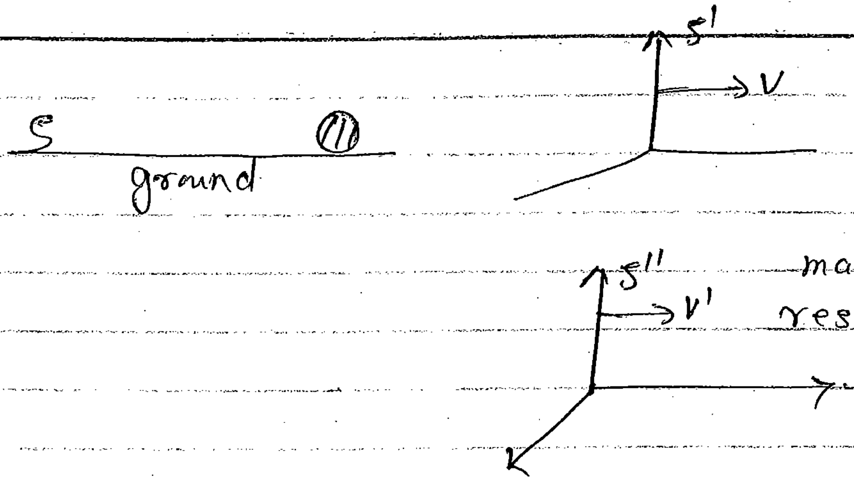
* Lorentz Invariant quantities :- What it means is that the value of quantity will be same in S and S' frame.

(i) $\underline{\Delta r'^2 - c^2 \Delta t'^2} = \Delta r^2 - c^2 \Delta t^2$

$$ds^2 = \Delta r^2 - c^2 \Delta t^2 \Rightarrow \text{Distance in space-time (Distance in Minkowski space)}$$

Distance in space time remains invariant under Lorentz transformation

(ii) Rest mass, proper time, proper length.



mass = m

rest mass $= m\sqrt{1-v^2/c^2}$

mass = m'

rest mass $= m'\sqrt{1-v'^2/c^2}$

(iii) Charge, Maxwell Equations are invariant.

$\vec{E}\cdot\vec{B}$, $E^2 - c^2B^2$

Charge density $= \dfrac{\text{Charge}}{\text{Volume}}$

$$\rho_0 = \frac{Q_0}{V_0}$$

$$\rho = \frac{Q_0}{V} = \frac{Q_0}{V_0\sqrt{1-v^2/c^2}}$$

$$\boxed{\rho = \frac{\rho_0}{\sqrt{1-v^2/c^2}}}$$

Charge density is not ~~in~~ Lorentz invariant.

* $E^2 - c^2p^2$ is also Lorentz invariant:-

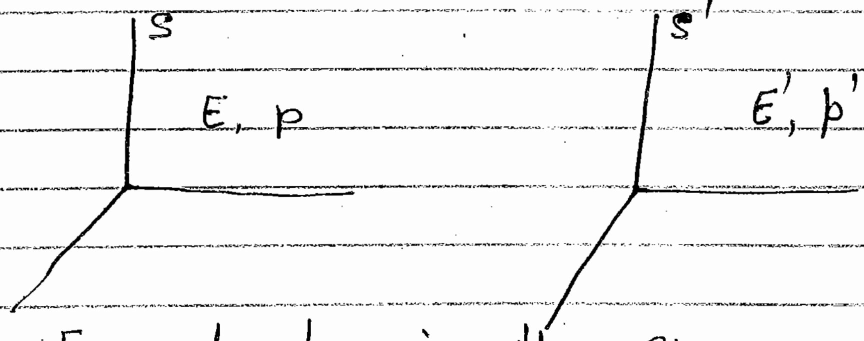
$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

$$E^2 - p^2c^2 = \underbrace{m_0^2c^4}_{\text{Lorentz Invariant.}}$$

So it also.

Taken by :- J. D. Jackson {Electrodynamics}.

* Energy Momentum Transformation Under Lorentz Transformation :-



Let E and p is the energy and momentum in S
and E' and p' is the energy and momentum in S'.

$$E = mc^2$$

$$= \frac{m_0 c^2}{\sqrt{1-V^2/c^2}} \times \frac{dt}{dt} \qquad \left\{ dt = \frac{dt'}{\sqrt{1-V^2/c^2}} \right\}$$

$$E = \frac{m_0 c^2}{dt'} dt \qquad \left\{ dt = \frac{dt'}{m_0} \frac{E}{c^2} \right.$$

Proper time.

Invariant under Lorentz transformation.

$$E \propto dt$$

So energy is transforms as same way time transforms.
E will transform in same way as 'dt' does.

Momentum :-

$$p_x = m v_x$$

$$= \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \times \frac{dx}{dt}$$

$$p = \frac{m_0}{dt'} \, dx \qquad dx = \frac{dt'}{m_0} p$$

Ivariant under lorentz transformation

So we can say p will transforme in same way as 'dx' does.

* Energy Transformation Relation:-

$$\because \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$t \rightarrow k \frac{E}{c^2}$, $x \rightarrow k p$. when $K = \frac{dt'}{m_0}$ ← Invariant.

So

$$\frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{v p_x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{E' = \frac{E - p_x V}{\sqrt{1 - \frac{v^2}{c^2}}}} \leftarrow \text{JEST}$$

* Momentum Transformation Relation:-

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

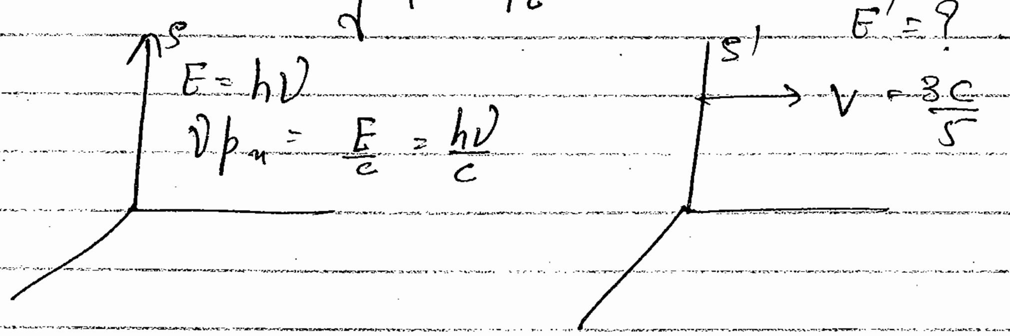
$$\boxed{p_x' = \frac{p_x - V \cdot \frac{E}{c^2}}{\sqrt{1 - v^2/c^2}}}$$

$$E' = \frac{m_0c^2}{\sqrt{1 - v_x^2/c^2}}$$ ← Not applicable in case of photon becoz it rest mass is zero, and v is same in all frame.

Prob: A photon of frequency ν is moving along x-direcⁿ in S frame. what is energy of photon in S' frame which is also moving in x-direcⁿ with speed $\frac{3c}{5}$.

Solⁿ

$$E' = \frac{E - p_x v}{\sqrt{1 - v^2/c^2}}$$

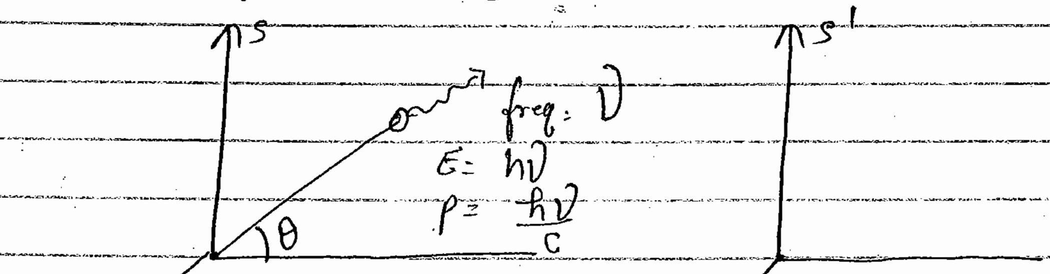


So $$E' = \frac{h\nu - \frac{h\nu}{c}\cdot\frac{3c}{5}}{4/5} = \frac{h\nu - \frac{3h\nu}{5}}{4/5}$$

$$\boxed{E' = \frac{h\nu}{2}}$$

Q. In Lab frame a photon is moving at angle θ in x-direcⁿ if its freq. in lab frame is ν. What is freq. of photon in S' frame which is moving with speed $\frac{4c}{5}$ in X-direcⁿ.

Solⁿ



So $p_x = p\cos\theta$

$p_x = \frac{h\nu}{c}\cos\theta$

Last two question is like a Doppler's Effect.

$$E' = \frac{E - p_x v}{\sqrt{1 - v^2/c^2}} = \frac{h\nu - \frac{h\nu}{c}\cos\theta \, \frac{4c}{5}}{3/5}$$

$$E' = \frac{h\left(1 - \frac{4}{5}\cos\theta\right)\nu}{3/5} = \frac{h\nu(5 - 4\cos\theta)}{3}$$

$$\therefore \nu' = \frac{E'}{h}$$

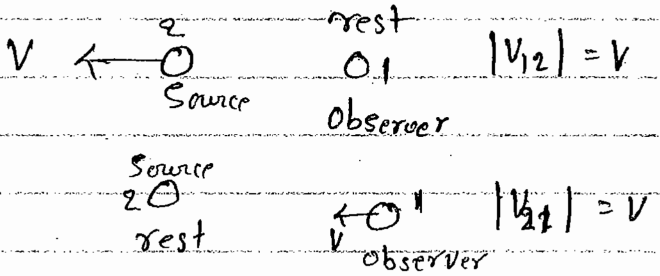
So $$\boxed{\nu' = \frac{\nu(5 - 4\cos\theta)}{3}}$$ Ans

* Doppler Effect of Light:-

The phenomenon of change in frequency of light due to relative velocity or motion.

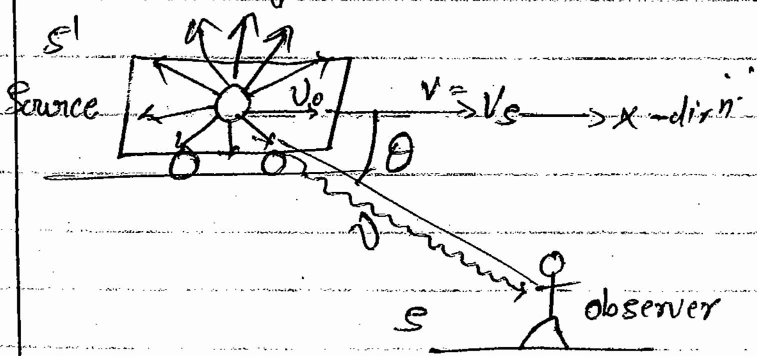
Let ν_0 is natural frequency of light emitted by ~~sound~~ source which is moving with speed V_s.

ν_0 is frequency measured in rest frame of source.



$|V_{12}| = |V_{21}|$

$\boxed{\nu_1 = \nu_2}$



Let ν = frequency in S frame (in observer frame)
θ = angle b/w dir^n of V_s and line joining the source and observer.

$$E' = \frac{E - p_x \cdot v}{\sqrt{1 - v^2/c^2}}$$

$$h\nu_0 = \frac{h\nu - \frac{h\nu}{c} \cos\theta \cdot v_s}{\sqrt{1 - v_s^2/c^2}}$$

$$\nu_0 = \frac{\nu - \frac{\nu}{c} \cos\theta \cdot v_s}{\sqrt{1 - v_s^2/c^2}}$$

So

$$\boxed{\nu = \frac{\nu_0 \sqrt{1 - v_s^2/c^2}}{1 - \frac{v_s}{c}\cos\theta}}$$

Special Case :-

When $\theta = 0$

$$\nu = \frac{\nu_0 \sqrt{1 - v_s^2/c^2}}{1 - v_s/c}$$

$$\boxed{\nu = \nu_0 \sqrt{\frac{1 + v_s/c}{1 - v_s/c}}} \quad \nu > \nu_0$$

If distance b/w source and observer is decreasing then $\nu > \nu_0$ and vice versa.

When $\theta = 180°$

$$\boxed{\nu = \nu_0 \sqrt{\frac{1 - v_s/c}{1 + v_s/c}}} \quad \nu < \nu_0$$

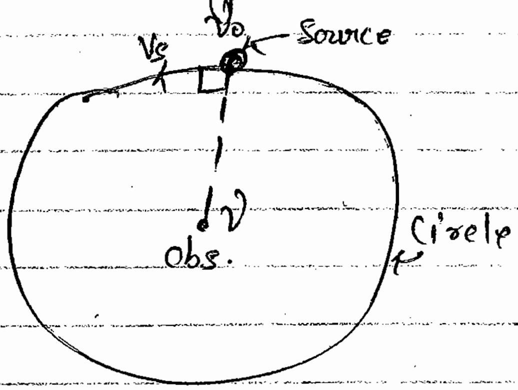
* If both Source and observer is moving:-

$$\boxed{\nu = \nu_0 \frac{\sqrt{1 - v^2_{relativistic}/c^2}}{1 - \frac{v_{relativistic}}{c}\cdot \cos\theta}}$$

$\theta = 90^\circ$

$$\boxed{\nu = \nu_0 \sqrt{1 - v_s^2/c^2}}$$ ← Transverse Doppler's Effect.

In case of sound no Transverse Doppler's Effect because at $\theta = 90^\circ$ $\nu = \nu_0$ in sound but in light $\nu \neq \nu_0$ at $\theta = 90^\circ$.



In case of sound

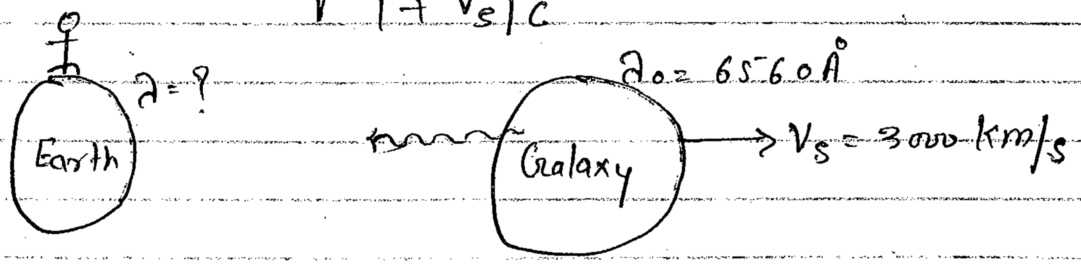
$\boxed{\nu = \nu_0}$ (No Transverse case in Doppler effect in sound.)

But In case of light.

$\boxed{\nu \neq \nu_0}$

59.

$$\nu = \nu_0 \sqrt{\frac{1 - V_s/c}{1 + V_s/c}} \qquad \nu < \nu_0$$



Here source is moving away from observer. So distance is increasing so freq. decreasing so wavelength is increasing.

So

$$\nu = \nu_0 \sqrt{\frac{1 - V_s/c}{1 + V_s/c}}$$

$$\frac{c}{\lambda} = \frac{c}{\lambda_0} \sqrt{\frac{1 - V_s/c}{1 + V_s/c}}$$

$$\lambda = \lambda_0 \sqrt{\frac{1 + V_s/c}{1 - V_s/c}} = \lambda_0 \sqrt{\frac{1 + 10^{-2}}{1 - 10^{-2}}}$$

$$= \cancel{\lambda_0}\ 6560 \left[1 + 10^{-2}\right]^{1/2} \cdot \left(1 - 10^{-2}\right)^{-1/2}$$

$$= 6560 \left[1 + \frac{10^{-2}}{2}\right] \left[1 + \frac{10^{-2}}{2}\right]$$

$$\cancel{= 6560 \left[1 - \frac{10^{-4}}{4}\right]}$$

$$= 6560 \left[1 + 10^{-2} + \underbrace{\frac{1}{4} \times 10^{-4}}_{\text{neglect}}\right]$$

$$= 6560 + 65.60$$

$$\boxed{\lambda = 6625.60}$$

A-8

56

$V = v\hat{n} = v\hat{i}$ { becoz in L.T. direⁿ of velocity is X so in this direcⁿ unit vector is $\hat{i}$ }

In S :-

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

In S' :-

$$\vec{r} = x'\hat{i} + y'\hat{j} + z'\hat{k}$$

Option (a) :-

$$\underbrace{\vec{r}'}_{\text{Vector}} = \gamma\,[\underbrace{(\hat{n}\cdot\vec{r})}_{\text{Scalar}}\ \underbrace{\hat{n}\cdot\vec{v}t}_{\text{Scalar}}]$$

So it can not be correct ans.

Option (b) :-

$$\vec{r}' = \gamma[(\hat{n}\cdot\vec{r})\hat{n} - \vec{v}t] + [\vec{r} - (\hat{n}\cdot\vec{r})\hat{n}]$$

$$(x'\hat{i} + y'\hat{j} + z'\hat{k}) = \gamma[x\hat{i} - v\hat{i}t] + [\cancel{x\hat{i}} + y\hat{j} + z\hat{k} - \cancel{x\hat{i}}]$$

Equating the cofficients of $\hat{i}, \hat{j}$ and $\hat{k}$ -

$$\boxed{x' = \gamma[x - vt]}$$

$$t' = \gamma[t - \vec{v}\cdot\vec{r}/c^2]$$

$1 - \tanh^2\theta = \sec^2\theta \quad , \quad 1 + \tan^2\theta = \sec^2\theta$

A.Q
2.38

$\frac{v}{c} = \tanh\theta \quad , \quad c = 1$

$v = \tanh\theta$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \tanh^2\theta}}$

$= \frac{1}{\sqrt{\operatorname{sech}^2\theta}} = \frac{1}{\operatorname{sech}\theta} = \cosh\theta$

$\gamma = \cosh\theta$

L. T. :- t, x, y, z or x, y, z, t.

$t' = \gamma\left(t - \frac{xv}{c^2}\right) \rightarrow t' = \cosh\theta\left(t - \frac{x \cdot \sinh\theta}{\cosh\theta}\right)$

$t' = \cosh\theta\, t + (-\sinh\theta)\, x$

$x' = \gamma(x - vt) \rightarrow x' = \cosh\theta\, x - \sinh\theta\, t$

$= -\sinh\theta\, t + \cosh\theta\, x$

$y' = y \qquad y' = y$

$z' = z \qquad z' = z$

So

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Transformation Matrix}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Rotation about z-axis :-

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

* Relation b/w Force and Acceleration :-

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} m\vec{v}$$

$$\boxed{\vec{F} = m_0 \frac{d}{dt}\left(\frac{\vec{V}}{\sqrt{1-v^2/c^2}}\right)} \qquad \because m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$\vec{F} = m_0 \left[\frac{\sqrt{1-v^2/c^2}\,\frac{d\vec{V}}{dt} - \vec{V}\frac{d}{dt}\sqrt{1-v^2/c^2}}{[1-v^2/c^2]}\right]$$

$$\frac{d}{dt}\sqrt{1-v^2/c^2} = \frac{1}{\cancel{2}\sqrt{1-\frac{v^2}{c^2}}}\left(-\frac{1}{c^2}\right)\frac{d}{dt}(v^2)$$

$$\frac{d}{dt}(\vec{v}^2) = \frac{d}{dt}(\vec{V}\cdot\vec{V}) = 2\vec{V}\left(\frac{d\vec{v}}{dt}\right)$$

$$\frac{d}{dt}\sqrt{1-\frac{v^2}{c^2}} = \frac{-\frac{1}{c^2}(\vec{V}\cdot\vec{a})}{\sqrt{1-v^2/c^2}}$$

$$\vec{F} = m_0 \frac{\left[\sqrt{1-v^2/c^2}\,\vec{a} + \frac{\vec{v}(\vec{V}\cdot\vec{a})}{\sqrt{1-v^2/c^2}}\right]}{(1-v^2/c^2)}$$

Case - I :- $\vec{v} \perp \vec{a}$

$$F = \frac{m_0}{\sqrt{1 - v^2/c^2}} \vec{a}$$

$$\boxed{\vec{F} = \gamma m_0 a} \quad \vec{a} \perp \vec{v}$$

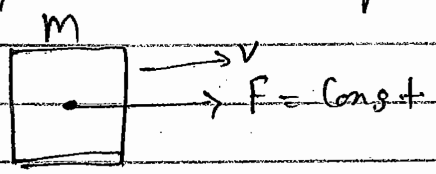
Case II :- $\vec{a} \parallel \vec{v}$

$$\boxed{\vec{F} = \gamma^3 m_0 \vec{a}} \quad \vec{a} \parallel \vec{v}$$

JEST

Q.34

State V/s Time Graph :-



Non Relativistic Case -

$$F = ma$$

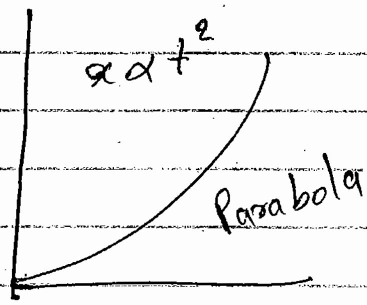
$$a = \frac{F}{m}$$

$$x = ut + \frac{1}{2} a t^2$$

$$x = 0 + \frac{1}{2} \frac{F}{m} t^2$$

$$\boxed{x = \frac{F}{2m} t^2}$$

equation of parabola



* ~~Lagrangian Formulation of Classical Mechanics~~:-

Case - II Relativistic Case :-

$$\vec{F} = \gamma^3 m_0 a \rightarrow a = \frac{F}{m_0 K} = \text{Not Const.}$$

not Const.

Constt.

$$\boxed{F = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \frac{dv}{dt}}$$

$$\int \frac{F}{m} dt = \int \frac{dv}{\left(1 - v^2/c^2\right)^{3/2}} \longrightarrow$$ See Griffith E.M.T. Relativistic.

Hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = K$

* **Density :-**

$$\text{Density} = \frac{\text{mass}}{\text{Volume}}$$

Rest :- $$\rho_0 = \frac{m_0}{V_0}$$

moving :- $$\rho = \frac{m}{V} = \frac{m_0\sqrt{1-v^2/c^2}}{V_0\sqrt{1-v^2/c^2}}$$

$$\boxed{\rho = \frac{\rho_0}{\left(1-\frac{v^2}{c^2}\right)}} \longrightarrow \text{mass density}$$

* **Charge Density :-**

$$\text{Density} = \frac{\text{Charge}}{\text{Volume}}$$

Rest $\Rightarrow$ $$\rho_0 = \frac{Q}{V_0}$$

moving $\Rightarrow$ $$\rho = \frac{Q}{V_0\sqrt{1-v^2/c^2}}$$

$$\boxed{\rho = \frac{\rho_0}{\sqrt{1-v^2/c^2}}} \leftarrow \text{Charge density.}$$

Q.32

Sol^n $K.E. = \text{Total energy} - m_oc^2$

$$K = mc^2 - m_oc^2$$

$$= \frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_oc^2 = m_oc^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{\lfloor 2} + \frac{n(n-1)(n-2)}{\lfloor 6}x^3 - - -$$

$$K = m_oc^2\left[\left(\cancel{1} + \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} - - -\right) - \cancel{1}\right]$$

$$K = \frac{1}{2}m_o v^2 - \frac{3}{8}\frac{m_o v^2}{c^2} - - - -$$

first lowest order correction.

Q.29

Hamiltonian of a free particle :-

Non Relativistic Case :-

$$H = \frac{p^2}{2m} = K.E.$$

Relativistic Case :-

$$H = \sqrt{p^2c^2 + m_o^2c^4} = T.E.$$

$$H = \sqrt{p^2c^2 + m_0^2c^4}$$

$$= m_0c^2\sqrt{\frac{p^2}{m^2c^2} + 1}$$

$$= m_0c^2\left[1 + \frac{p^2}{m_0^2c^2}\right]^{1/2}$$

$$= m_0c^2\left[1 + \frac{p^2}{2m_0^2c^2} - \frac{1}{8}\frac{p^4}{m_0^4c^4} + \cdots\right]$$

$$\Rightarrow m_0c^2 + \underbrace{\frac{p^2}{2m_0}}_{\text{non Relativistic}} - \frac{p^4}{8m_0^3c^2} \leftarrow \text{first correction}$$

So option (a) is correct.

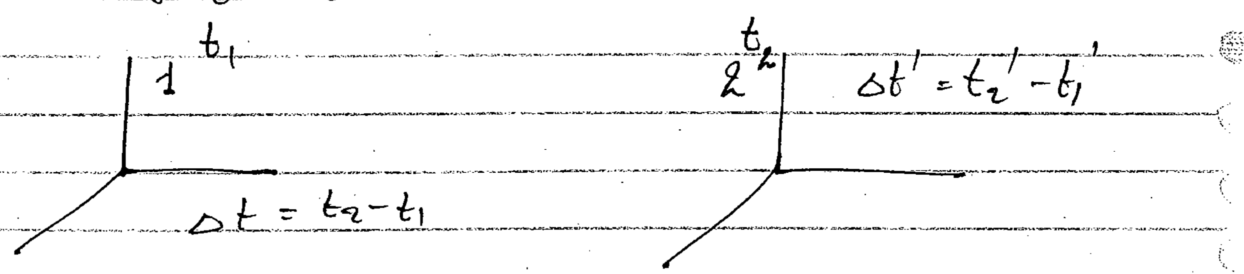
Lagrangian $\Rightarrow$ $\boxed{L = -m_0c^2\sqrt{1 - v^2/c^2}}$

Q.41 The ordering of two events in absolute (does not change) if the invariant interval is ?

(a) Space-like (b) time-like

(c) light-like (d) either space-like or time-like.

Let two events occurs -



(1) If $\Delta t > 0$ in S & $\Delta t' > 0$ in S' then ordering of event not changing.

(2) If $\Delta t > 0$ & $\Delta t' < 0$ then ordering changing.

$$\Delta t' = \frac{\Delta t - \frac{\Delta x v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

If $\Delta t > 0$ in s and $\Delta t' > 0$ in s' } ordering of event not changing.

$\Delta t > 0$, $\Delta t' < 0$ ordering of event changing.

$$\Delta t' = \frac{\Delta t \left[1 - \frac{\Delta x}{\Delta t} \frac{v}{c^2}\right]}{\sqrt{1 - v^2/c^2}}$$

$$\Delta t' = \frac{\Delta t \left[1 - \frac{\Delta x}{c\Delta t} \frac{v}{c}\right]}{\sqrt{1 - v^2/c^2}}$$ { ∵ here $\frac{v}{c} < 1$, ∵ $v < c$ }

$\frac{\Delta x}{c\Delta t} = 1$ ← Light like

$\frac{\Delta x}{c\Delta t} > 1$ ← Space like ← ordering is change.

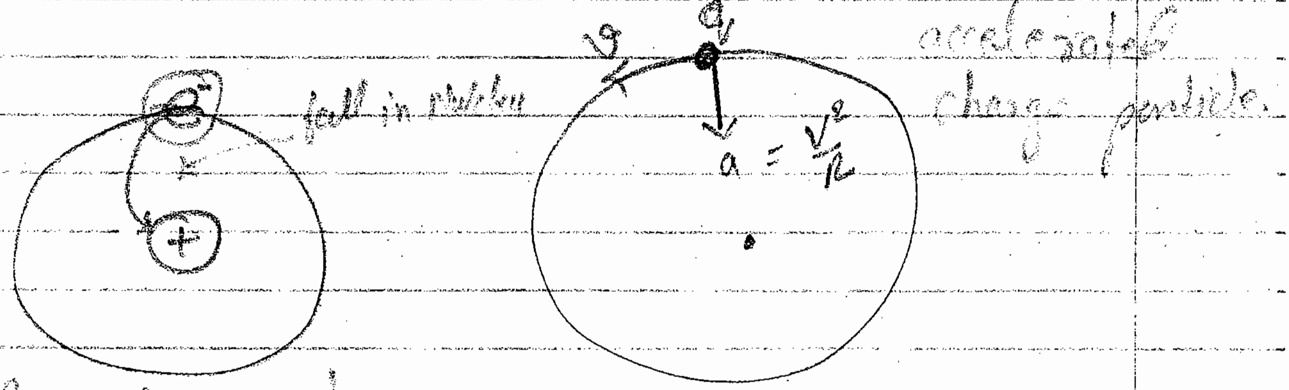
$\frac{\Delta x}{c\Delta t} < 1$ ← time like

Light like, time like ← ordering does not change

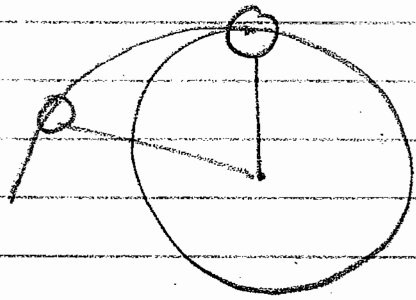
option C ← B

Q. 96 → b

Q. 97



Rutherford model



Ans. (C)

Q.53 Lorentz boost = Lorentz's transformation.

$u' = ct' + x'$

$= c\left(t - \frac{xv}{c^2}\right)\gamma + (x - vt)\gamma$

(b)

(51)

$\tan\theta = \frac{\tan\theta'}{\sqrt{1-\frac{v^2}{c^2}}}$ ← angle in Rest frame.

(5.1)

So $\tan\theta' = \frac{\tan\theta}{\sqrt{1-\frac{v^2}{c^2}}}$ $\quad 50$ ← Rest angle

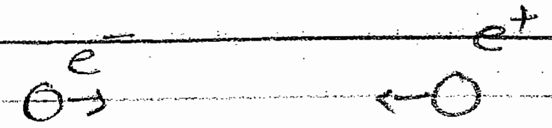
$\therefore \sqrt{1-\frac{v^2}{c^2}} < 1$

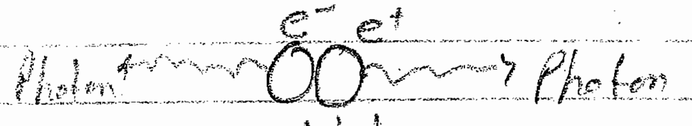
So $\tan\theta' = \gamma \tan\theta$

$\tan\theta' > \tan\theta$

$\boxed{\theta' > \theta}$

(33)





anihilation
(by taking in contact).

$T.K._i = T.E._f$

$m_0c^2 + m_0c^2 = h\nu + h\nu$

$2m_0c^2 = 2h\frac{c}{\lambda}$

2×0.5 M.eV $= \frac{2}{\lambda}\times 1242$ eV - nanometer

1×10^6 $\cancel{eV}$ $= \frac{2}{\lambda}\times 1242$ $\cancel{eV}$ - nanometer

$10^6 = \frac{2}{\lambda}\times 1242$ n.m.

$\lambda = 1242\times 2\times 10^{-6}$ - n.m

$= 2.484\times 10^{-12}$ pikometer

(18) $\Delta t' = \frac{\Delta t - \Delta x v}{\sqrt{1 - v^2/c^2}}$ $v \rightarrow$ relative velo.

* Lagrangian Formulation of Classical Mechanics :-

Degree of freedom :- {DOF} :- It is the number of independent coordinates required to describe dynamics of system. It is represented by 'f'.

For N-particles moving freely in 'd' dimensional world.

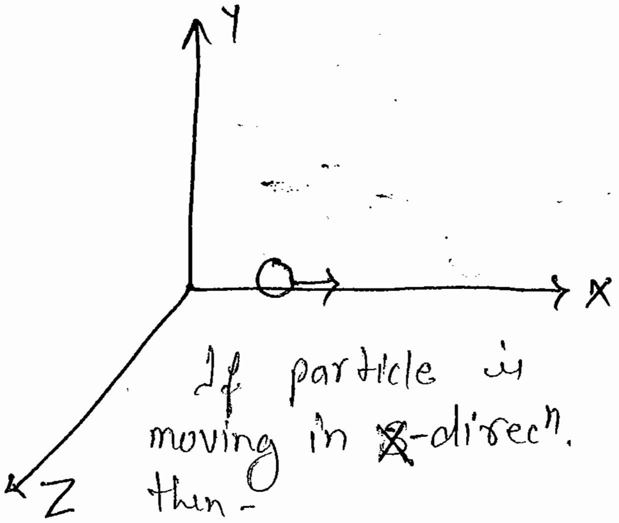
$$\boxed{f = Nd}$$

N = No. of particle
d = ~~dimensioned~~ dimension of particle.

If there is some constraints then -

$$\boxed{f = Nd - k}$$

where k = No. of Constraints.



If particle is moving in ~~x~~-direc^n then -

$d = 3,\ N = 1,\ K = 2$ (means it can't travel in y and z)

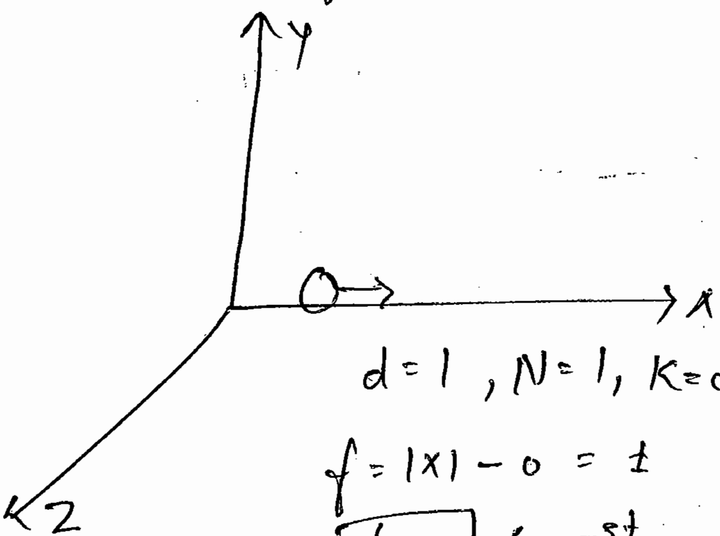
So $f = 1 \times 3 - 2$

$= 3 - 2$

$$\left\{\begin{matrix} y = 0 \\ z = 0 \end{matrix}\right\}$$

$$\boxed{f = 1}$$

IInd way of description.



$d = 1,\ N = 1,\ K = 0$

$f = 1 \times 1 - 0 = 1$

$$\boxed{f = 1}$$

Ist way of description

If 2 particle is moving freely in xy - plane.

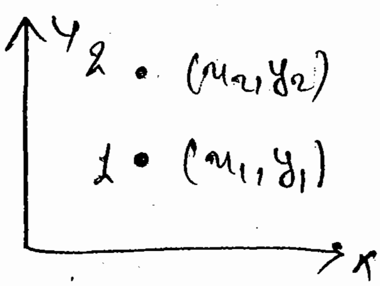
$N = 2,\ d = 2$

$K = 0$

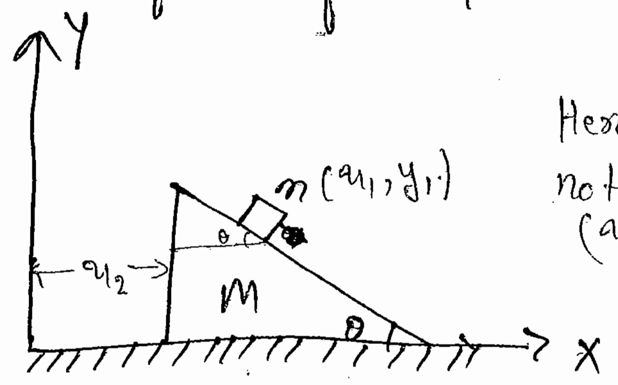
$f = Nd - K$

$f = 2 \times 2 - 0$

$$\boxed{f = 4}$$



Q. Block and Wedge move in vertical plane. what is D.O.f. of system is ?



Here (x_1, y_1) and x_2 are not independent becoz here (x_1, y_1) are dependent on each other

Sol^n

$$f = Nd - K$$

$N = 2$, $d = 2$, $K = 2$

$$f = 2 \times 2 - 2$$

$$\boxed{f = 2}$$

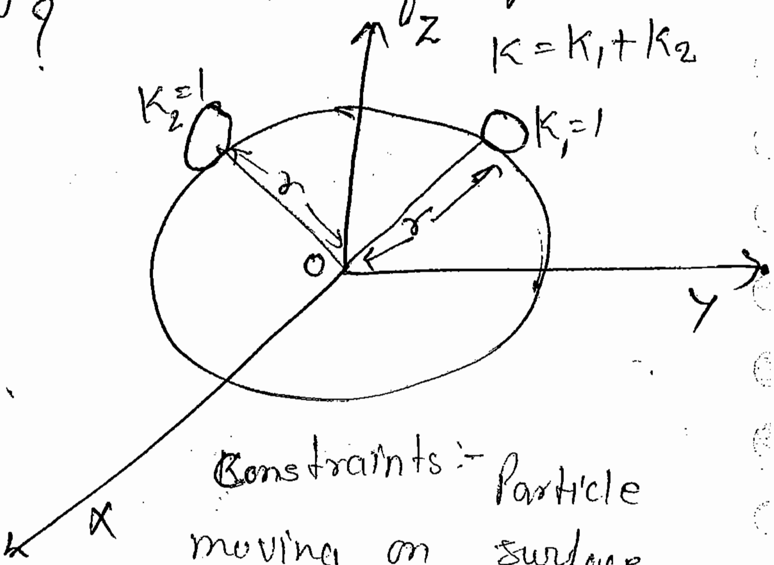
Q. Two particles are moving on surface of sphere find degree of freedom ?

Sol^n

$$f = Nd - K$$

$N = 2$, $d = 3$, $K = 2$

$$f = 2 \times 3 - 2$$
$$= 6 - 2$$
$$\boxed{f = 4}$$

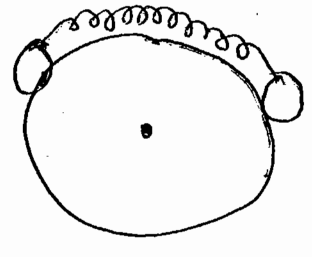


Constraints :- Particle moving on surface any where but distance from the center is always r.

* Particle Connected by spring :-

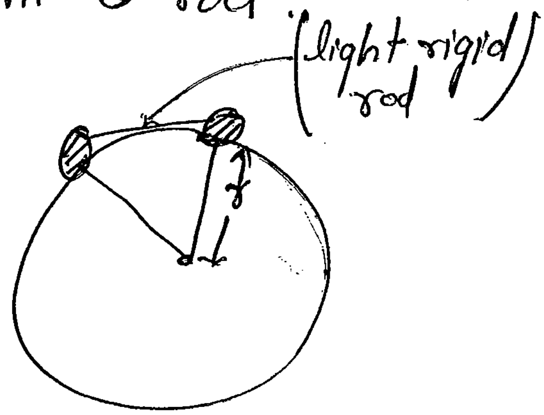
Spring does not put any condition (Constraints).

So. $\boxed{f = 4}$



* Particle Connected by light rod :-

Here there is extra condition (constraints) arise that distance between particle is fixed.

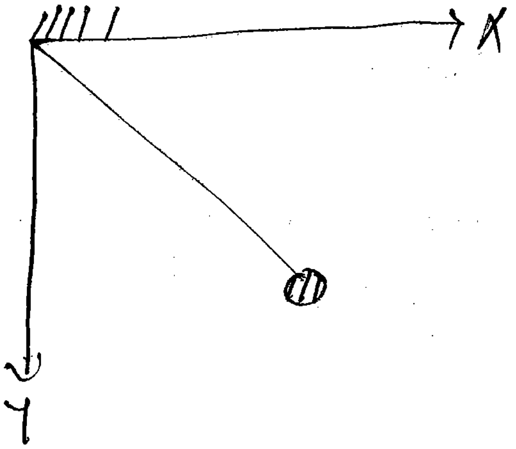


So $K = 3$

$$f = Nd - K$$
$$= 2 \times 3 - 3$$
$$= 6 - 3$$
$$\boxed{f = 3}$$

* Simple Pendulum :-

In simple pendulum, motion is confined in one plane.



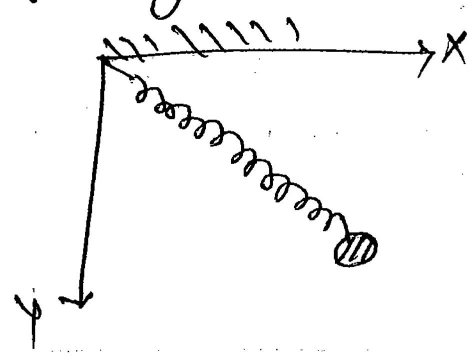
$N = 1$	$N = 1$
$d = 2$	$d = 3$
$K = 1$	$K = 2$
$f = Nd - K$	$f = Nd - K$
$= 2 \times 1 - 1$	$= 1 \times 3 - 2$
	$= 3 - 2$
$\boxed{f = 1}$	$\boxed{f = 1}$

① Confined in one plane
② Distance from point of suspension is l.

① If string is replaced by spring (or flexible string) :-

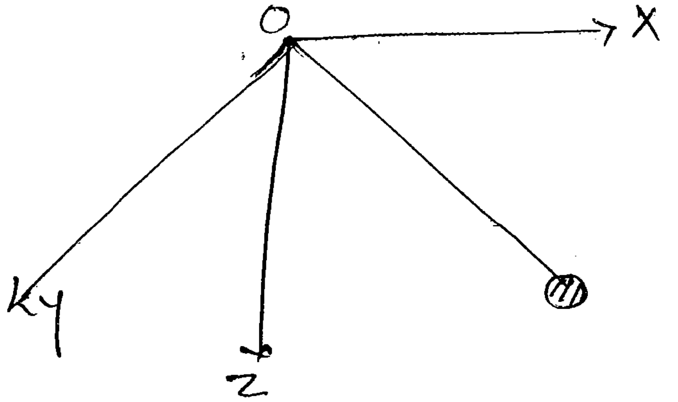
$d = 2$
$K = 0$
$N = 1$

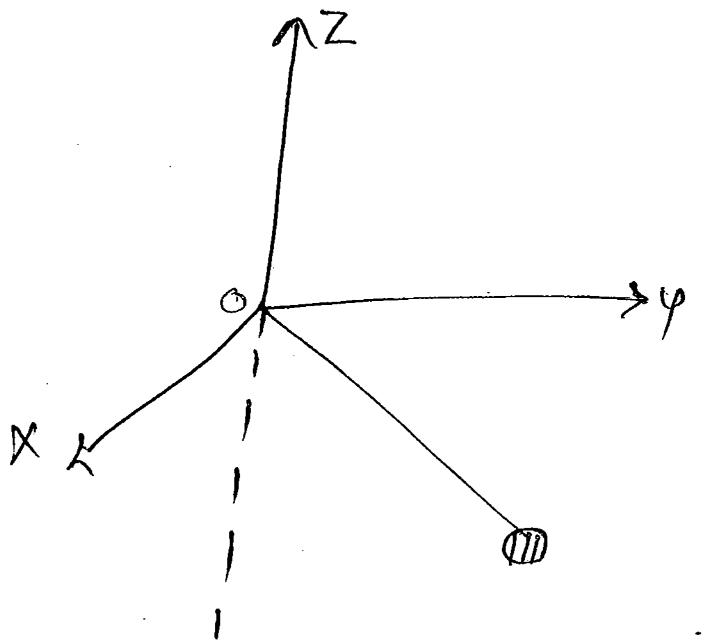
$\boxed{D.O.f. = 2}$



* Spherical Pendulum :-

Motion is not confined in one plane.





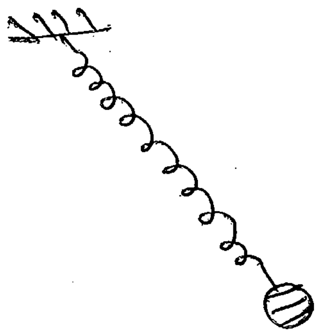
$d = 3$

$N = 1$

$K = 1$ (distance from O is fixed.)

$\boxed{D.O.F. = 2}$

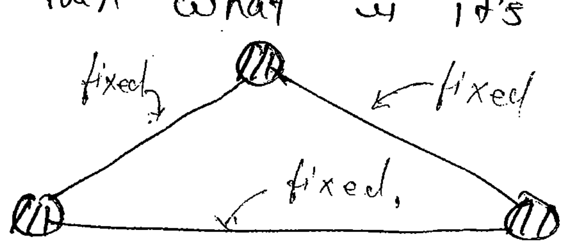
If string is replaced by flexible string or spring :-



$d = 3$
$K = 0$
$N = 1$

$\boxed{DOF = 3}$

Q. Three particle are connected by a light rod to each other as shown in figure. If system is moving in 3-Dimension then what is it's degree of freedom.



Sol^n $K = 3$, $N = 3$, $d = 3$

$f = Nd - K = 3 \times 3 - 3$

$\boxed{f = 6}$.

If moving in 2-D.

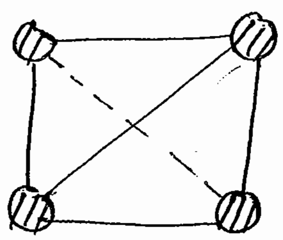
$d = 2$
$N = 3$
$K = 3$

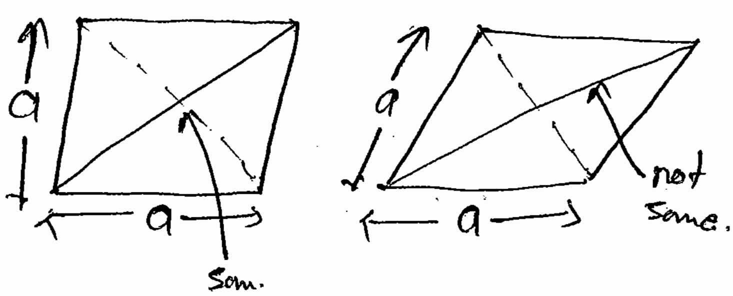
$$Dof = 3 \times 2 - 3 = 6 - 3$$
$$\boxed{Dof = 3}$$

Q. 10 particles are connected to each other by light rod and system is moving in 3-D what is degree of freedom.

Sol^n If $N \geq 3$ and distance b/w the particle is fixed. And the system is moving in 3-D then degree of freedom is equal to 6.
If it is in 2-D it is equal to 3.
If it is in 1-D it is equal to 1.
True for any rigid body

for $N = 4$:-



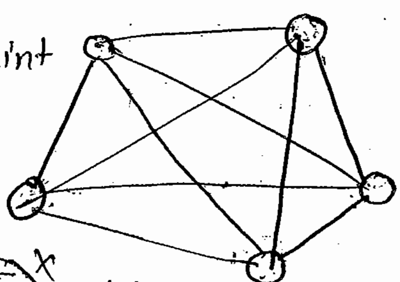


$$DOF = Nd - K = 4 \times 3 - 6 = 12 - 6$$
$$\boxed{DOF = 6}$$

for $N = 5$:-

$N = 5$
$K = 10$
$d = 3$

One constraint is redundant (not require).



$DoF = 5 \times 3 - 10 = 15 - 10 = 5$ not true becoz by statemen it is $= 6$

* Degree of freedom of rigid Body :-

$$\boxed{f = \frac{d(d+1)}{2}}$$

d = dimentionality of World.

3-D → $f = 6$ = 3 rotation + 3 translation
2-D → $f = 3$ = 2 translation + 1 rotation
1-D → $f = 1$ = 1 translation.

---

* Constraints (Conditions)

or

Geometrical Condition on Coordinates

{There is a relation between Coordinates}

* Types of Constraints :-

Constraints

Holonomic (Complete)

⇒ Relation b/w Coordinates is algebric equation

e.g. ⇒ $x^2 + y^2 = l^2$

⇒ If relation b/w coordinates is differential equation then it must be reducible to algebraic equation.

e.g. $x\,dy + y\,dx = 0$

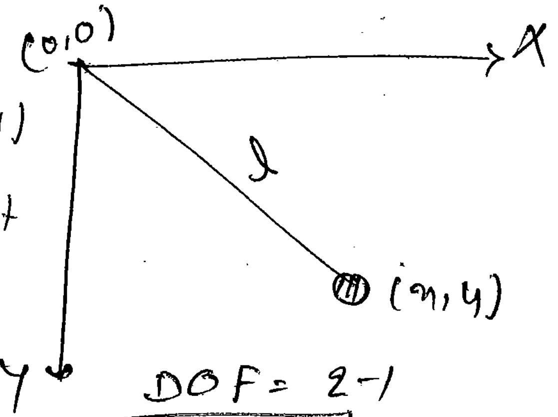
$\int d(xy) = 0 \Rightarrow xy = \text{Const}.$ ← Algebraic eqn.

Non-Holonomic

If relation b/w coordinates is differential equation

* Holonomic Constraints :-

1. Simple Pendulum :-



Variable Coordinates = 2 ⇒ (x,y)

$x^2 + y^2 = l^2$ → Constraint equation. (Holonomic)

$\boxed{y^2 = l^2 - x^2}$

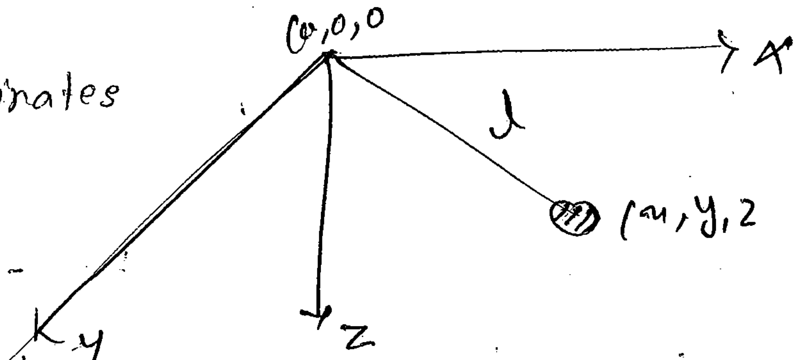
So if we know 'x' we can find 'y'. So it is dependent.

DOF = 2-1

$\boxed{DOF = 1}$

$\boxed{DOF = \text{No. of variable Coordinate} - \text{No. of Constraint equation (Holonomic)}}$

2. Spherical Pendulum :-



No. of variable Co-ordinates = 3 (x,y,z)

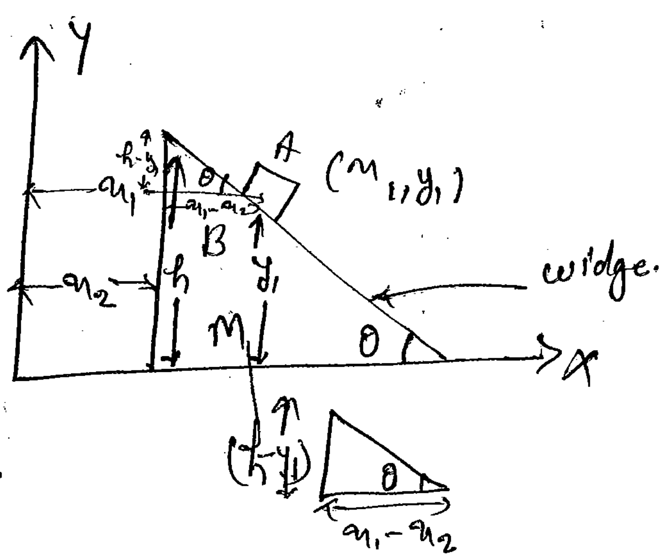
Constraints equation -

$x^2 + y^2 + z^2 = l^2$ (Holonomic)

$\boxed{DOF = 3-1 = 2}$

3. Block & Wedge :-

Here h and θ are fixed.



$$\tan\theta = \frac{h - y_1}{x_1 - x_2}$$

$$h - y_1 = (x_1 - x_2)\tan\theta$$

$$\boxed{y_1 = h - (x_1 - x_2)\tan\theta}$$ ← Constraint equation

$DOF = 3 - 1$

$$\boxed{DOF = 2}$$

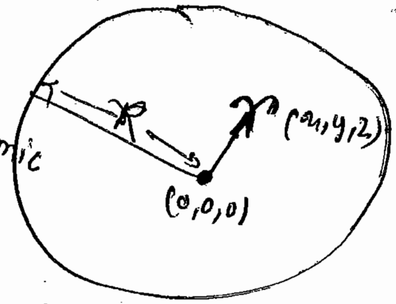
Non Holonomic Constraintes

Constraint equations are either inequality or non integrable differential equations.

* It does not reduce degree of freedom.

Ex- A particle (fly) moving ~~inside~~ inside a sphere.

$$\boxed{\sqrt{x^2+y^2+z^2} \le R}$$ ← Non Holonomic



$d = 3$
$N = 1$
$K = 0$

$DOF = Nd - K$
$= 1 \times 3 - 0$

$$\boxed{DOF = 3}$$

Here there is one constraint ⇒ the fly can not go out of the sphere. but it is non-Holonomic.

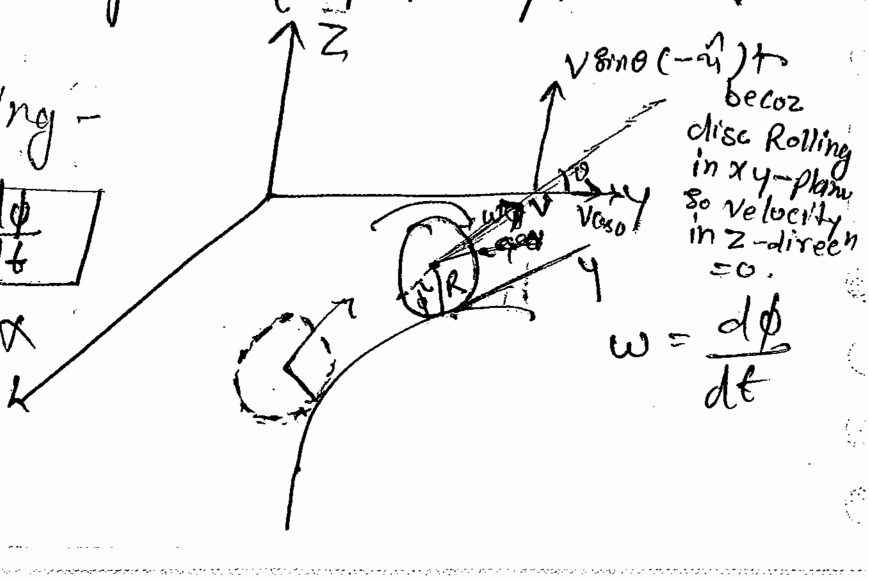
Ex - Disk Rolling on a plane (XY-Plane) :-

Condition ~~of~~ for Rolling -

$$\boxed{V = R\omega} \quad \boxed{V = R\frac{d\phi}{dt}}$$

$V_y = V\cos\theta$

$$\frac{dy}{dt} = R\frac{d\phi}{dt}\cos\theta$$



$$\omega = \frac{d\phi}{dt}$$

$$\boxed{dy = R\,d\phi \cos\theta} \quad \text{(i)}$$

$$V_x = -V\sin\theta$$

$$\boxed{dx = -R\,d\phi \sin\theta} \quad \text{(ii)}$$

Divide (i) by (ii)

$$\frac{dy}{dx} = -\cot\theta$$

$$\boxed{\int dy = -\int \cot\theta \, dx}$$

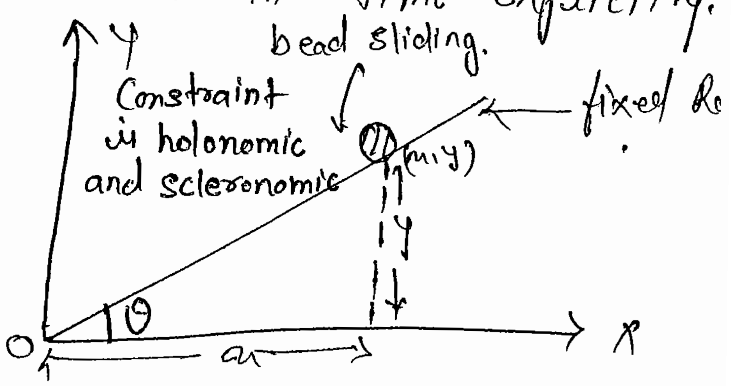
"Differential equation for x and y θ is variable So it can not be integrated.

∴ Rolling is a non-holonomic constraint when motion is along a curved line. and it is holonomic constraint, if motion is along a straight line!"

* Holonomic and Non-Holonomic Constraints have two types.

1. Scleronomic (rigid) : Time Independent:- Therefore constraint equation do not contain time explicitly.

Example :- Bead sliding on a fixed rod.



Constraint equation -

$$\tan\theta = \frac{y}{x}$$

$\boxed{y = x\tan\theta}$ → does not contain time.

$$\text{Dof} = \text{No. of variable} - (\text{No of Constraint eq}^n)$$

$$= 2 - 1$$

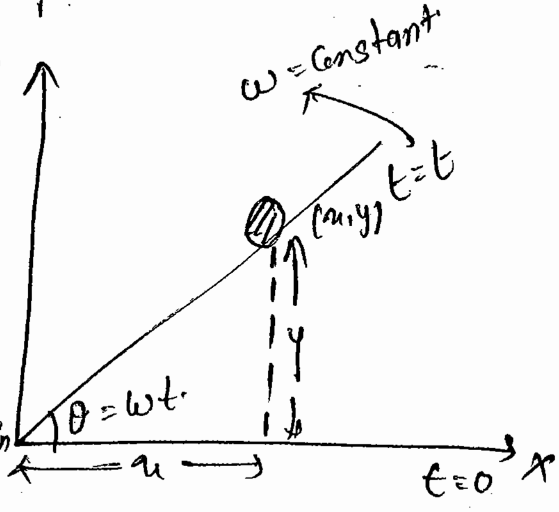
$$\boxed{\text{Dof} = 1}$$

2. Rheonomic Constraint: (Non Rigid)

{Time Dependent}

Constraint equation explicitly contain time.

Ex:- Bead sliding on a rotating rod.



Equation of Constraint -

$$\tan\theta = \frac{y}{x}$$

$$\boxed{y = x \tan\omega t} \Rightarrow \text{It contain time.}$$

⇓

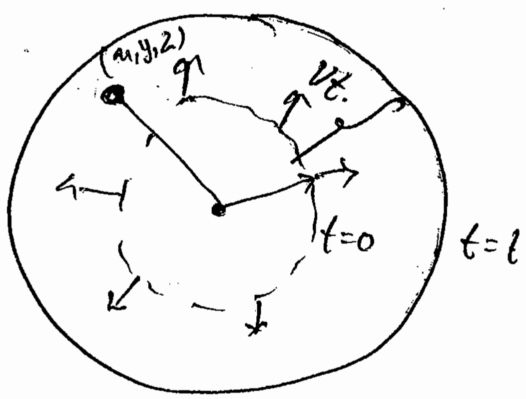
Holonomic and Rheonomic.

Ex:- Particle moving inside a expanding sphere.

$$\boxed{\sqrt{x^2+y^2+z^2} \leq R_0 + vt}$$

⇓

Non - Holonomic and Rhonomic.



* Generalised Coordinates : Convenient Coordinates

Note :- Do not use concepts about dynamics of system lernt from Newtonian Mechanics.

* "Convenient Coordinates chosen to simplify problem are called generalised coordinates."

"In most cases number of generalised coordinates is equal to Degree of freedom."

2. A particle moving along Curve line (fixed) what is degree of freedom of particle?

Sol^n

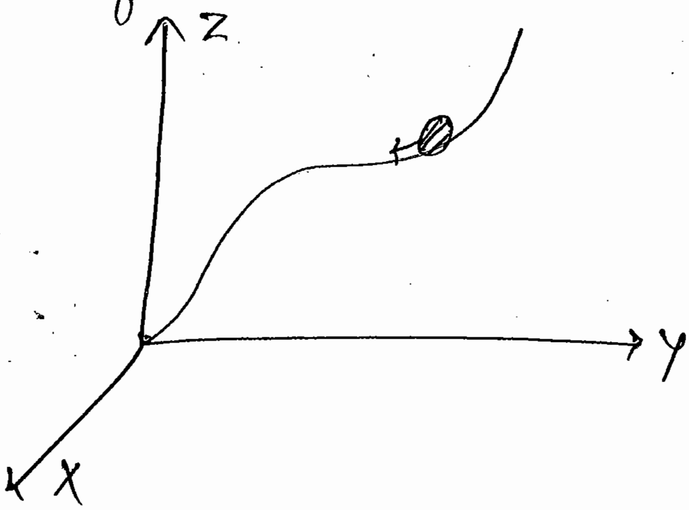
$N = 1$
$k = 1$
$d = 3$

$f = 3\times1 - 1 = 2$ ✗

but it is not true

Here degree of freedom

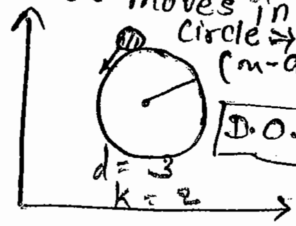
$\boxed{f = 1}$



because A line is formed by intersection of two planes.

So ~~degree of freedom~~ of one plane = 1
and for another plane = 1
So total Constraint = 1+1 = 2
$f = 3\times1 - 2 = 1$ ✓

If Particle moves in Constrained Circle ⇒ $(x-a)^2+(y-a)^2=R$



$\boxed{\text{D.O.F.} = 1}$

Note :- If Particle moves along a fixed line (it may be a curved line or straight line) then $\boxed{\text{D.O.F} = 1}$

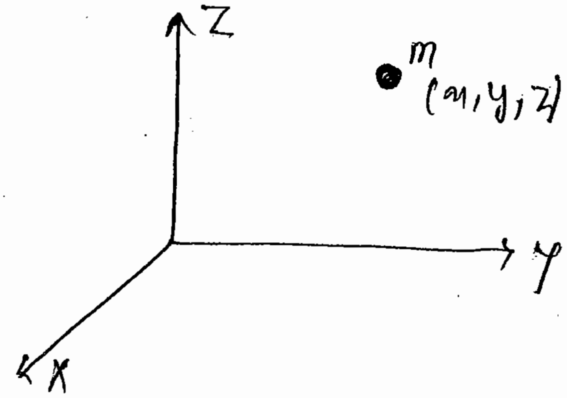
Because a line is formed by intersection of two planes.

* Kinetic Energy in Different Coordinate System :-

(1) Cartesian Co-ordinate :-

$$K.E. = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

(Most Useful)



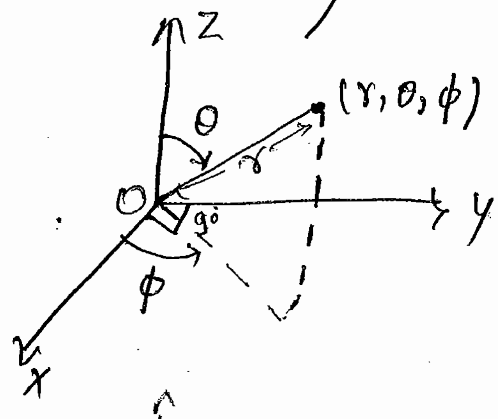
(2) Spherical Polar :-

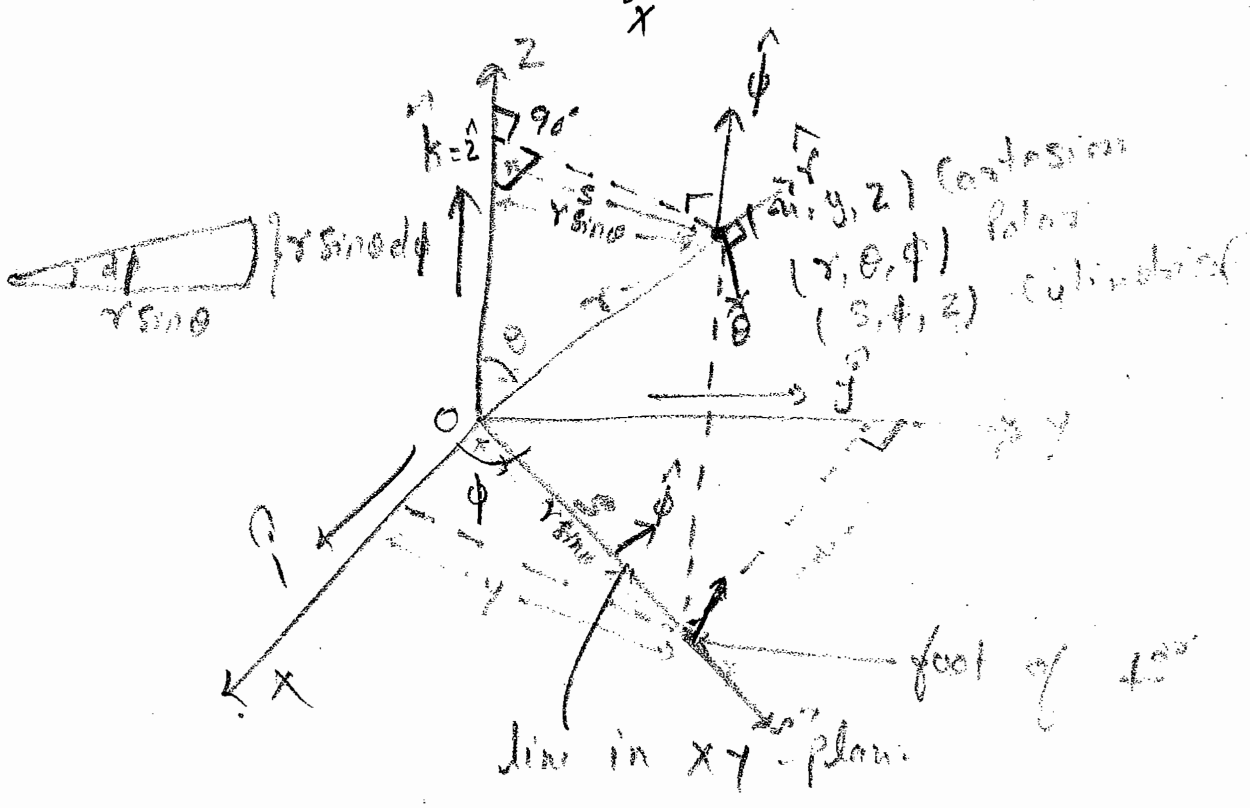
$$K.E. = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$





* How Write K.E. in Different Coordinate System:

Small displacement of particle in different coordinates.

$\Rightarrow$ $\boxed{d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}}$ In Cartesian Coordinate.

$\Rightarrow$ In Spherical Polar :-

$$\boxed{d\vec{s} = dr\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}}$$

$\Rightarrow$ In Cylindrical Coordinate :-

$$\boxed{d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}}$$

* Velocity :-

$$\vec{v} = \frac{d\vec{s}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\vec{v} = \frac{d\vec{s}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

$$v = \frac{d\vec{l}}{dt} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

* Kinetic Energy :-

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin\theta\,\dot{\phi}^2)$$

$$K = \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2)$$

Note:- If a particle is moving in a plane/space and force on it is always directed towards a point or particle is attached to a point by a string/spring or rod then use of plane polar/spherical polar is convenient.

⇒ If case is not so in a plane then we can use Cartesian.

⇒ Plane Polar :-

If we remove z from cylindrical coordinate system then we get plane polar.

* How to write K.E. for more than one particle Cases:-

1. Fist Method :- More General Method.

$$K.E. = \frac{1}{2} m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)$$

2. Second Method :-

Use Superposition method to calculate speed of second particle if speed of first particle is given. (less General).

[Use mostly for motion in one plane]

* Generalised Coordinate :-

Independent Coordinates (Varia... chosen to simplify the problem.

Notation :- q_i

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}$$

$$\boxed{\frac{\partial q_1}{\partial q_2} = 0}$$

Generalised Velocity :-

$$\boxed{\dot{q}_i = \frac{dq_i}{dt}}$$

$\Rightarrow$ In lagrangean formulation q_i and $\dot{q}_i$ are taken to be independent while solving the problem.

$$\boxed{\frac{\partial \dot{q}_i}{\partial q_i} = 0 \qquad \frac{\partial q_i}{\partial \dot{q}_i} = 0}$$

Before problem has been solved

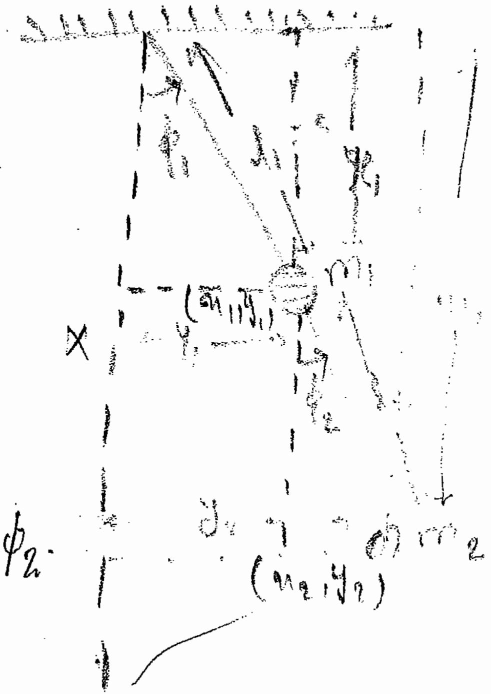
$$\boxed{\frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \delta_{ij}}$$

After problem is solved q_i and $\dot{q}_i$ may turn out to be dependent (actually).

Imp

* K.E. of Double Pendulum :- {both are moving in same plane}

Degree of freedom of system:-



$N = 2$, $d = 2$, $K = 2$

$f = Nd - K$

$f = 2 \times 2 - 2 = 4 - 2$

$\boxed{f = 2}$

Generalised Co-ordinates are ϕ_1 and ϕ_2.

$x_1 = l_1 \cos\phi_1$, $\dot{x}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$

$y_1 = l_1 \sin\phi_1$, $\dot{y}_1 = l_1 \cos\phi_1 \dot{\phi}_1$

$x_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$, $\dot{x}_2 = -[l_1 \sin\phi_1 \dot{\phi}_1 + l_2 \sin\phi_2 \dot{\phi}_2]$

$y_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$, $\dot{y}_2 = [l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2]$

We write K.E. of the system by **first method** -

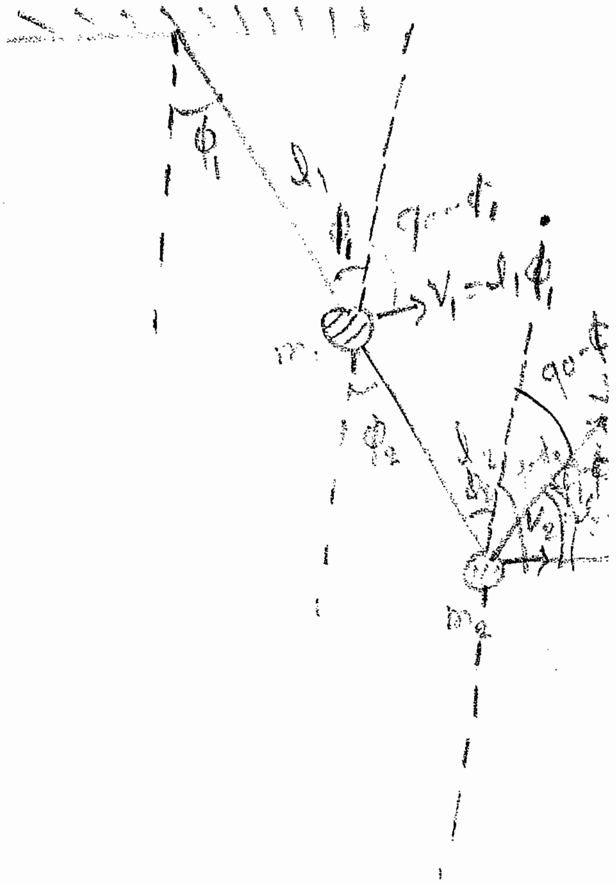
$$K.E. = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

So $$K.E. = \frac{1}{2} m_1 [l_1^2 \dot{\phi}_1^2] + \frac{1}{2} m_2 [l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)]$$

So $$\boxed{T = \frac{1}{2} \left[m_1 l_1^2 \dot{\phi}_1^2 + m_2 \left(l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right) \right]}$$

* Second Method :-

Velocity Superposition Method :-



$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}}$$

$$\vec{v}_1 = 0 + l_1\dot{\phi}_1\hat{\phi}_1$$

$$\boxed{\vec{V}_1 = l_1\dot{\phi}\hat{\phi}_1} \quad \boxed{\vec{V}_2 = l_2\dot{\phi}_2\hat{\phi}_2}$$

Motion of m_2 depends upon motion of m_1.

So K.E. of $m_1 = \frac{1}{2} m_1 v_1^2$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$$

Net velocity of $m_2 = \sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos(\phi_1 - \phi_2)}$

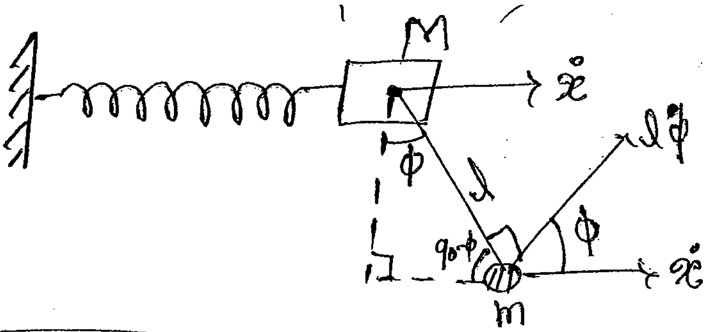
So K.E. of $m_2 = \frac{1}{2} m_2 (\text{net velocity})^2$

$$\text{K.E. of } m_2 = \frac{1}{2} m_2 \left[l_1^2\dot{\phi}_1^2 + l_2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right]$$

So Total K.E. of the system is -

$$\boxed{T = \frac{1}{2}\left[m_1l_1^2\dot{\phi}_1^2 + m_2\left(l_1^2\dot{\phi}_1^2 + l_2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right)\right]}$$

Q.



What is K.E. of the system?

Soln

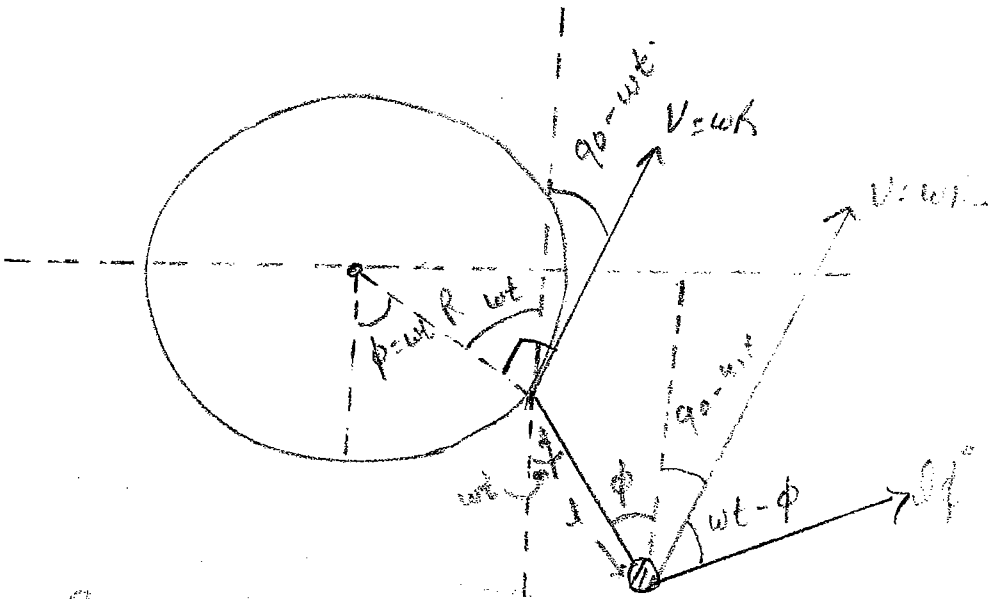
$$\boxed{DOF = 2}$$

$V_1 = \dot{x}$

$V_2 = l\dot{\phi}$

$$\boxed{K.E. = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi\right)}$$

A-10
Q.20



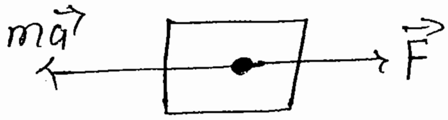
$$\boxed{K.E. = \frac{1}{2} m\left(l^2\dot{\phi}^2 + \omega^2 r^2 + 2l\dot{\phi}\omega r\cos(\omega t-\phi)\right)}$$

* Lagrangian Formulation :-

It is baised on following two approaches -

1. D'Alembert's Principle. +
   Principle of virtual work. [Baised on Physics]

D'Alembert's principle converts Dynamic system into static system. by assuming a revese force.



$$\vec{F} - m\vec{a} = 0$$

Principle of Virtual Work:-

$$\sum_{i=1}^{3N} \vec{f_i} \cdot \delta\vec{r_i} = 0$$

2. Variation Calculus (principle) Approach :- [Baised on Mathematics]
   (Derivation of Schrodinger eq^n).

$$\text{Action } (S) = \int_{t_1}^{t_2} L\,dt$$

dynamics of system is such that action is extremum (only that dynamics is allowed in which action is extrimum).

$\delta S = 0$ [Condition for extremum]

$\delta \int L\,dt = 0$ [Condition for extremum]

$\downarrow$

This gives as Lagrangian's Equation.

* Lagranges' Equation :-

First form: $\boxed{\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_i}\right) - \dfrac{\partial T}{\partial q_i} = Q_i}$

$T = K.E.$

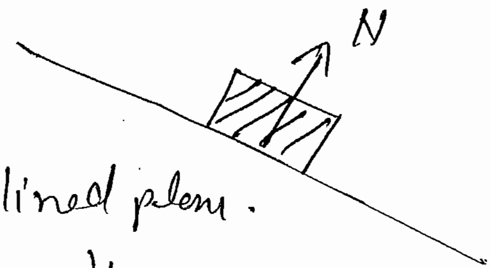
$Q =$ Generalised force (It includes all forces except constraint forces)

* Constraint forces :-

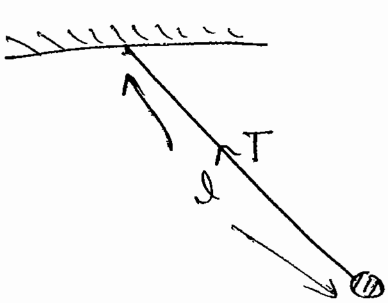
forces arising due to some constraints.

Example :- Normal reation, tension etc.

Normal reaction is arised when object is started to slide down on a inclined plane.



It is due to contact between the inclined plane surface and the object ~~betwee~~ is produced the normal reaction.



* Second form:-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i}$$

Q_i — Generalised force (It includes those forces which potential energy can not be written.).

$L = T - V$ = Lagrangian

V — Potential Energy

* For Constructive system (monogenic system):-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

(Monogenic system is that system for which potential energy can not be written.)

* Relation between Q_i and f_i (actual force's component)

Q_i is may or may not be component of actual force.

$$\boxed{Q_i = \sum_i f_j \frac{\partial r_i}{\partial q_i}} \Rightarrow$$ Used to find Q_i if f_i is given.

$$\boxed{Q_\theta = \sum F_j \frac{\partial r_j}{\partial \theta}}$$

Imp.

* Relation b/w Generalised force and Generalised potential :- $\{U(q_i, \dot{q}_i)\}$:-

$$\boxed{Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right)}$$

Generalised force may or may not be unit of actual force.

Same as:-

$$\vec{F} = -\vec{\nabla} V(x, y, z)$$

$$\vec{F}_x = -\frac{\partial V(x)}{\partial x}$$

* Correspondence b/w Q_i and q_i :-

"Product Q_i, q_i always has dimension of work. Although q_i and Q_i may not have dimension of length and force."

q_i	Q_i
(actual cordinate) x	F_x (actual component of force)
(Angle) θ	F_θ (torque)
(Thermodynamic co-ordinate (variable)) Volume	P (Pressure)
temp.	entropy

* Generalised Momentum :-

$$\boxed{\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{\text{dimension}} - \underbrace{\frac{\partial L}{\partial q_i}}_{\text{dimension}} = \underbrace{Q_i}_{\text{generalised force.}}}$$

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$$

$$\vec{F} = \frac{d(\vec{p})}{dt}$$

It may or may not depend on q_i and $\dot{q}_i$

$\frac{\partial p_i}{\partial q_i} = 0$ may or may not be equal to zero.

$\frac{\partial p_i}{\partial \dot{q}_i} = 0$ or may not be equal to zero.

p_i may or may not be component of actual momentum.

However product p_i and q_i always dimension of angular momentum.

	q_i	p_i
(actual Co-ordinate)	x	p_x (actual momentum)
(angle)	θ ↓ dimensionless	p_θ (angular momentum)

CSIR 2013 Dec

Q. $V = x^2+y^2+\frac{z^2}{2}$ which compount of angular momentu is conserved.?

Soln

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - \left[x^2+y^2+\frac{z^2}{2}\right]$$

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$L = \frac{1}{2}m\{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\} - \left[r^2\sin^2\theta\cos^2\phi + r^2\sin^2\phi\sin^2\theta + \frac{r^2\cos^2\theta}{2}\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right] - \left[r^2\sin^2\theta + r^2\frac{\cos^2\theta}{2}\right]$$

eqn of r :- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$

$$m\ddot{r} - \left[m r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 - 2r\sin^2\theta + r\cos^2\theta\right]$$

eqn of θ :-

ϕ = Cyclic

p_ϕ = Conserved

$$\boxed{L_z = \text{Constant}}$$

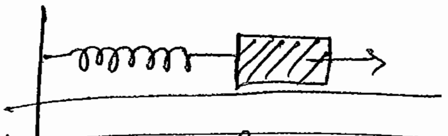
* Cyclic Coordinate / ignorable / removable Coordinate:-

If Lagrangian does not depend on a coordinate then that coordinate is called cyclic coordinate.

Ex.- For a particle is moving in 3-D space (x, y, z).

$$\boxed{L = f(\dot{x}, \dot{y}, \dot{z}, x, y, t)}$$

z = is Cyclic



$$\boxed{L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2}$$

No need to say that y and z are cyclic.

* Conservation theorem / Principle:- If the system is conservative (monogenic).

Potential can be written for all forces.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

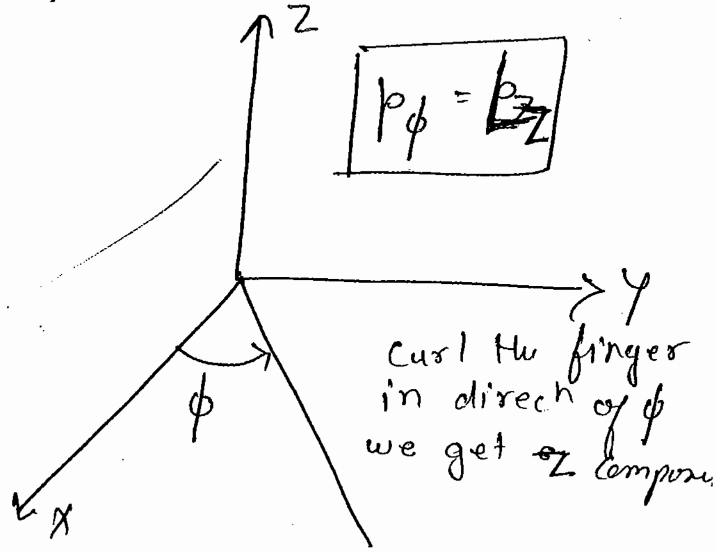
If L is no function of q_i (Cyclic)

$$\frac{\partial L}{\partial q_i} = 0$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\rightarrow \boxed{\frac{\partial L}{\partial \dot{q}_i} = \text{Constant}} \Rightarrow \boxed{p_i = \text{Constant}}$$

Generalised momentum corresponding (conjugate) to cyclic generalised coordinate is conserved.
If x is cyclic then p_x is conserved or if ϕ (in spherical polar coordinate) is cyclic then p_ϕ = Constant. $\{ P_\phi = L_z \}$.



* Problem Based On Lagrangian :-

$$L = T - V$$

$$\text{Equation of motion} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 .$$

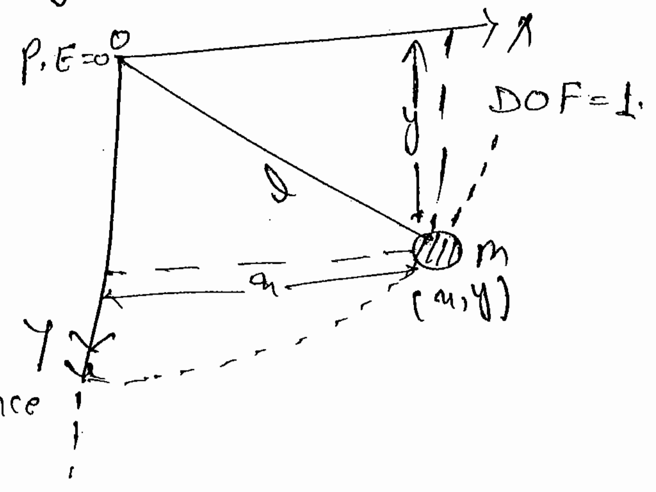
(1) Simple Pendulum :-

(1) Write Lagrangian in terms of x coordinate taken as shown in fig.

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$V = -mgy$$

Here -ve sign comes becoz of particle below the reference level.



$$L = T - V$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy$$

$\because$ $x^2 + y^2 = l^2$

$\therefore$ $y^2 = l^2 - x^2$

$$\boxed{y = \sqrt{l^2 - x^2}}$$

$$\dot{y} = \frac{1}{2\sqrt{l^2 - x^2}} (-2x\dot{x}) \quad \left\{ \text{Diff. w.r.to time} \right)$$

$$\boxed{\dot{y} = \frac{-x\dot{x}}{\sqrt{l^2 - x^2}}}$$

So $L = \frac{1}{2} m \dot{x}^2 \left[1 + \frac{x^2}{l^2 - x^2} \right] + mg\sqrt{l^2 - x^2}$

$$\boxed{L = \left[\frac{1}{2} m \dot{x}^2 + \frac{m\dot{x}^2 l^2}{l^2 - x^2} + mg\sqrt{l^2 - x^2} \right]}$$

Here $L = f(x)$ $\qquad$ x is not cyclic

$\Rightarrow$ p_x is not conserved.

To know whether p_y is conserved or not we must express L in terms of y. (becoz y is also changing when pendulum moves).

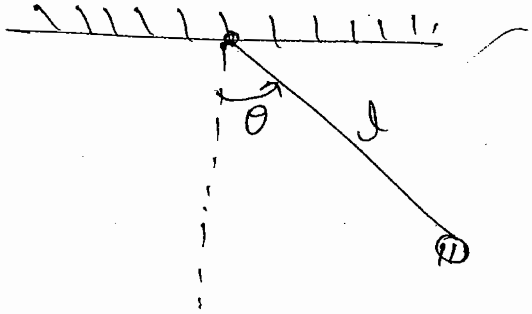
$\Rightarrow$ As pendulum oscillates in x-y plane only then z component of coordinate can not change It can not be considered $\therefore$ z is cyclic

$\therefore$ p_z is conserved.

* Simple Pendulum in terms of θ Coordinate:

We use here plane polar coordinate

$PE = 0$



DOF = 1

$$q = \theta$$

$$T = \frac{1}{2}(\dot{s}^2 + s^2\dot{\phi}^2)$$

$$s = l = \text{Constant} \therefore \dot{s} = 0$$

$$\phi = \theta$$

$$\therefore T = \frac{1}{2}m(l^2 \dot{\theta}^2)$$

$$V = mgl\cos\theta$$

$$\therefore \boxed{L = \frac{1}{2}m(l^2\dot{\theta}^2) + mgl\cos\theta}$$

θ is not cyclic as angular momentum is not conserved.

Equation of motion -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(ml^2\dot{\theta}) - mgl\sin\theta = 0$$

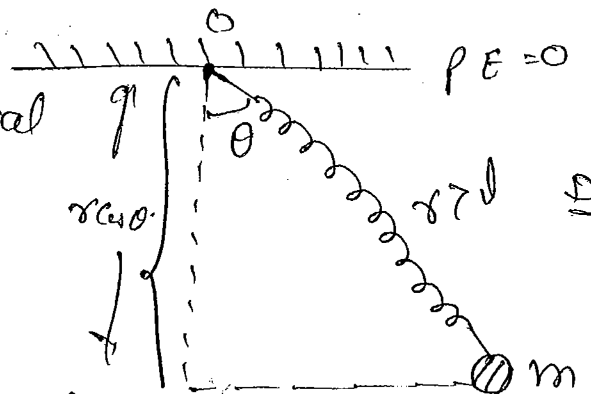
$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

Q. A simple pendulum consists of a small bob of mass 'm' suspended from light spring of ~~natural length 'l'~~ with spring constant 'k'. Write Lagrangian of the system and eqⁿ of motion using plane polar coordinate (r,θ). Natural length of spring is l.

Solⁿ Case :- string ⟶ spring

It is not a vertical oscillation it is a planar oscillation



DOF = 2

spring does not put any constraint.

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = mgr\cos\theta + \frac{1}{2} k (r-l)^2$$

$$L = T - V$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r-l)^2$$

Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$q_1 = r, \quad q_2 = \theta$$

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}, \quad \frac{\partial q_1}{\partial q_2} = 0$$

r - quation $\rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot r) - m r\dot\theta^2 - mg\cos\theta + K(r-l) = 0$$

$$\boxed{\ddot r - r\dot\theta^2 - g\cos\theta + \frac{k}{m}(r-l) = 0} \longrightarrow (*)$$

θ - quation $\rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(m r^2\dot\theta) + mgr\sin\theta = 0$$

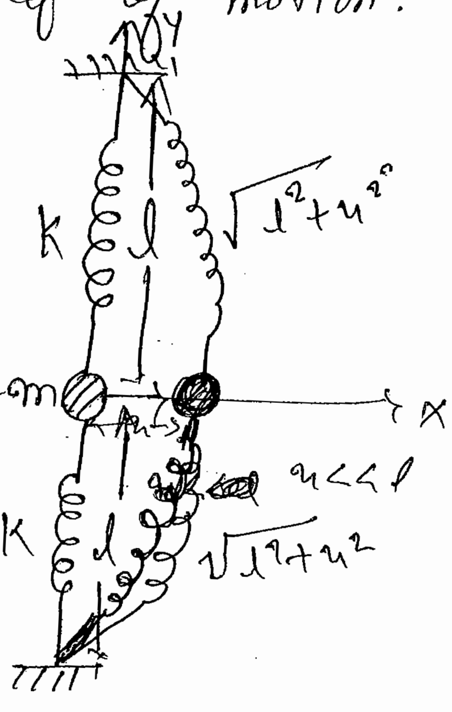
$$2r\dot r\dot\theta + r^2\ddot\theta + gr\sin\theta = 0$$

divide by r

$$\boxed{2\dot r\dot\theta + r\ddot\theta + g\sin\theta = 0} \longrightarrow (**)$$

NET-2013

Q. If the particle is slightly displaced along X-axis write its eqn of motion.



$\boxed{DOF = 1}$

Elongation $= \sqrt{l^2+x^2} - x$

$L = T - V$

$= \frac{1}{2}m\dot x^2 - \cancel{2}\cdot\frac{1}{\cancel{2}}K\left[\sqrt{l^2+x^2} - l\right]^2$

$$\boxed{L = \frac{1}{2}m\dot x^2 - K\left[\sqrt{l^2+x^2} - l\right]^2}$$

Equation of motion :- Here $\boxed{q = x}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}[m\dot{x}] + 2k\left(\sqrt{l^2+x^2} - l\right)\cdot \frac{(\cancel{2}x)}{\cancel{2}\sqrt{l^2+x^2}} = 0$$

$$\Rightarrow m\ddot{x} + 2k\left(\sqrt{l^2+x^2} - l\right)\cdot \frac{x}{\sqrt{l^2+x^2}} = 0$$

$$\Rightarrow m\ddot{x} + 2k\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right]x = 0$$

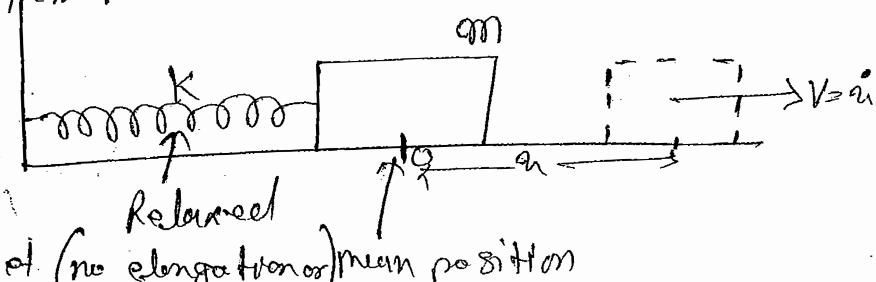
$$\Rightarrow \boxed{m\ddot{x} + 2kx\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right] = 0}$$

$$\Rightarrow m\ddot{x} + 2kx\left[1 - \left(1 + \frac{x^2}{l^2}\right)^{-1/2}\right] = 0 \qquad \left\{\because (1+x)^n = (1+nx) \text{ neglect higher term}\right.$$

$$\Rightarrow m\ddot{x} + 2kx\left[1 - \left(1 - \frac{x^2}{2l^2}\right)\right]$$

$$\Rightarrow \boxed{m\ddot{x} + \frac{kx^3}{l^2} = 0} \longleftarrow \text{Eq}^n \text{ of motion for small } x.$$

* Spring Mass System :-



Here elongation $= x$ because the spring is oscillating along parallel to its length.

So $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2}Kx^2$

x = displacement from mean position.

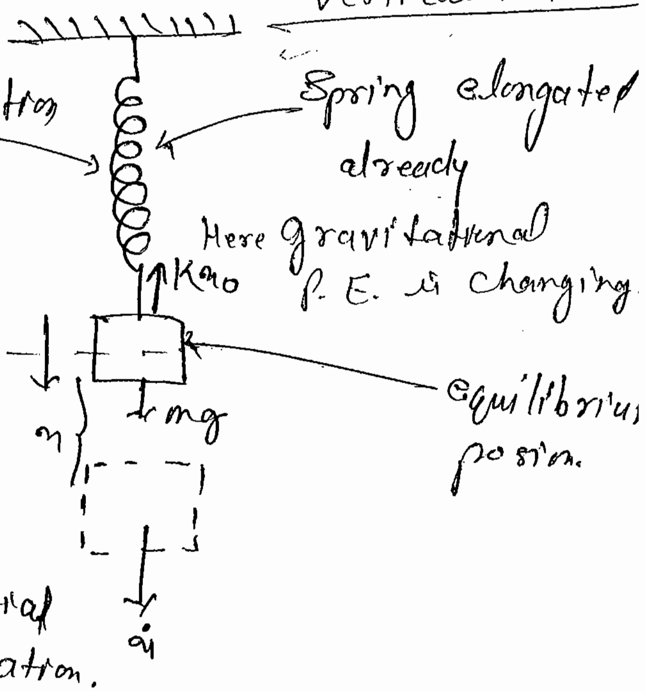
Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\boxed{m\ddot{x} + kx = 0}$$

Q.

Vertical motion



$L = T - V$

$T = \frac{1}{2} m\dot{x}^2$

$V = -mgx + \frac{1}{2}k(x_0 + x)^2$

initial elongation.

$$L = \frac{1}{2} m\dot{x}^2 - \left[-mgx + \frac{1}{2}k(x_0+x)^2\right]$$

$$\boxed{L = \frac{1}{2} m\dot{x}^2 + mgx - \frac{1}{2}k(x+x_0)^2}$$

Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - mg + k(x + x_0) = 0$$

$$\boxed{m\ddot{x} + kx - mg + kx_0 = 0} \quad \text{—— (1)}$$

At equilibrium position –

$$mg = Kx_0 \quad \text{—(II)}$$

from (I) and (II)

$$\boxed{m\ddot{x} + Kx = 0} \longrightarrow \text{Equation of motion at Equilibrium position.}$$

x is displacement from mean position.

<u>Note</u>: "For spring mass system if we can neglect initial elongation and gravitational potential energy then also we get correct equation of motion!" ✓

NET-2013

Q. A particle moving in a potential

$$V(x, y, z) = x^2 + y^2 + \frac{z^2}{2}$$

If L_x, L_y and L_z be component of angular momentum then which is constant or conserved.

(a) L_x (b) L_y (c) L_z (d) None of these.

Sol^n Let us write Lagrangian in spherical polar coordinate.

$$L = T - V$$

$$\boxed{L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - \left[r^2\sin^2\theta + \frac{r^2\cos^2\theta}{2}\right]}$$

$\because$ $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$\because$ ϕ is not coming in L so ϕ is cyclic

So p_ϕ is conserved. || $\boxed{L_z \text{ is conserved}}$ Ans

* Lagrangian of a system is -

$$\boxed{L(x,\dot{x}) = \int_0^{\dot{x}} e^{-\alpha^2}\, d\alpha + \int_0^{x} e^{\alpha^2}\, d\alpha}$$

find eqn of motion.

Soln Eqn of motion:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(e^{-\dot{x}^2}\right) - e^{x^2} = 0$$

$$e^{-\dot{x}^2}\cdot\left[-2\dot{x}\ddot{x}\right] - e^{x^2} = 0$$

$$\boxed{2\dot{x}\ddot{x}e^{-\dot{x}^2} + e^{x^2} = 0}$$

TIFR 2014

Q. If $f(x) = \int_0^{\infty} f(t)\, dt$ plot $f(x)$

$$\frac{df(x)}{dx} = f(x)$$

$$\int \frac{df(x)}{f(x)} = \int dx$$

$$\log f(x) = x + C$$

$$\boxed{f(x) = Ke^{x}}$$

* Lagrangian Dynamics:-

Spherical Polar Co-ordinate :-

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$

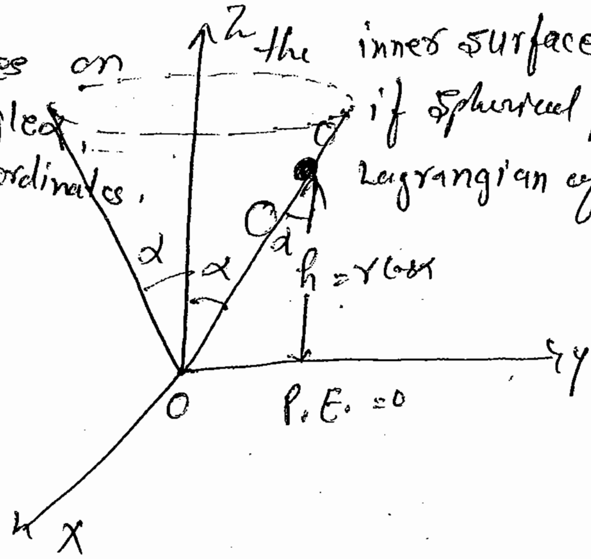
In Cylindical Co-ordinate -

$$T = \frac{1}{2} m (\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2)$$

Plane Polar :-

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

A-10
Q.9 A particle of mass 'm' moves on the inner surface of inverted cone of half vertex angle α, if spherical polar co-ordinate are taken as generalised co-ordinates, Lagrangian of the system particle is -



Spherical polar (r, θ, ϕ)

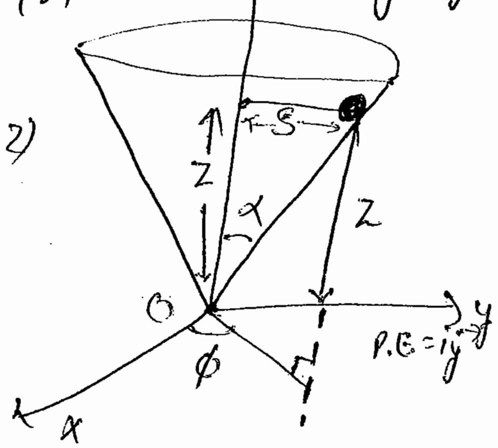
$\theta = \alpha$, $\phi = 0$

$\dot{\theta} = 0$

$L = T - V$

$$\boxed{L = \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2) - mgr\cos\alpha}$$

A-10
Q.10 In cylinderical co-ordinate System Lagrangian of bead in previous question is -



Cylindrical Coordinate (s, ϕ, z)

$\tan\alpha = \frac{s}{z}$

$s = z\tan\alpha$

$$\dot{s} = \dot{z}\tan\alpha$$

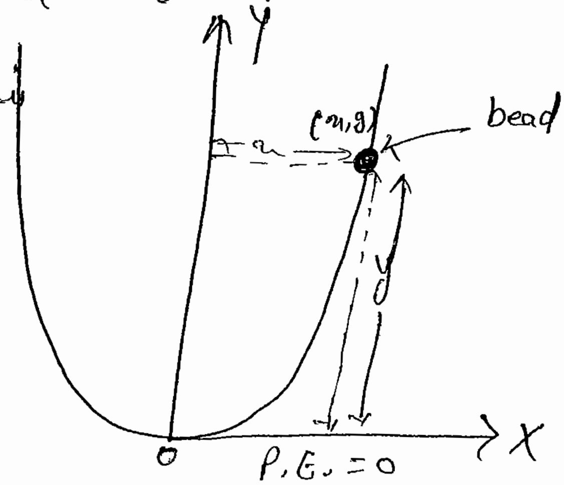
$$L = T - V$$

$$L = \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

$$\boxed{L = \frac{1}{2}m(\dot{z}^2\tan^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2 + \dot{z}^2) - mgz}$$

A-10

Q.1 A bead of mass 'm' slides along a wire kept in vertical pla[ne] as shown in fig. The eq^n of wire is $y = \alpha x^2$. Lagrangian of the bead is - ?
$y = \alpha x^2$



Lagrangian of bead

$$\therefore \dot{y} = 2\alpha x\dot{x}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \frac{1}{2}m(\dot{x}^2 + 4\alpha^2x^2\dot{x}^2) - mg\cdot\alpha x^2$$

$$\boxed{L = \frac{1}{2}m\dot{x}^2(1 + 4\alpha^2x^2) - mg\alpha x^2}$$

Write equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

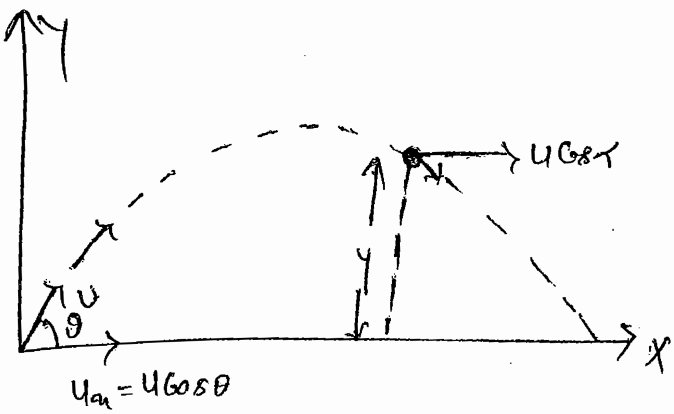
$$\frac{d}{dt}\left[m\dot{x}(1 + 4\alpha^2x^2)\right] - 4\alpha^2m\dot{x}^2x + 2mg\alpha x = 0$$

$$\ddot{x}(1 + 4\alpha^2x^2) + 8\alpha^2x\dot{x}^2 - 4\alpha^2\dot{x}^2x + 2mg\alpha x = 0$$

$$\boxed{\ddot{x}(1 + 4\alpha^2x^2) + 4\alpha^2x\dot{x}^2 + 2mg\alpha x = 0}$$

<u>Note</u> :- "Lagrangian is always written at any instant of time."
In Lagrangian initial velocity does not consider.

A-10
<u>Q.2</u>



$L = T - V$

$$\boxed{L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy}$$

Write eqⁿ of motion :-

x - equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - 0 = 0$$

$$\ddot{x} = 0$$

$$\boxed{\dot{x} = \text{Constant}}$$

y - equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$m\ddot{y} + mg = 0$$

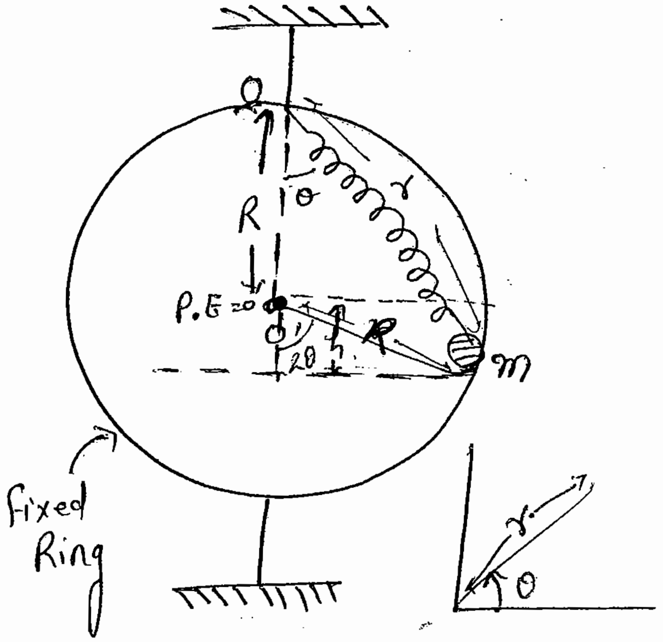
$$m(\ddot{y} + g) = 0$$

$$\ddot{y} + g = 0$$

$$\boxed{\ddot{y} = -g}$$

A-10

Q.4

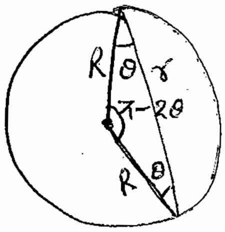


$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -mgh + \frac{1}{2}K(r - 2R)^2$$

Use ~~Sine~~ rule

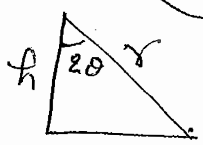
$$\frac{R}{\sin\theta} = \frac{r}{\sin(\pi - 2\theta)}$$



$$\frac{R}{\cancel{\sin\theta}} = \frac{r}{\sin 2\theta} = \frac{r}{2\,\cancel{\sin\theta}\cos\theta}$$

$$\therefore\ r = 2R\cos\theta$$

$$\dot{r} = 2R\sin\theta.\dot{\theta}$$



$$h = R\cos 2\theta$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgh - \frac{1}{2}K(r-2R)^2$$

$$= \frac{1}{2}m\left(4R^2\sin^2\theta.\dot{\theta}^2 + 4R^2\cos^2\theta.\dot{\theta}^2\right) + (mg.R\cos 2\theta) - \frac{1}{2}K(2R\cos\theta - 2R)^2$$

$$= \frac{1}{2}m\left(4R^2\dot{\theta}^2\right) + (mgR\cos 2\theta) - \frac{1}{2}K\left(4R^2\cos^2\theta + 4R^2 - 8R^2\cos\theta\right)$$

$$= \frac{1}{2}m(4R^2\dot{\theta}^2) + mgR\cos 2\theta - 2KR^2\cos^2\theta + 2R^2K + 4R^2K\cos\theta$$

$$= 2mR^2\dot{\theta}^2 + mgR\cos 2\theta - 2KR^2\cos^2\theta + 2R^2K + 4R^2K\cos\theta$$

$$\boxed{L = 2mR^2\dot{\theta}^2 - 2KR^2(1-\cos\theta)^2 + mgR\cos 2\theta}$$

Ans

Q.5 In previous question, equation of motion of bead for small value of θ is ?

~~(a)~~ $\ddot{\theta} + \frac{g}{R}\theta = 0$ (b) $\ddot{\theta} + \frac{2g}{R}\theta$

(c) $\ddot{\theta} + \frac{g}{2R}\theta = 0$ (d) $\ddot{\theta} + \frac{4g}{R}\theta = 0$

Sol^n

for small values of θ

$\cancel{\sin\theta} = \theta$

$\cos\theta = 1$

$$L = 2mR^2\dot{\theta}^2 - 2kR^2(1-\cos\theta)^2 + mgR\cos 2\theta$$

$$L = 2mR^2\dot{\theta}^2 + mgR\cos 2\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow 4mR^2\ddot{\theta} + mgR \cdot 2\sin 2\theta = 0$$

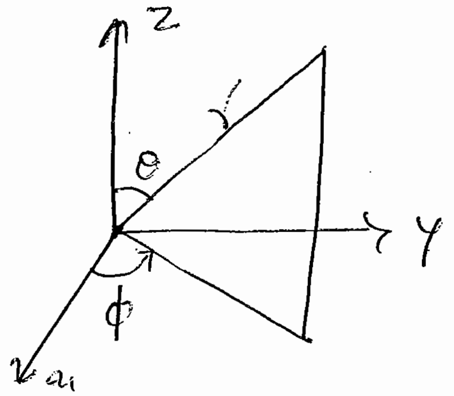
$$4mR^2\ddot{\theta} + 4mgR\theta = 0$$

$$4mR^2\left(\ddot{\theta} + \frac{g}{R}\theta\right) = 0$$

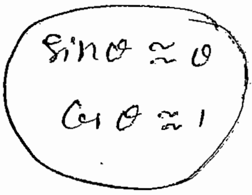
$$\boxed{\ddot{\theta} + \frac{g}{R}\theta = 0}$$

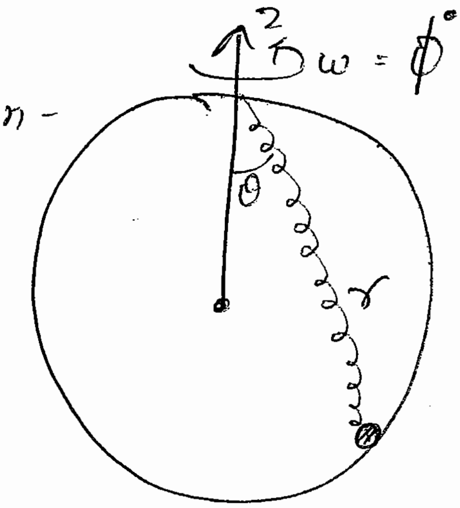
Ans

* If the ring is rotated with angular speed ω as shown :-



Write Lagrangian -





$p_\phi = L_z$

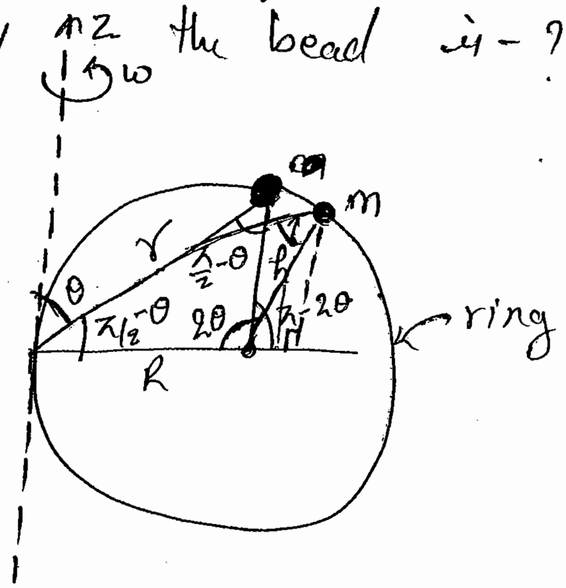
ϕ represents rotation about z axis (on spherical & cylindrical coordinate)

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2\right)$$

$$\boxed{T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \omega^2\right)}$$

A-10

Q.11 A bead of mass 'm' is sliding on a vertical circular loop of radius R. The loop is rotated with constant angular velocity ω about a tangential axis shown in figure. Lagrangian of the bead is - ?



Sol^n

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right)$$

$$V = mgh = mgR \sin(\pi - 2\theta)$$

$$V = mgR \sin 2\theta$$

Use sine Rule

$$\frac{R}{\sin(\frac{\pi}{2} - \theta)} = \frac{r}{\sin 2\theta}$$

$$\boxed{r = 2R \sin\theta}$$

$$\dot{r} = 2R \cos\theta\, \dot{\theta}$$

$$\dot{\phi} = \omega,$$

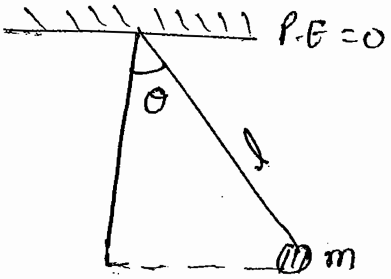
$$T = \frac{1}{2} m \left(4R^2 \cos^2\theta\, \dot{\theta}^2 + 4R^2 \sin^2\theta\, \dot{\theta}^2 + 4R^2 \sin^4\theta\, \omega^2\right)$$

$$T = \frac{1}{2} m \left(4R^2 \dot{\theta}^2 + 4R^2 \sin^4\theta\, \omega^2\right)$$

$$L = T - V$$

* Simple Pendulum :-

Oscillations is in plane. $r = l$, $\dot{r} = 0$



$$\boxed{L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta}$$

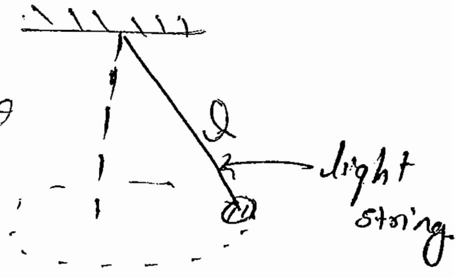
* Spherical Pendulum :-

Plane of oscillation is not fixed (there is oscillation about z-axis)

Use spherical polar,

~~$r = l$, θ~~ $\dot{r} = 0$

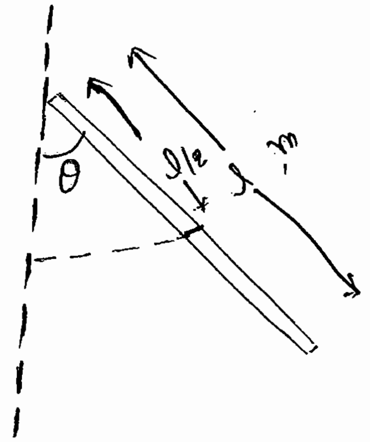
$$L = \frac{1}{2} m \cancel{l^2} \left(\dot{\theta}^2 l^2 + l^2 \sin^2\theta \, \dot{\phi}^2\right) + mgl \cos\theta$$



$$\boxed{L = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) + mgl \cos\theta}$$

* A thin rod suspended from one end :

$$L = \frac{1}{2}\left(\frac{ml^2}{3}\right)(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) + mg\frac{l}{2}\cos\theta$$



$$\boxed{L = \frac{-ml^2}{6}(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) + \frac{mgl\cos\theta}{2}}$$

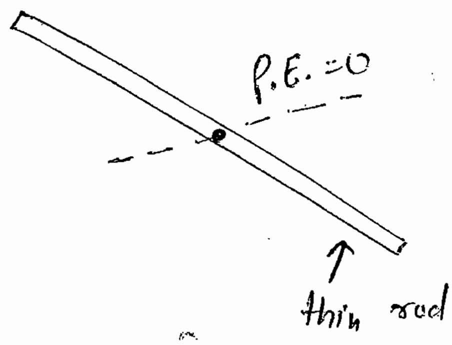
* A rod is suspended from middle (free to rotate in any orientation) :

Centre of is not moving.

$$K = \frac{1}{2} I \omega^2$$

$$K = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

$$L = \frac{1}{2}\left(\frac{ml^2}{12}\right)\left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right)$$



$$\boxed{L = \frac{ml^2}{24}(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2)}$$

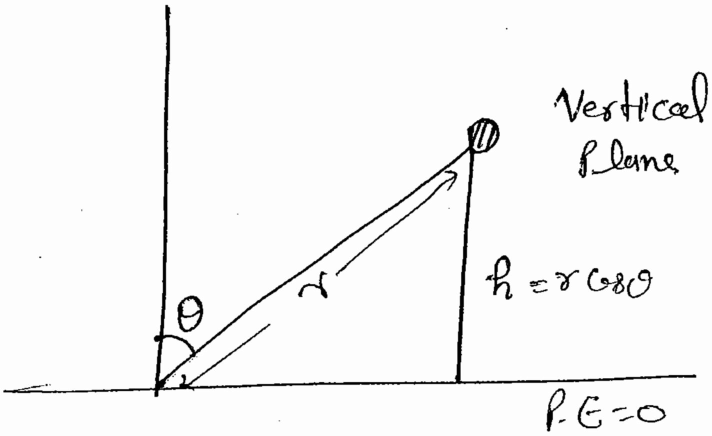
A-10

Q.12

Sol^n

∵ Here θ and h both variable

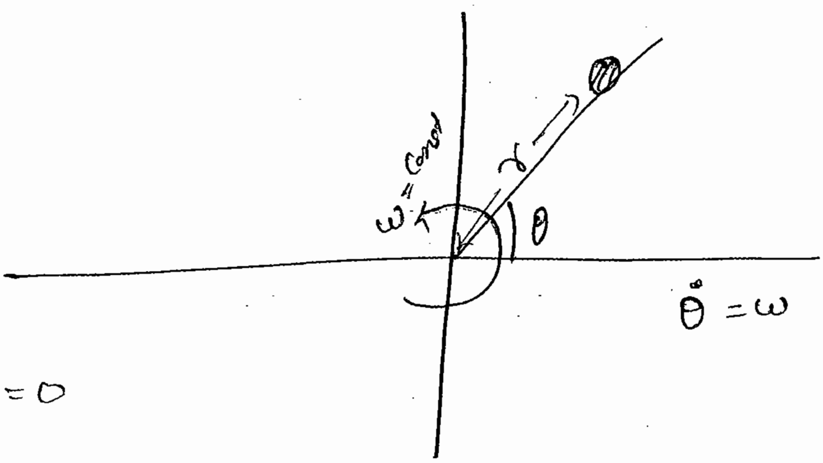
∵ It is plane polar co-ordinate



So $\boxed{L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta.}$

A-10
Q. 13

Sol^n



$L = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2)$

r - eq^n

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} - mr\omega^2 = 0$$

$$\ddot{r} - r\omega^2 = 0$$

$$\frac{d^2r}{dt^2} - \omega^2 r = 0$$

$$(D^2 - \omega^2) r = 0$$

roots $= \pm\omega$

$\boxed{r = Ae^{\omega t} + Be^{-\omega t}}$ Ans

A-10 F.

Q.25 Lagrangian of a system is $L = a\dot{x}^2 - bx^2$ then

(a) $x = c_1 t + c_2 t^2 + c_3$ (b) $x = c_1 e^{-c_2 t} + c_3$

(c) $x = c_1 \sin(c_2 t + c_3)$ (d) $x = c_1 e^{-c_2 t} \sin(c_2 t + c_3)$

Sol^n

Equation of motion -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$2a\ddot{x} + 2bx = 0$$

$$\ddot{x} + \frac{b}{a}x = 0$$

$$\boxed{\text{roots} = \pm i\sqrt{\frac{b}{a}}}$$ Purely imaginary.

$$\boxed{x = c_1 \sin(c_2 t + c_3)}$$

B-A-2

Q.47 NET 2012 Dec.

Sol^n

$$L = \frac{1}{2}m\dot{x}^2 - bx$$

Equation of motion :-

$$m\ddot{x} + b = 0$$

$$\ddot{x} = -\frac{b}{m}$$

$$\frac{d^2x}{dt^2} = -\frac{b}{m}$$

Integrate -

$$\frac{dx}{dt} = -\frac{b}{m}t + c_1$$

$$\boxed{x = -\frac{b}{m}t^2 + c_1 t + c_2}$$

Ans

Imp. #10

Q.24 A planet of mass 'm' revolves around the Sun of mass M in an elliptical orbit. If motion of planet is confined in one plane, Lagrangian of the system is -?

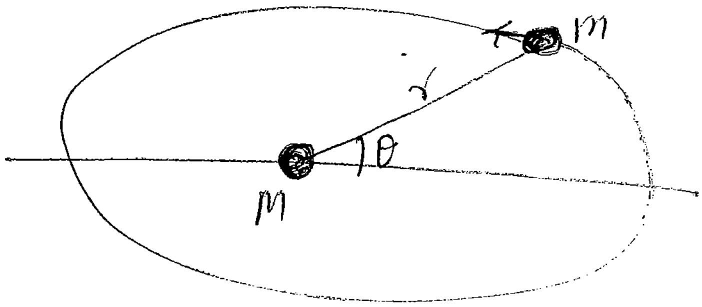
(a) $\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}$ (b)

(c) (d)

Sol^n

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -\frac{GMm}{r}$$



$$L = T - V$$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}}$$

Q.22

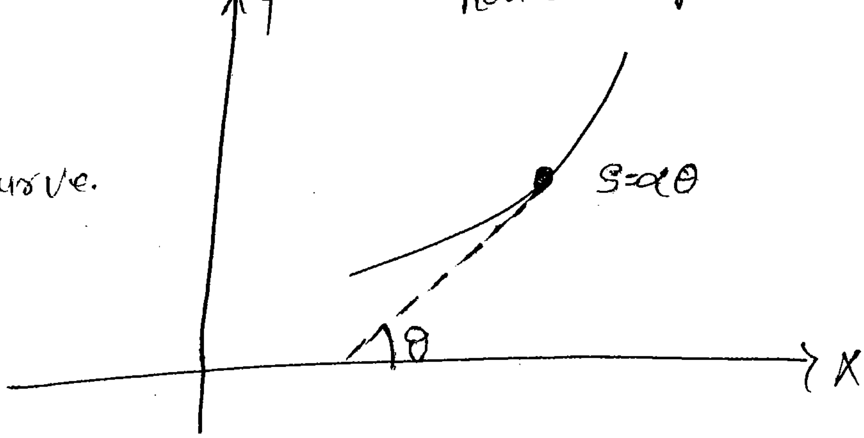
Sol^n Equation of curve

s = dist along curve.

$$L = \frac{1}{2}mv^2 \quad -0$$

$$= \frac{1}{2}m\left(\frac{ds}{dt}\right)^2$$

horizontal plane.

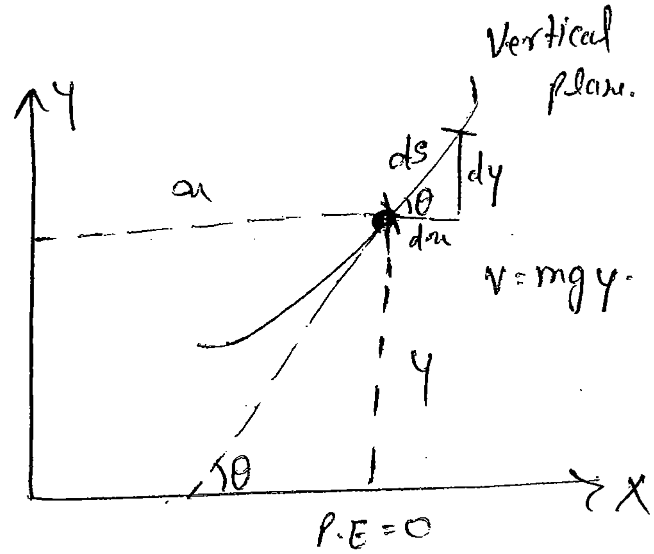


$\therefore \; S = \alpha\theta$

$$\boxed{L = \frac{1}{2} m \alpha^2 \dot{\theta}^2} \qquad \frac{ds}{dt} = \alpha\dot{\theta}$$

Q. In previous question plane is vertical write lagrangian in θ co-ordinate.

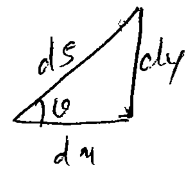
Sol^n



$L = T - V$ $\qquad V = mgy$

$L = \frac{1}{2} m \alpha^2 \dot{\theta}^2 + mg\alpha\cos\theta$

$\sin\theta = \frac{dy}{ds}$



$S = \alpha\theta$

$ds = \alpha d\theta$

$\sin\theta = \frac{dy}{\alpha d\theta}$

$\int dy = \int \alpha \sin\theta \, d\theta$

$\boxed{y = -\alpha\cos\theta}$

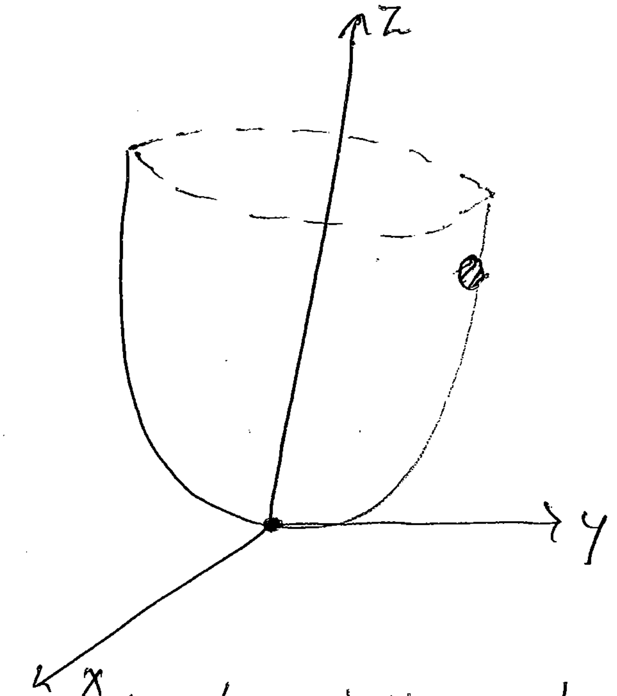
NET-2011

Q.38 A particle of mass 'm' moves inside a bowl. If the surface of the bowl is given by the equation $z = \frac{1}{2} a(x^2+y^2)$, where 'a' is a constant, the Lagrangian of the particle is.

Sol^n

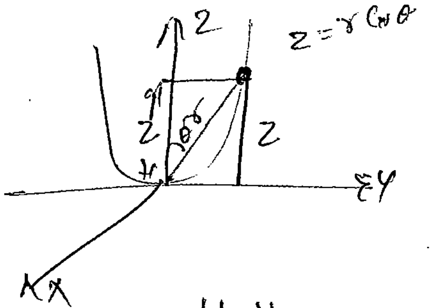
$$\boxed{Z = \frac{1}{2} a (x^2 + y^2)}$$

a = Constant.



He co-ordinate is not given so we check which co-ordinate system we use -

When we ~~see~~ option we confused whether it is spherical ~~pla~~ polar of cylindrical ~~po~~ co-ordinate.
So we check potential energy in both region so identify the co-ordinate system.



$V = mgZ$

$\boxed{V = mgr\cos\theta}$ not match

Whether we check $z = \frac{1}{2} a (x^2 + y^2)$ given

So $z = \frac{1}{2} a r^2 \sin^2\theta$ where $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$

which is also mismatch.

So we check in cylindrical co-ordinate:-

$V = mgZ$

$x = r\cos\phi$

$y = r\sin\phi$

$z = z$

So $\boxed{V = mg\frac{1}{2}ar^2}$ match in 3 options.

Here options are independent of z so we have to remove z.

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right)$$

$$z = \frac{1}{2} a r^2$$

$$\dot{z} = a r \dot{r}$$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\phi}^2 + a^2 r^2 \dot{r}^2\right)$$

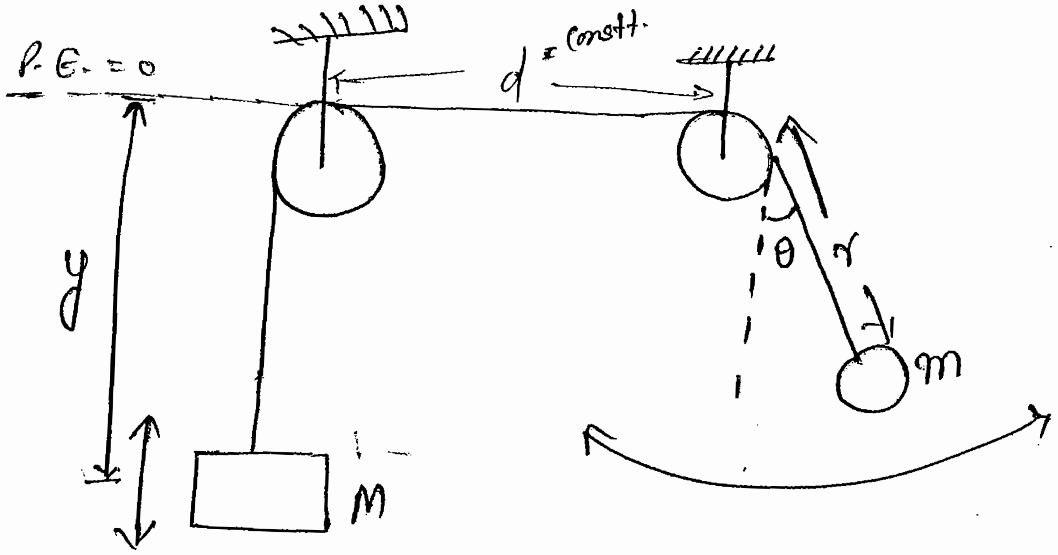
$$\boxed{T = \frac{1}{2} m \left(\dot{r}^2(1 + a^2 r^2) + r^2\dot{\phi}^2\right)}$$

$$L = T - V$$

$$L = \frac{1}{2} m \left(\dot{r}^2(1 + a^2 r^2) + r^2\dot{\phi}^2\right) - \frac{1}{2} m g a r^2$$

$$\boxed{L = \frac{1}{2} m \left(\dot{r}^2(1 + a^2 r^2) + r^2\dot{\phi}^2 - g a r^2\right)}$$ Ans

Q.



Write the eqn of motion of the two blogs.

Soln. DOF = 2.

Length of string is constant. if it is inextensible

$y + d + r =$ Constant $=$ length of string $= l$.

$$\dot{y} + 0 + \dot{r} = 0$$

$$\boxed{\dot{y} = -\dot{r}}$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m \dot{y}^2$$

$$\boxed{T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m \dot{r}^2}$$

$$V = -Mgy - mgr\cos\theta$$

$$\boxed{V = -Mg(l-d-r) - mgr\cos\theta.}$$

$$\boxed{L = \frac{1}{2}\left[(m+M)\dot{r}^2 + mr^2\dot{\theta}^2\right] + Mg(l-d-r) + mgr\cos\theta.}$$

Eq^n of motion -

r - eq^n :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

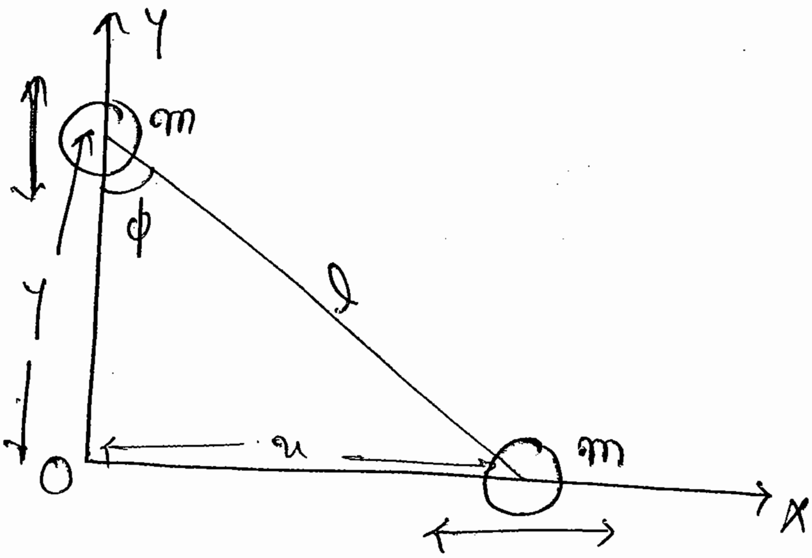
$$\boxed{(m+M)\ddot{r} + Mg - Mg\cos\theta \; \cancel{m} \; mr\dot{\theta}^2 = 0}$$ eq^n of motion.

$\cancel{\ddot{r} = g(m\cos\theta - M)}$

Q. Two particle connected by a light rod. Particles are constrained to move along two ⊥^{or} lines. If ϕ is generalied co-ordinate What is K.E. of the system.

Sol^n

$$\boxed{DOF = 1}$$



$$K.E. = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

$$K.E. = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

from Δ

$$x = l \sin\phi \Rightarrow \dot{x} = l \cos\phi \, \dot{\phi}$$

$$y = l \cos\phi \Rightarrow \dot{y} = -l \sin\phi \, \dot{\phi}$$

So $$\boxed{K.E. = \frac{1}{2} m l^2 \dot{\phi}^2}$$

* Small Oscillation :-

Normal Modes :

It is a special type of oscillation in which all particles of system oscillates with same frequency.

Finding frequency of Normal modes :-

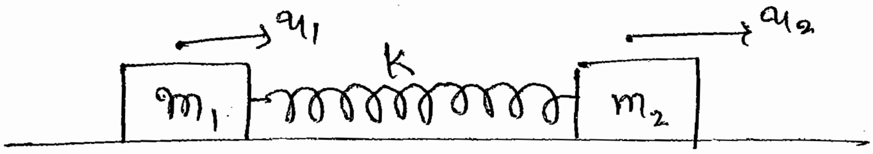
It is obtained by solving following matrix equation or determinant.

$$\left| \hat{V} - \omega^2 \hat{T} \right| = 0$$

where $\hat{V}$ is a matrix obtained from ~~kinetic~~ potential energy.
$\hat{T}$ " " " " kinetic "

To obtain $\hat{V}$ we take cofficients of q_i, q_j in P.E.
" $\hat{T}$ " " " K.E.

Q. Two masses connected by a light spring. masses are free to oscillate.



find frequency (angular) of Normal modes?

Sol^n DOF = 2.

Let x_1 and x_2 be displacement of m_1 and m_2 from their mean position

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2}\left(m_1 \dot{x}_1 \dot{x}_1 + m_2 \dot{x}_2 \dot{x}_2\right)$$

$\therefore$ $$\hat{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$ Here $\frac{1}{2}$ is not consider becoz it is canceled out finally.

$$V = \frac{1}{2} K (x_2 - x_1)^2 = \frac{1}{2} k (x_1^2 + x_2^2 - 2 x_1 x_2)$$

$$V = \frac{1}{2}\left(k x_1^2 + K x_2^2 - K x_1 x_2 - K x_2 x_1\right)$$

$\therefore$ $$\hat{V} = \begin{pmatrix} K & -k \\ -K & k \end{pmatrix}$$

Secular Equation :-

$$|\hat{V} - \omega^2 \hat{T}|$$

$$\begin{vmatrix} K - m_1\omega^2 & -K \\ -K & K - m_2\omega^2 \end{vmatrix} = 0$$

$$(K - m_1\omega^2)(K - m_2\omega^2) - K^2 = 0$$

$$\cancel{K^2} - K(m_1+m_2)\omega^2 + m_1 m_2 \omega^4 - \cancel{K^2} = 0$$

$$\omega^2 \left[-K(m_1+m_2) + m_1 m_2 \omega^2 \right] = 0$$

$$\omega^2 = 0 \longrightarrow \omega = 0$$

$$\omega^2 = \frac{K(m_1+m_2)}{m_1 m_2}$$

$$\omega = \sqrt{\frac{K}{\left(\frac{m_1 m_2}{(m_1+m_2)}\right)}}$$

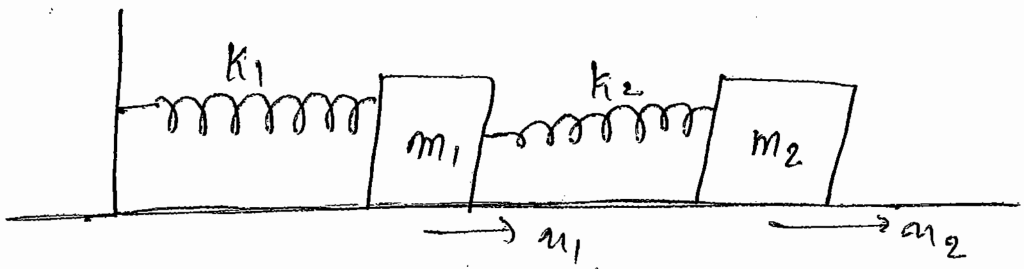
Here -ve term is not consider becoz freq. is never negative

$$\boxed{\omega = \sqrt{\frac{K}{\mu}}}$$

Where $\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$

Note :- If system is not rigidly fixed then one of freq. comes out to be zero.

Q.



If x_1 and x_2 are displacement from mean positions.

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$$

$$\hat{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

<u>Note</u> :- If x and θ are involved in a question then to calculate ω we will consider $(l\theta)$ and x as two coordinates [$\because$ so that the two co-ordinates so that the two co-ordinates have same dimension.

*** Approximation :-**

$$mgl\cos\theta \simeq mgl\left(1 - \frac{\theta^2}{2}\right)$$

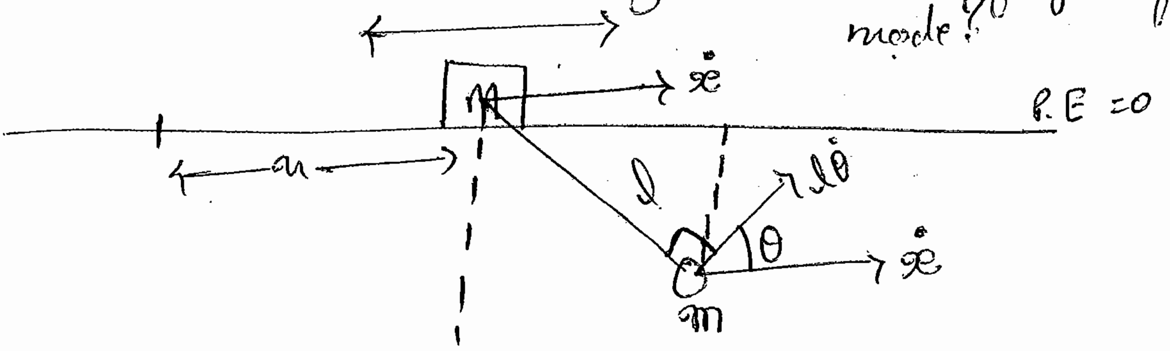
$$\dot{x}\dot{\theta}\cos\theta \simeq \dot{x}\dot{\theta}\left(1 - \frac{\theta^2}{2}\right)$$

$$\simeq \dot{x}\dot{\theta} - \dot{x}\dot{\theta}\frac{\dot{\theta}^2}{2} \rightarrow \text{very small} \approx 0$$

for small oscillat[ions]

$$\simeq \dot{x}\dot{\theta}$$

<u>Q</u> Block can move forward and backward and pendulum is oscillating. What is freq. of normal mode?



$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left[\dot{x}^2 + (l\dot{\theta})^2 + 2\dot{x}(l\dot{\theta})\cos\theta\right]$$

for small oscillations

$$T \simeq \frac{1}{2}\left[M\dot{x}^2 + m\left[\dot{x}^2 + (l\dot{\theta})^2 + 2\dot{x}(l\dot{\theta})\right]\right]$$

$x \rightarrow$ first coordinate

$l\theta \rightarrow$ second co-ordinate.

$$\hat{T} = \begin{pmatrix} M+m & m \\ m & m \end{pmatrix}$$

$$V = -mgl\cos\theta$$

$$= -mgl\left(1 - \frac{\theta^2}{2}\right) = -mgl + \frac{1}{2} mgl\,\theta^2$$

$$= -mgl + \frac{1}{2} \frac{mg}{l} (l\theta)^2$$

So $\hat{V} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{mg}{l} \end{pmatrix}$

So secular equation -

$$\left|\hat{V} - \omega^2 \hat{T}\right| = 0$$

$$\left|\omega^2 \hat{T} - \hat{V}\right| = 0$$

$$\begin{vmatrix} \omega^2(M+m) & m\omega^2 \\ m\omega^2 & m\omega^2 + \frac{mg}{l} \end{vmatrix} = 0$$

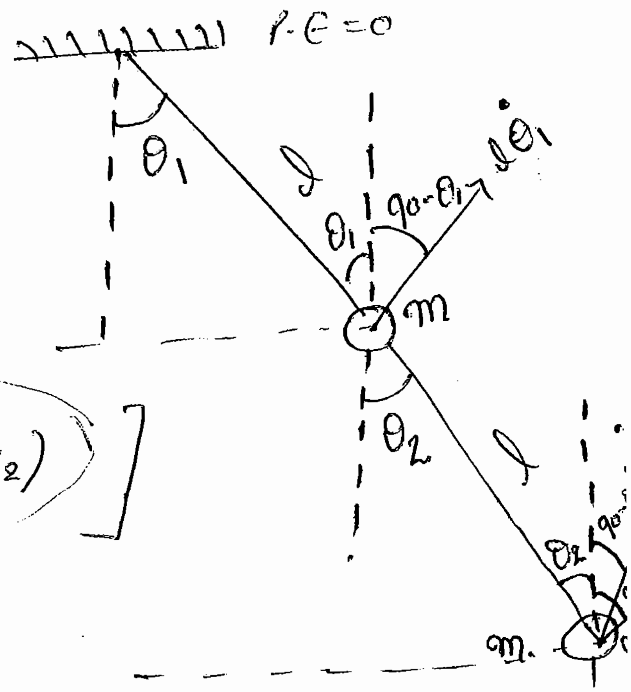
$$\omega^2(M+m)\cancel{m}\left(\omega^2 - \frac{g}{l}\right) - m^2\omega^4 = 0$$

$$\omega^2\left[(M+m)\left(\omega^2 - \frac{g}{l}\right) - m\omega^2\right] = 0$$

$$\Rightarrow \boxed{\begin{array}{l} \omega = 0 \\ \omega = \sqrt{\frac{g}{l}\left(1 + \frac{m}{M}\right)} \end{array}}$$

If $M \to \infty$ then we get result of simple pendulum.

Q. Write $\hat{T}$ and $\hat{V}$.



$$T = \frac{1}{2}m(l\dot{\theta}_1)^2 + \frac{1}{2}m\Big[(l\dot{\theta}_1)^2 + (l\dot{\theta}_2)^2 + 2(l\dot{\theta}_1)(l\dot{\theta}_2)\underbrace{\cos(\theta_1 - \theta_2)}_{\Downarrow 1}\Big]$$

$$\hat{T} = \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix}$$

θ_1 = first Co-ordinate

θ_2 = Second Co-ordinate.

$$V = -mgl\cos\theta_1 - mgl(\cos\theta_1 + \cos\theta_2)$$

$$= -mgl\,[2\cos\theta_1 + \cos\theta_2]$$

$$= -mgl\left[2\left(1 - \frac{\theta_1^2}{2}\right) + \left(1 - \frac{\theta_2^2}{2}\right)\right]$$

$$V = -3mgl + \frac{1}{2}\left[2mgl\,\theta_1^2 + mgl\,\theta_2^2\right]$$

$$\hat{V} = \begin{pmatrix} 2ml & 0 \\ 0 & mgl \end{pmatrix}$$

Hamiltonian Dynamics

A transition from Lagrangian function to Hamiltonian function.

$$L \xrightarrow[\text{Transformation}]{\text{Legendre}} H$$

$$\boxed{H(q_i, p_i) = p_i \dot{q}_i - L(q_i, \dot{q}_i)}$$

Hamiltonian.

In hamiltonian dynamics q_i & p_i are taken as independent variable.

$$\frac{\partial p_i}{\partial q_i} = 0 \quad , \quad \frac{\partial q_i}{\partial p_i} = 0$$

$\dfrac{\partial \dot{q}_i}{\partial q_i}$ may not be zero

$\dfrac{\partial \dot{q}_i}{\partial p_i}$ may not be zero.

$$\frac{\partial p_i}{\partial p_j} = \delta_{ij} \quad , \quad \frac{\partial q_i}{\partial q_j} = \delta_{ij}$$

* Hamilton's Equation of motion:-

It is observed from -

1. Legendre Transform
2. Variational principle.

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}}$$

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

* Dynamical Variable :- A function of any quantity which is function of q_i, p_i and t. If A is dynamical variable, Then -

$$A = A(q_i, p_i, t)$$

* Possion's Equations of Motion :- It gives the rate of change of a dynamical variable.

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial t}$$

$$= \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial A}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right) + \frac{\partial A}{\partial t}$$

$$= \underbrace{\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}}_{\text{Possion Bracket of } A \text{ with } H.} + \frac{\partial A}{\partial t}$$

$$\boxed{\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}}$$

This is possion's eqn of motion.

If A does not explicitly depends on time t.

$$\frac{\partial A}{\partial t} = 0$$

$$\frac{dA}{dt} = [A, H]$$

$$\boxed{\dot{A} = [A, H]}$$

If A is constant

$$\dot{A} = 0$$

$$\boxed{[A, H] = 0}$$

If a quantity is constant then its possion's Bracket with H is zero.

* Hamilton's Equation in terms of possion's bracket:-

$$\boxed{\begin{aligned} \dot{q}_i &= [q_i, H] \\ \dot{p}_i &= [p_i, H] \end{aligned}}$$

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or

$$\boxed{\begin{aligned} [q_i, H] &= \frac{\partial H}{\partial p_i} \\ [p_i, H] &= -\frac{\partial H}{\partial q_i} \end{aligned}}$$

* Conversion of Lagrangian (L) into Hamiltonian (H):

Steps :-

(I) Use $p_i = \frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}$ and find $\dot{q}_i$

(II) Use $H = p_i \dot{q}_i - L$ and put the value of $\dot{q}_i$

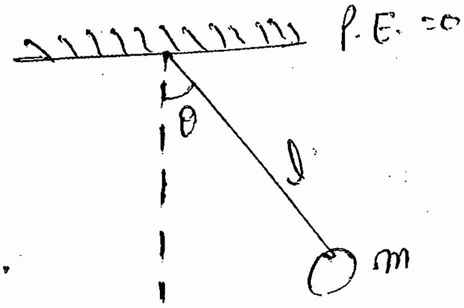
* Conversion of Hamiltonian (H) into Lagrangian (L) :-

Steps :-

(1) Use $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and find p_i

(2) Use $L = p_i \dot{q}_i - H$ and put value of p_i

Simple Pendulum



$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$

Suppose it is given find H.

Sol^n

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

$$H = p_\theta \dot{\theta} - L$$

So $H = p_\theta \dot{\theta} - \frac{1}{2} ml^2 \dot{\theta}^2 - mgl\cos\theta$

put $\dot{\theta}$

$$H = \frac{p_\theta^2}{ml^2} - \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta$$

$$\boxed{H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta}$$

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Q. $L = \frac{1}{2} m\dot{q}^2 - \frac{\lambda}{2} q\dot{q}^2$ find H?

Soln

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$p = m\dot{q} - \lambda q\dot{q} = \dot{q}(m - \lambda q)$$

$$\boxed{\dot{q} = \frac{p}{m - \lambda q}}$$

$$H = p\dot{q} - L = p\dot{q} - \frac{1}{2} m\dot{q}^2 + \frac{\lambda}{2} q\dot{q}^2$$

put $\dot{q}$

$$H = \frac{p^2}{m - \lambda q} - \frac{1}{2}\dot{q}^2(m - \lambda q)$$

$$= \frac{p^2}{m - \lambda q} - \frac{1}{2}\frac{p^2}{(m - \lambda q)^2}\cancel{(m - \lambda q)}$$

$$\boxed{H = \frac{p^2}{2(m - \lambda q)}}$$

Q. $L = \frac{1}{2}mv^2 + \vec{a}\cdot\vec{V}$. where $\vec{V}$ = velocity
$\vec{a}$ = Constant vector.
find $H = ?$

Sol^n Some mathematical methods -

$$\vec{a} \longleftrightarrow a_i$$

$$\vec{a}\cdot\vec{b} \longleftrightarrow a_i b_i$$ {when repetation of index then we assume Sumation}

$$\vec{a}^2 \longleftrightarrow a_i^2$$

Summation over repeated index (i) is assumed.

$$(\vec{a}\times\vec{b})_i = \underbrace{\varepsilon_{ijk}}_{\text{Levicivita Tensor}} a_j b_k$$

$$\varepsilon_{ijk} = \hat{i}\cdot(\hat{j}\times\hat{k})$$

if two index are equal then $\varepsilon = 0$

$$\varepsilon_{iik} = \hat{i}\cdot(\hat{i}\times\hat{k}) = 0$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$\varepsilon_{ijj} = 0$$

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

Now come in question -

$$L = \frac{1}{2}mv_i^2 + a_i v_i$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v_i}$$

generalised velocity

$$p_i = m v_i + a_i$$

$$v_i = \frac{p_i - a_i}{m}$$

$$H = p_i \dot{q}_i - L$$

$$= p_i v_i - \frac{1}{2} m v_i^2 - a_i v_i$$

$$H = p_i \left(\frac{p_i - a_i}{m}\right) - \frac{1}{2} \frac{(p_i - a_i)^2}{m} - \frac{a_i (p_i - a_i)}{m}$$

$$= \frac{(p_i - a_i)}{m} (p_i - a_i) - \frac{(p_i - a_i)^2}{2m}$$

$$= \frac{(p_i - a_i)^2}{2m}$$

$$\boxed{H = \frac{(\vec{p} - \vec{a})^2}{2m}}$$

* Some Standard Lagrangian and Corresponding Hamiltonian :-

① Free Particle ($V = 0$)

(i) Non- Relativistic :-

$$L = \frac{1}{2} m v^2$$

$$H = \frac{p^2}{2m}$$

(ii) Relativistic Case :-

$$L = - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}$$ ← It gives correct expression for momentum.

$$H = \sqrt{p^2 c^2 + m_0^2 c^4}$$

② Particle moving in electromagnetic field (ϕ, $\vec{A}$)

ϕ ↑ Electric Potential; $\vec{A}$ ↑ mag. Vector poten.

(i) Non - Relativistic :-

$$L = \frac{1}{2} m v^2 - q\phi + q \vec{A} \cdot \vec{v}$$

$$H = \frac{(\vec{P} - q\vec{A})^2}{2m} + q\phi$$

$$P.E(V) = \underbrace{q\phi - q\vec{A} \cdot \vec{v}}$$

{ q = charge. }

↓ It gives correct Lorentz force.

(ii) Relativistic Case :-

$$L = - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - q\phi + q\vec{A} \cdot \vec{v}$$

$$H = \sqrt{(\vec{P} - q\vec{A})^2 c^2 + m_0^2 c^4} + q\phi$$

$$L = T - V$$

$$L = mc^2 - m_0c^2$$

$$\times \quad L = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 \longrightarrow$$ We can not write Lagrangian because it can not give correct expression for momentum.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v} = \frac{m_0 v^2}{(1 - v^2/c^2)^{3/2}}$$ but this is not correct.

Correct expression for momentum.

$$p = mv. \ \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

but $L = -m_0 c^2 \sqrt{1 - v^2/c^2}$

$$\frac{\partial L}{\partial v} = p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$ This is the correct expression of Lagrangian.

*** Conversion of Lagrangian into Hamiltonian for relativistic case :-**

$$L = -m_0 c^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

$$L = -m_0 c^2 \sqrt{1 - \frac{v_i^2}{c^2}} \qquad \{\because \vec{a} \rightarrow a_i\}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v_i}$$

$$p_i = \frac{-m_0 \cancel{c^2}}{\cancel{2}\sqrt{1 - \frac{v_i^2}{c^2}}} \left(-\frac{\cancel{2} v_i}{\cancel{c^2}}\right)$$

$$p_i = \frac{m_0 v_i}{\sqrt{1 - \frac{v_i^2}{c^2}}}$$

Now squaring both side

$$p_i^2 \left(1 - \frac{v_i^2}{c^2}\right) = m_0^2 v_i^2$$

$$p_i^2 = v_i^2 \left[m_0^2 + \frac{p_i^2}{c^2} \right]$$

$$v_i^2 = \frac{p_i^2 c^2}{m_0^2 c^2 + p_i^2}$$

$$H = p_i v_i - L$$

$$= p_i v_i + m_0 c^2 \sqrt{1 - \frac{v_i^2}{c^2}}$$

$$= \frac{p_i p_i c}{\sqrt{m_0^2 c^2 + p_i^2}} + m_0 c^2 \sqrt{1 - \frac{p_i^2}{m_0^2 c^2 + p_i^2}}$$

$$= \frac{p_i^2 c}{\sqrt{m_0^2 c^2 + p_i^2}} + \frac{m_0 c^2 \, m_0 c}{\sqrt{m_0^2 c^2 + p_i^2}}$$

$$= \frac{c \, (p_i^2 + m_0^2 c^2)}{\sqrt{p_i^2 + m_0^2 c^2}} = c \sqrt{p_i^2 + m_0^2 c^2}$$

$$= \sqrt{p_i^2 c^2 + m_0^2 c^4}$$

So $\boxed{H = \sqrt{p^2 c^2 + m_0^2 c^4}}$

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Q. If $L = -\sqrt{1-\dot{q}^2} - V(q)$ find $H = ?$

Sol^n

$$\boxed{H = \sqrt{p_q^2 + 1} + V(q)}$$

Standared form. { here $m_0 = 1$, $c = 1$

* Question on $H \longrightarrow L$:-

Q. $H = \frac{p^2}{2m} + \vec{a}\cdot\vec{p}$ find Lagrangian ?

Sol^n

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \Rightarrow \because H = \frac{p_i^2}{2m} + a_i p_i$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = v_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} + a_i$$

$$p_i = (v_i - a_i)m$$

$$L = p_i\dot{q}_i - H = p_i\dot{q}_i - \frac{p_i^2}{2m} - a_i p_i$$

put p_i

$$L = (v_i - a_i)m v_i - \frac{m}{2}(v_i - a_i)^2 - a_i m (v_i - a_i)$$

$$= m(v_i - a_i)(v_i - a_i) - \frac{m}{2}(v_i - a_i)^2$$

$$= m(v_i - a_i)^2 - \frac{m}{2}(v_i - a_i)^2$$

$$= \frac{1}{2}m(v_i - a_i)^2$$

So $$\boxed{L = \frac{1}{2}m(\vec{v} - \vec{a})^2}$$

Ans

Q. Hamiltonian of the system is $H = ap_r^2 + \frac{bp_\theta^2}{r^2} + c\cos\theta$

Where a, b, c are constants find lagrangian?

Sol^n Here two co-ordinates are used which is r and θ. So velocity corresponding to r is $\dot{r}$ and corresponding to θ is $\dot{\theta}$.

$$q_i = r, \theta$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = 2ap_r$$

$$\boxed{p_r = \frac{\dot{r}}{2a}}$$

Now

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2bp_\theta}{r^2} \Rightarrow \boxed{p_\theta = \frac{r^2\dot{\theta}}{2b}}$$

$$L = p_i\dot{q}_i - H$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - ap_r^2 - \frac{bp_\theta^2}{r^2} - c\cos\theta$$

Put p_r and p_θ

$$L = \frac{\dot{r}^2}{2a} + \frac{r^2\dot{\theta}^2}{2b} - \frac{\dot{r}^2}{4a} - \frac{r^2\dot{\theta}^2}{4b} - c\cos\theta$$

$$\boxed{L = \frac{\dot{r}^2}{4a} + \frac{r^2\dot{\theta}^2}{4b} - c\cos\theta}$$

* Lagrangian and Hamiltonian of a particle in different co-ordinate system:-

(I) Plane Polar (r, θ) :-

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - \text{P.E. term}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \text{P.E. term.}$$

(II) Spherical Polar (r, θ, ϕ) :-

$$L = \frac{1}{2} m (1 \cdot \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \dot{\phi}^2) - \text{P.E. term}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + \text{P.E.}$$

(III) Cylindrical Coordinate (s, ϕ, z) or (r, ϕ, z) :-

$$L = \frac{1}{2} m (\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - \text{P.E. term.}$$

$$H = \frac{p_s^2}{2m} + \frac{p_\phi^2}{2ms^2} + \frac{p_z^2}{2m}$$

Q. If $H = a p_\theta^2 + b\cos\theta$ write eq^n of motion of system (2nd order} differential eq^n).

Sol^n Write Hamilton's Equation.

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = 2ap_\theta$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = b\sin\theta$$

$$\boxed{\begin{aligned}\dot{\theta} &= 2ap_\theta \\ \dot{p}_\theta &= b\sin\theta\end{aligned}} \rightarrow \text{first order differential Equation.}$$

To get second order diff. eq^n in θ put $\dot{p}_\theta$ in I^{st} Equation.

diff. w.r.t. time.

first eq^n - $\ddot{\theta} = 2a\dot{p}_\theta$

$$\ddot{\theta} = 2ab\sin\theta$$

$$\boxed{\ddot{\theta} - 2ab\sin\theta = 0}$$ ← second order differential eq^n in θ.

Q. If $H = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$ find second order diff. eq^n in θ.

Sol^n

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2p_\theta}{2ml^2}$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = -mgl\sin\theta$$

$$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = -\frac{mgl\sin\theta}{ml^2}$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

* Poisson's Bracket :- {P.B.}

If A and B are two dynamical variables -

i.e. $A = A(p_i, q_i, t)$, $B = B(p_i, q_i, t)$

possion Bracket of A with B is defined as -

$$\boxed{\{A, B\}_{q_i, p_i} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}}$$

Summation over i.

$$\left.\begin{aligned} &\{q_i, p_i\} = 1 \quad , \quad \{x, p_x\} = 1 \\ &\{q_i, p_j\} = \delta_{ij} \quad , \quad \{x, p_y\} = 0 \end{aligned}\right\} \leftarrow \text{Fundamental P.B.}$$

Properties :-

* $\{A, B\} = -\{B, A\}$

* $\{A, B+C\} = \{A, B\} + \{A, C\}$

* $\{A, BC\} = \{A, B\}C + B\{A, C\}$ → Leibnitz Identity

* $\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$ ← Jacobi Identity

* Poisson's Bracket with angular momentum:-

$[x, L_y] = z$, $[y, L_z] = x$

$[p_x, L_z] = -p_y$

$[p_z, L_x] = p_y$

$[L_x, L_y] = L_z$

$[p_y, L_y] = 0$

$[q, L_q] = 0$

If we replace $i\hbar \to 1$, then commutator bracket changes into P.B. and vice-versa.
Here in P.B. we can use { } or [ ]

* General Relation :-

$$[A_i, L_j] = \varepsilon_{ijk} A_k$$

$$[A_i, L_j] = \varepsilon_{ijl} A_l$$

Levicivita Tensor.

Here l is repeated index so we have summation over it, so we can change variable.

* A and B are constant of motion (Conserved) then their possion bracket is also constant of motion.

Q. If $A = ap_1 + bq_2$, $B = cq_1 + dp_2$ find the value of P.B. $\{A, B\}$.

Note :- Here A and B depends upon position and momentum co-ordinate so these two are dynamical variable. formula $\Rightarrow \{A, B\}_{q_i, p_i} = \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$

$$\{A, B\}_{q_i, p_i} = \{\underbrace{ap_1 + bq_2}_{A}, \underbrace{cq_1 + dp_2}_{B}\}_{q_1, p_1} + \{\underbrace{ap_1 + bq_2}_{A}, \underbrace{cq_1 + dp_2}_{B}\}_{q_2, p_2}$$

$$= (0 - a.c) + (bd - 0)$$

$$= bd - ac \quad \text{Ans}$$

Q. Explicit calculation of $[x, L_y]$

Sol^n

$$[x, zp_x - xp_z] = [\underbrace{x}_{A}, \underbrace{zp_x - xp_z}_{B}]_{x,p_x} + [\underbrace{x}_{A}, \underbrace{zp_x - x}_{B}]_{z}$$

$$= 1\cdot z + 0 + 0 + 0$$

$$= z \quad \underline{\text{Ans}}$$

B.A.

Q. 45

Sol^n

$$\{x_i, p_j\} = \delta_{ij}$$

$$C_1 = x_2 p_3 + x_3 p_2$$

$$C_2 = x_1 p_2 - x_2 p_1$$

$$C_3 = x_1 p_3 + x_3 p_1 = \{C_1, C_2\}$$

(a) $\{C_1, C_3\} = C_1$ & $\{C_1, C_3\} = C_2$

(b) $\{C_2, C_3\} = -C_1$ & $\{C_3, C_1\} = -C_2$

$$\{C_2, C_3\} = \{x_1 p_2 - x_2 p_1 \,,\, x_1 p_3 + x_3 p_1\}$$

$$= \{\quad\}_{x_1,p_1} + \{\quad\}_{x_2 p_2} + \{\quad\}_{x_3 p_3}$$

$$= p_2 x_3 + x_2 p_3 - p_1 \times 0 - x_1 \times 0 + 0 - 0$$

$$\{C_2, C_3\} = C_1$$

B.A. note.

Q. 23

Sol^n $\quad L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

P.B $\quad \{\theta, \dot{\theta}\} = ? \quad \because \{\theta, p_\theta\} = 1 \quad$ {stand.}

Convert $\dot{\theta}$ into p_θ.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

$$\{\theta, \dot{\theta}\} = \{\theta, \frac{p_\theta}{ml^2}\} = \frac{1}{ml^2}\{\theta, p_\theta\} = \frac{1}{ml^2} \times 1$$

$$\boxed{\{\theta, \dot{\theta}\} = \frac{1}{ml^2}}$$ Ans

A-11
Q.9

Sol^n

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{qB}{2}(x\dot{y} - y\dot{x})$$

$$[\dot{x}, \dot{y}] = ?$$

$$\because \{\dot{x}, p_x\} = 1$$

Express $\dot{x}$ and $\dot{y}$ in terms of co-ordinate and momentum.

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB}{2}y$$

$$\dot{x} = \frac{\left(p_x + \frac{qB}{2}y\right)}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB}{2}x$$

$$\dot{y} = \frac{p_y - \frac{qB}{2}x}{m}$$

$$[\dot{x}, \dot{y}] = \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]$$

$$= \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]_{x, p_x} + \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]_{y p_y}$$

Q. Evaluate $\{(\vec{a}\cdot\vec{r}), \vec{p}\}$ Where $\vec{a}$ is constant

Sol^n $\therefore$ $\{(\vec{a}\cdot\vec{r}), \vec{p}\} = \{a_i r_i, p_j\}$

$$= a_i \{r_i, p_j\}$$

$$= a_i \delta_{ij} = a_j \times 1$$

$$= a_j$$

$$= \vec{a}. \quad \text{Ans}$$

Q. $\{(\vec{a}\cdot\vec{r}), \vec{r}\}$ evaluate ?

Sol^n $\{(\vec{a}\cdot\vec{r}), \vec{r}\} = 0$ ($\because$ Here no momentum term ~~absent~~)

Q. Evaluate $\{(\vec{a}\cdot\vec{r})^2, \vec{p}\}$

Sol^n

$$\{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = \{(\vec{a}\cdot\vec{r})(\vec{a}\cdot\vec{r}), \vec{p}\}$$

$\because$ $\{AB, C\} = A\{B, C\} + B\{A, C\}$

$$\therefore \{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = (\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\} + (\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\}$$

$$= 2(\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\}$$

$$= 2(\vec{a}\cdot\vec{r})\vec{a} \qquad \{\because \{(\vec{a}\cdot\vec{r}), \vec{p}\} = \vec{a}\}$$

Ans

Second Method:-

$$\{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = \{a_i^2 r_i^2, p_j\}$$

$$= a_i^2 \{r_i^2, p_j\}_{r_i, p_i}$$

$$= a_i^2 \left[2r_i \frac{\partial p_j}{\partial p_i} - 0\right]$$

$$= a_i^2 \, 2r_i \delta_{ij}$$

$$= 2a_j^2 r_j \qquad \{\because \delta_{ij} = 1 \text{ when } i=j\}$$

$$= 2a_j(a_j r_j)$$

$$= 2a_j(\vec{a}\cdot\vec{r})$$

$$= 2\vec{a}(\vec{a}\cdot\vec{r})$$

Ans

Q. Evaluate $[\vec{a}.\vec{r}, \vec{L}]$

angular momentum not Lagrangian becoz Lagrangian is scalar quantity

Sol^n $[a_i r_i, L_j] = a_i[r_j, L_j]$

$= a_i \varepsilon_{ijk} r_k$

$= \varepsilon_{ijk} a_i r_k$

$= -\varepsilon_{jik} a_i r_k$

$= -(\vec{a}\times\vec{r})_j \quad \left\{ \because (\vec{a}\times\vec{b})_i = \varepsilon_{ijk} a_i b_k \right\}$

$= (\vec{r}\times\vec{a})_j$

$= (\vec{r}\times\vec{a})$ Ans

Q. Evaluate $[\vec{a}.\vec{L}, \vec{b}.\vec{L}]$

$\Rightarrow [a_i L_i, b_j L_j]$

$\Rightarrow a_i b_j [L_i, L_j]$

$= a_i b_j \varepsilon_{ijk} L_K$

$= a_i \varepsilon_{ijk} b_j L_K$

$= a_i (\vec{b}\times\vec{L})_i$

$= \vec{a}.[\vec{b}\times\vec{L}] = [\vec{a}\times\vec{b}).\vec{L}$

Q. Evaluate $[x, L_y] = z$, $[L_x, y] = ?$

$$[y p_z - z p_y, y] = [\quad]_{y, p_y} + [\quad]_{z, \prime}$$

$$= p_z \times 0 + z \times 1 + (-p_y) \times 0 - y = 0$$

$$= z$$

$[p_x, L_y] = p_z$

$[L_x, L_y] = p_z$

$[L_x, y] = z$

Q. Evaluate $[\vec{r}, \vec{a}.\vec{L}]$ where $\vec{a}$ = Constant vector.

$$[r_i, a_j L_j] = a_j [r_i, L_j]$$

$$= a_j \varepsilon_{ijk} r_k$$

$$= \varepsilon_{ijk} a_j r_k$$

$$= (\vec{a} \times \vec{r})_i = \boxed{\vec{a} \times \vec{r}}$$ Ans

create

Q. Evaluate $\{\vec{a}.\vec{r}, \vec{b}.\vec{p}\}$

$$= \{a_i r_i, b_j p_j\}$$

$$= a_i b_j \{r_i, p_j\}$$

$$= a_i b_j \delta_{ij}$$

$$= a_j b_j \times 1 = \vec{a}.\vec{b}$$ Ans

Q. Evaluate $[|\vec{r}|, |\vec{p}|]$

Sol^n $\because r = \sqrt{x^2+y^2+z^2}$

So $\left[\sqrt{x^2+y^2+z^2}, \sqrt{p_x^2+p_y^2+p_z^2}\right]$

$$= [\quad]_{xp_x} + [\quad]_{yp_y} + [\quad]_{zp_z}$$

$$= \left(\frac{x}{|\vec{r}|}\cdot\frac{p_x}{|\vec{p}|} + 0\right) + \left(\frac{y}{|\vec{r}|}\frac{p_y}{|\vec{p}|} + 0\right) + \left(\frac{z}{|\vec{r}|}\frac{p_z}{|\vec{p}|} + 0\right)$$

$$= \frac{xp_x + yp_y + zp_z}{|\vec{r}||\vec{p}|} = \frac{\vec{r}\cdot\vec{p}}{|\vec{r}||\vec{p}|}$$

$$= \frac{\vec{r}}{|\vec{r}|}\cdot\frac{\vec{p}}{|\vec{p}|} = \boxed{\hat{r}\cdot\hat{p}}$$ Ans

Q. If $x' = x\cos\theta - p_x\sin\theta$, $p_x' = x\sin\theta + p_x\cos\theta$.
evaluate P.B. $\{x', p_x'\}_{x, p_x}$

Sol^n

$$\{x\cos\theta - p_x\sin\theta, x\sin\theta + p_x\cos\theta\}_{x, p_x}$$

$$= \cos\theta \times \cos\theta + \sin\theta \times \sin\theta$$

$$= \cos^2\theta + \sin^2\theta$$

$$= 1.$$

* Poission's Equation of Motion :-

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

If $A = H$

$$\frac{dH}{dt} = [H, H] + \frac{\partial H}{\partial t}$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

If H does not depends on t explicitly.

$$\frac{\partial H}{\partial t} = 0$$

$$\frac{dH}{dt} = 0 \Rightarrow \boxed{H = \text{Constant or Conserved.}}$$

If H does not depends on time explicitly then H is conserved.

A-11
Net-2012
Dec.

Q. A system is governed by the Hamiltonian $H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2$ where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved.

Sol^n $H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2$

$p_x - 3y$, $p_y + 2x$ conserved.

Let $A = p_x - 3y$ it does not depends on time explicitely

* Imp. So $\frac{\partial A}{\partial t} = 0$ so it is conserved

So $\frac{dA}{dt} = 0$

$\therefore [A, H] = 0$

$\therefore$ $B = p_y + \cancel{2x}$ it is also it does not depends on time explicitely

So $\frac{\partial B}{\partial t} = 0$ So it is also conserved

So $\frac{dB}{dt} = 0$

$\therefore [B, H] = 0$

$$\Big[\underbrace{p_x - 3y}_{A}, \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - bx)^2}{2}}_{B}\Big] = 0$$

$$\Rightarrow [\quad]_{x, p_x} + [\quad]_{y, p_y} = 0$$

$$0 - 1.(p_y - bx)(-b) + (-3)(p_y - bx) - 0 = 0$$

$$\Rightarrow (p_y - bx)(b - 3) = 0$$

$$b - 3 = 0$$

$$\boxed{b = 3}$$

$$[B,H] = [\underbrace{p_y + 2x}_{A},\ \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - bx)^2}{2}}_{B}] = 0$$

$$[B,H] = \Big[\underbrace{(p_y+2x)}_{A},\ \underbrace{\frac{(p_x-ay)^2}{2} + \frac{(p_y-bx)^2}{2}}_{B}\Big]_{x,p_x} + \Big[\underbrace{(p_y+2x)}_{A},\ \underbrace{\frac{(p_x-ay)^2}{2} + \frac{(p_y-bx)^2}{2}}_{B}\Big]_{y,p_y}$$

$$\left[\frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}\cdot\frac{\partial B}{\partial x}\right] \qquad \left[\frac{\partial A}{\partial y}\cdot\frac{\partial B}{\partial p_y} - \frac{\partial A}{\partial p_y}\cdot\frac{\partial B}{\partial y}\right]$$

$$= \Big[\{2\cdot(p_x - ay)\} - \{0\}\Big] + \Big[\cancel{0 + 1\cdot(p_y - b)}\Big]$$

$$[0 + 1\cdot(p_x - ay)(-a)$$

$$\cancel{= 2(p_x - ay) - (p_y - b)}$$

$$2(p_x - ay) + (p_x - a_y)(+a)$$

$$(p_x - ay)(2+a) = 0$$

$$a = -2$$

Ans

(*) Conservation Laws :-

① If L or H is translationally invariant then linear momentum is conserved in that direction.

$$p_x = \frac{\partial L}{\partial \dot{x}} \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(p_x) - \frac{\partial L}{\partial x} = 0$$

$$\dot{p}_x = \frac{\partial L}{\partial x}$$

If L does not explicitly depend on x

$$\frac{\partial L}{\partial x} = 0$$

$$\dot{p}_x = 0 \Rightarrow p_x = \text{Constant}$$

Hamiltonian eq^n of motion -

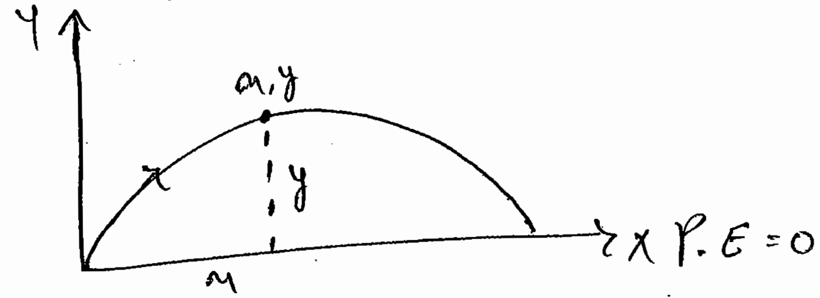
$$\dot{p}_x = -\frac{\partial H}{\partial x}$$

If H is not explicit function of x

$$\frac{\partial H}{\partial x} = 0$$

$$\dot{p}_x = 0 \quad , \quad p_x = \text{Constant}$$

Ex-



$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

x is cyclic

Translation $x \longrightarrow x + \alpha$, then $L \longrightarrow L$

Here L is invariant w.r.to translation x-direcⁿ

$\therefore$ p_x is conserved.

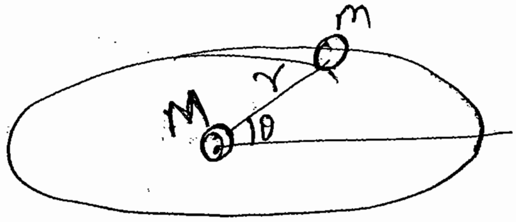
* Homoginity of space leads to conservation of momentum.

Second Theorem:- If L or H is Rotationally invariant, Angular momentum is conserved in that direction.

$$\theta \longrightarrow \theta + \alpha, \quad L \rightarrow L$$

Ex- Sun - Planet System -

$$L = \frac{1}{2} m (\dot{r}^2 + r\dot{\theta}^2) + \frac{GMm}{r}$$



$\theta \rightarrow$ is cyclic, $\frac{\partial L}{\partial \theta} = 0$

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$\dot{p}_\theta = 0$$

$$\boxed{p_\theta = \text{Constant}}$$

So $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$

θ is cyclic $\frac{\partial H}{\partial \theta} = 0$

Hamilton's equation –

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\boxed{p_\theta = \text{Conserved}}$$

Angular momentum is Conserved.

$$\boxed{p_\theta = L_z}$$

* Isotropy of space leads to Conservation of angular momentum.

* Third Theorem :-

If L or H does not explicitly depends on time then Hamiltonian is conserved.

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

$$A = H$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}} \rightarrow 0$$

$$\frac{dH}{dt} = 0$$

$$\boxed{H = \text{Constant}}$$

(a) If potential energy does not depends on velocity then H = total energy.

(b) If potential energy depends on velocity then

$$\boxed{H \neq \text{total energy}}$$

Imp.
* If L or H does not depends on t explicitly and potential does not depends on velocity then total energy is conserved.

** If potential depends on velocity then H is conserved but total is not conserved.

*** Homogeneity of time leads to conservation of Hamiltonian (Energy).

Q. $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}K(x^2+y^2)$ Which of the following is conserved.

p_x, p_y, L_z, E = energy

Sol^n x and y are not cyclic so p_x and p_y are not conserved.

To know about L_z write L in plane polar co-ordi

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}Kr^2$$ here θ is not come

so θ is cyclic.

$\therefore$ p_θ = Constant

$\therefore$ L_z = Constant

L does not explicitly depends on t.

so H = Conserved.

$\because$ Potential does not depend on velocity

$\therefore$ $H = E$

So $\boxed{E = \text{Conserved}}$

* How to identify the K.E. term and P.E. term in L.

$\Rightarrow$ Say $L = \underbrace{\frac{1}{2} m (\dot{x}^2 + \dot{y}^2)}_{\text{K.E. term}} - \underbrace{\frac{1}{2} K (x^2 + y^2) + \omega (x\dot{y} - y\dot{x})}_{\text{P.E. term.}}$

$\because$ K.E. term is alway quadratic in velocity.

like $\dot{x}^2$, $\dot{y}^2$ or $(\dot{x}\dot{y})$ etc.

in $\dot{q}_i$ i.e. $\dot{q}_i^2$ or $\dot{q}_i \dot{q}_j$

$H = \text{Constant}$

$H \neq E$

$\therefore$ E is not conserved.

A-11

Q.11 Lagrangian of a System is $L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qEy - \frac{qB}{c} y\dot{x}$

Which of the following is not correct.

Soln $L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qEy - \frac{qB}{c} y\dot{x}$

(a) $m\dot{x} - \frac{qBy}{c} = \text{Const.}$

(b) $m\dot{z} = \text{Const.}$

× (c) $m\dot{y} + \frac{qBx}{c} + qEt = \text{Const.}$ ✓

(d) $m\dot{y} + \frac{qBy}{c} - qEt = \text{Const.}$

$\because$ x and z are constant.

$p_x = \text{Constant}$, $p_z = \text{Constant}$.

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

$$p_x = m\dot{x} - \frac{qB}{c}y = \text{Constant}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} = \text{Constant}$$

~~If $m\dot{y}$~~ say $A = m\dot{y} + \frac{qBx}{c} + qEt$

If $A = \text{Constant}$

then $\frac{dA}{dt} = 0$

$$\frac{dA}{dt} = m\ddot{y} + \frac{qB\dot{x}}{c} + qE \quad \text{--- (I)}$$

write Lagrangian's eq^n in y -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$m\ddot{y} - qE + \frac{qB}{c}\dot{x} = 0$$

$$m\ddot{y} + \frac{qB}{c}\dot{x} = qE \quad \text{--- (II)}$$

So from (I)

$$\therefore \frac{dA}{dt} = 2qE \neq 0$$

So (c) is wrong. Ans

Q. 92

Solⁿ

$$L = \underbrace{\mu(\dot{x}_1^2 + \dot{x}_2^2)}_{K.E.} - \underbrace{\gamma(x_1 + x_2)}_{P.E.}$$

∵ L is not explicitly depend on time so

H = Conserved

∵ in P.E. part is independent of velocity

so H = total Energy

∴ H = Conserved

$$\therefore \boxed{E = \text{Conserved}}$$

Here x_1 & x_2 are not cyclic

∴ p_1, p_2 are not conserved.

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = 2\mu \dot{x}_1$$

$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = 2\mu \dot{x}_2$$

$\dot{x}_1$ & $\dot{x}_2$ are not constant. (b) E & $\dot{x}_1 - \dot{x}_2$ is conserved

∵ (a) E & $\dot{x}_1 + \dot{x}_2$ is conserved

let's check say $A = \dot{x}_1 - \dot{x}_2$

$$\frac{dA}{dt} = \ddot{x}_1 - \ddot{x}_2 \quad \text{——} \ (*)$$

Say $B = \dot{x}_1 + \dot{x}_2$

$$\frac{dB}{dt} = \ddot{x}_1 + \ddot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$2M\ddot{x}_1 + \gamma = 0$$

$$\boxed{\ddot{x}_1 = \frac{-\gamma}{2M}}$$

Put in (*) get $\frac{dA}{dt} = 0$

$$\text{||} \quad \boxed{\ddot{x}_2 = \frac{-\gamma}{2M}}$$

So option (b) is correct.

A-11 Net
Q. 19

Sol^n

$$H = \underbrace{\frac{p_\theta^2}{2ml^2}}_{T} + \underbrace{mgl(1-\cos\theta)}_{V}$$

If L = Lagrangian & $\frac{dL}{dt} = ?$

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

$$\frac{dL}{dt} = [L, H] + \frac{\partial L}{\partial t} = 0$$

becoz L is does not depend on time explicitely

$$\boxed{\frac{dL}{dt} = [L, H]}$$

$$L = p_i \dot{q}_i - H$$

$$L = p_\theta \dot{\theta} - H$$

$$\because \quad L = T - V$$

$$L = \frac{p_\theta^2}{2ml^2} - mgl(1-\cos\theta)$$

$$\frac{dL}{dt} = [L, H] = \Big[\underbrace{\frac{p_\theta^2}{2ml^2} - mgl(1-\cos\theta)}_{A}, \underbrace{\frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)}_{B}\Big]$$

$$= \left[\frac{p_\theta^2}{2ml^2} - mgl(1-\cos\theta), \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)\right]_{\theta, p_\theta}$$

$$= -\cancel{m}gl\sin\theta \times \frac{p_\theta}{\cancel{m}l^2} - \frac{p_\theta}{\cancel{m}l^2} \cdot \cancel{m}gl\sin\theta$$

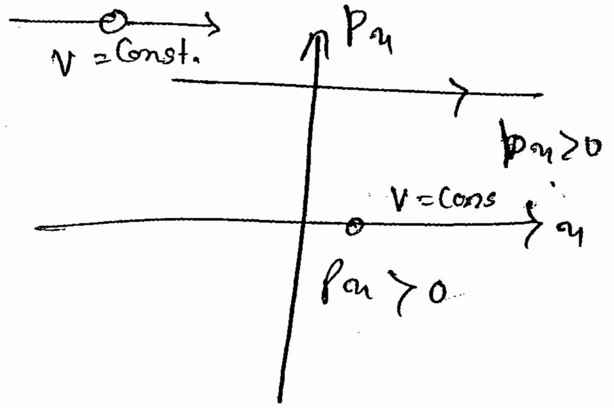
$$\boxed{\frac{dL}{dt} = -\frac{2g\,p_\theta \sin\theta}{l}}$$

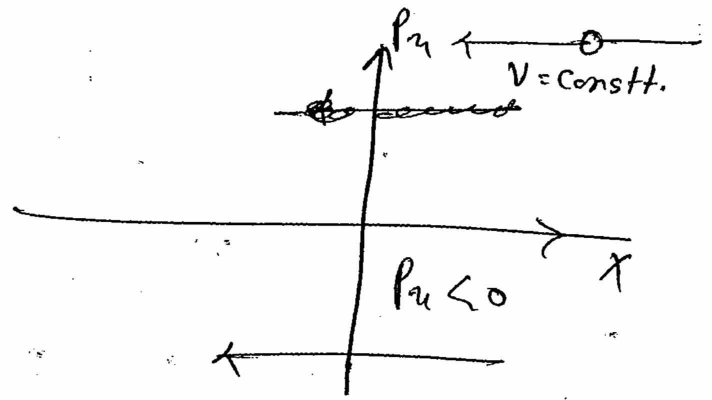
* Phase Space Dynamics :- How momentum varies with coordinate (graph b/w p_i, q_i)

Phase Space : It consists of co-ordinates and momenta. for a system having f - D.O.F. phase space is a 2f dimensional space (f - coordinate + f momenta).

How to draw phase space line (phase space trajectory) [for one dimensional problem]

* If particle is moving towards +ve x direc^n then $p_x > 0$ (+ve)
* If particle is moving towards -ve x direc^n $p_x < 0$ (-
* If $p_x > 0$ then arrow should be towards +ve X
* If $p_x < 0$ then arrow should be towards -ve X.





* Cauchy Lipschitz Condition :- The two phase space trajectory can not intersect each other.

* For Conservative System (Energy = Constant) :-

If $V(x)$ [P.E.] is increasing then K.E. must decrease [p_x must decrease] and vice-versa.

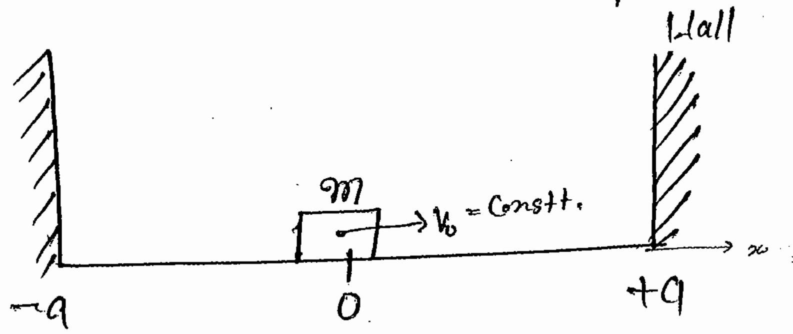
Therefore we first draw $V(x)$ and then using if we draw phase space lines for conservative system.

* Types of questions :-

(1) Potential Energy is Given

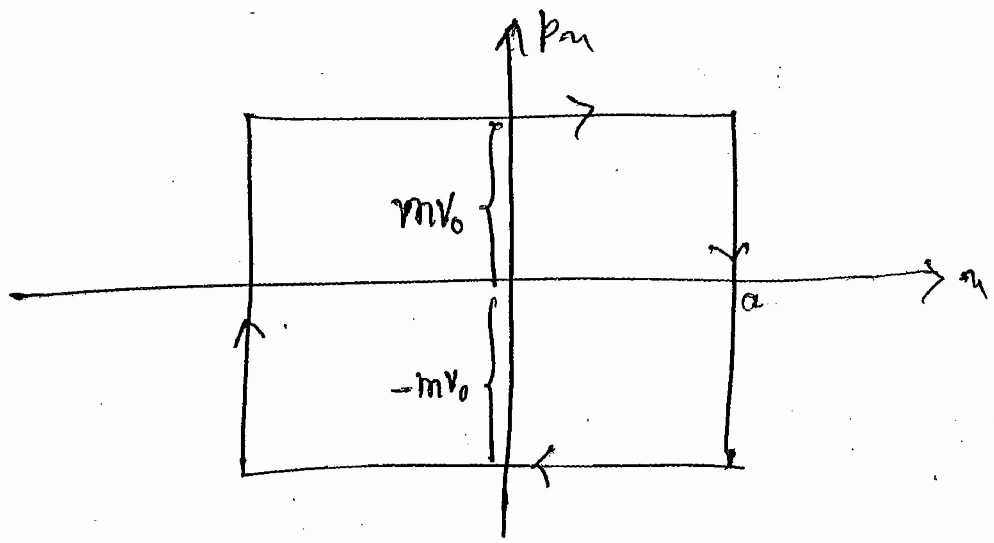
(2) Some information about dynamics is given

Q.

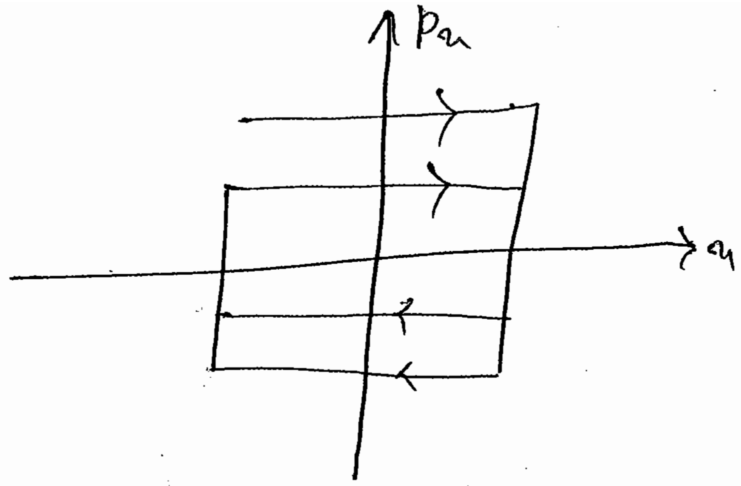


If collisions are elastic draw phase space line.

Sol^n

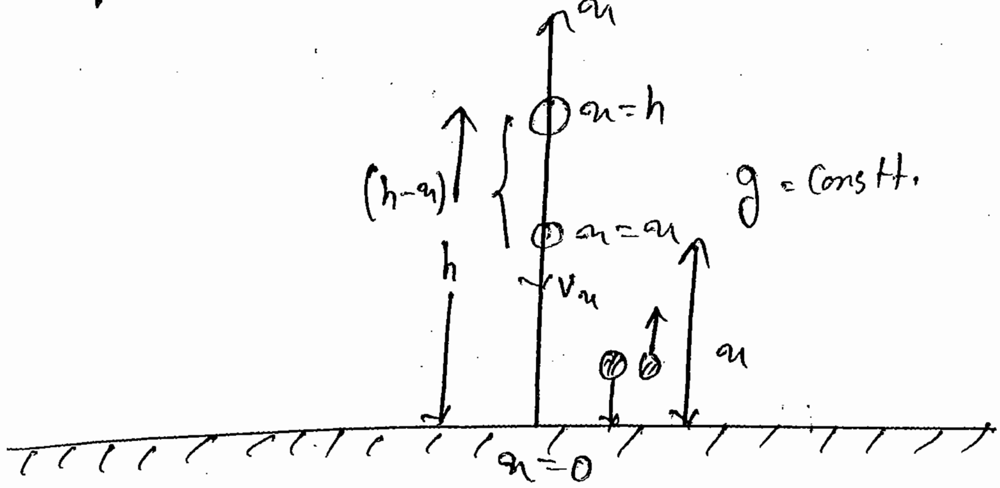


(ii) If Collisions are inelastic :-



P.

Q. A ball dropped from some height on horizontal surface.



$v^2 = u^2 + 2as$

$V_x^2 = 0 + 2g(h-x)$

$V_x^2 = 2g(h-x)$

$$\boxed{\frac{p_x^2}{m^2} = 2g(h-x)} \leftarrow \text{parabola}$$

$p_x^2 = 2g(h-x)m^2$

put $h - x = x'$

$$\boxed{p_x^2 = 2gm^2x'}$$

$V_x^2 = 2g(h-x)$

$V_x = \frac{p_x}{m}$

$p_x^2 = 2gm^2(h-x)$

$p_x \frac{dp_x}{dx} = -2gm^2$

$\frac{dp_x}{dx} = -\frac{2gm^2}{p_x}$

$\frac{d^2p_x}{dx^2} = \frac{2gm^2}{p_x^2}\cdot$

$\frac{d^2p_x}{dx^2} = -\frac{4g^2m^4}{p_x^3}$

$p_x < 0$ so $\frac{d^2p_x}{dx^2} > 0$

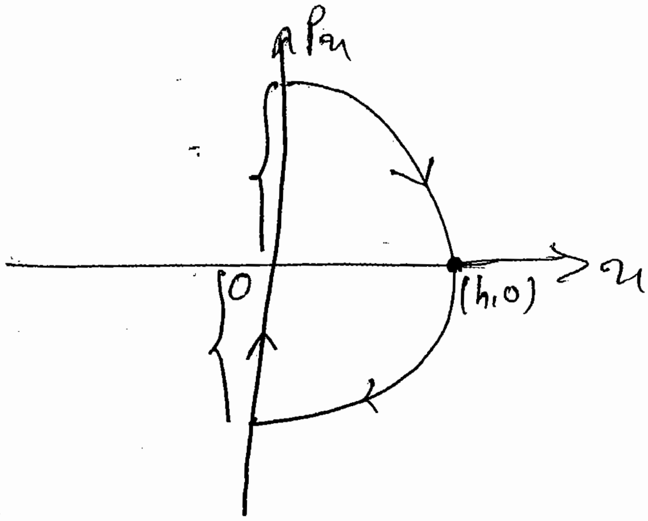
So trajectory is +ve.

Rule :-

$$E = \frac{p_x^2}{2m} + mgx$$

$p_x \longrightarrow -p_x$

E does not change then Symmetry about x-axis.



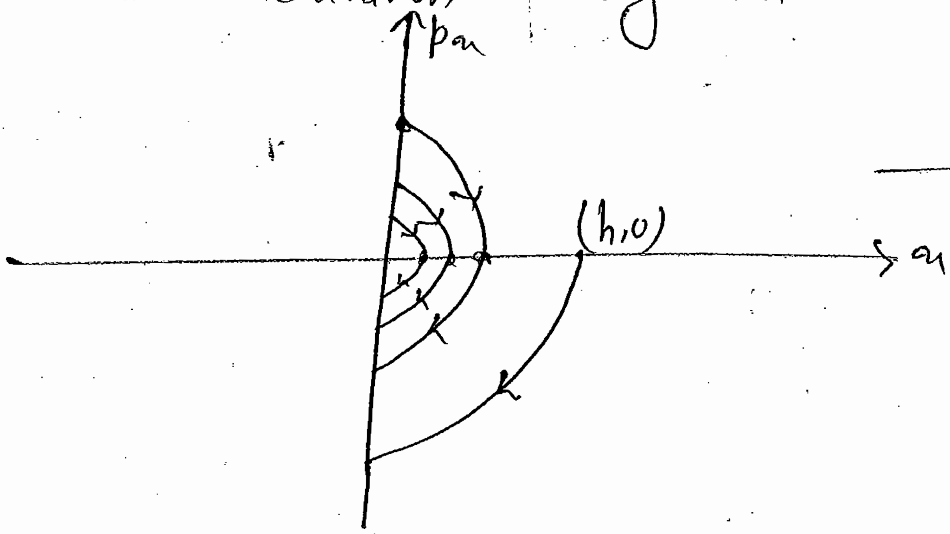
Elastic

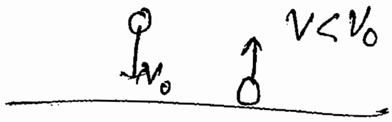
When $x \longrightarrow -x$

E does not change then Symmetry about p_x axis.

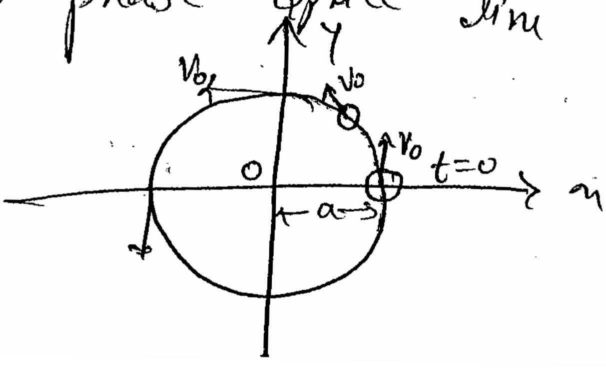
(II) If Collision is inelastic :-

Inelastic collision with ground.



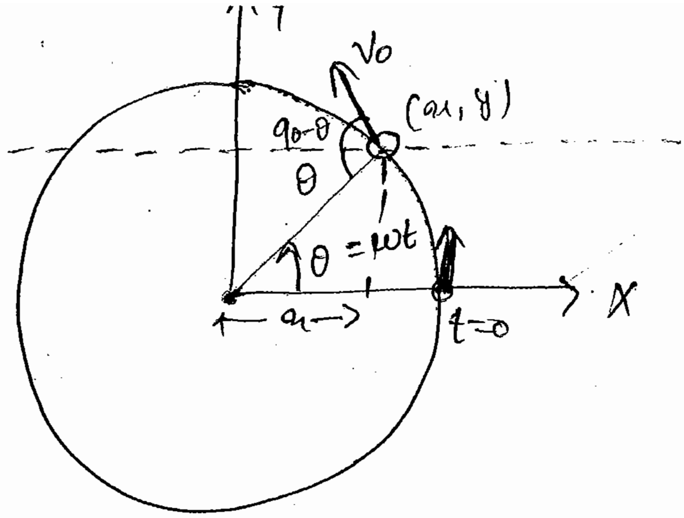


Q. A particle is moving in a circle with constant speed, draw phase space line in $x-p_x$ space.



$\therefore \quad \omega = \frac{d\theta}{dt}$

$$\boxed{\omega = \frac{V_0}{a}}$$



$x = a\cos\theta$

$x = \cos\omega t$

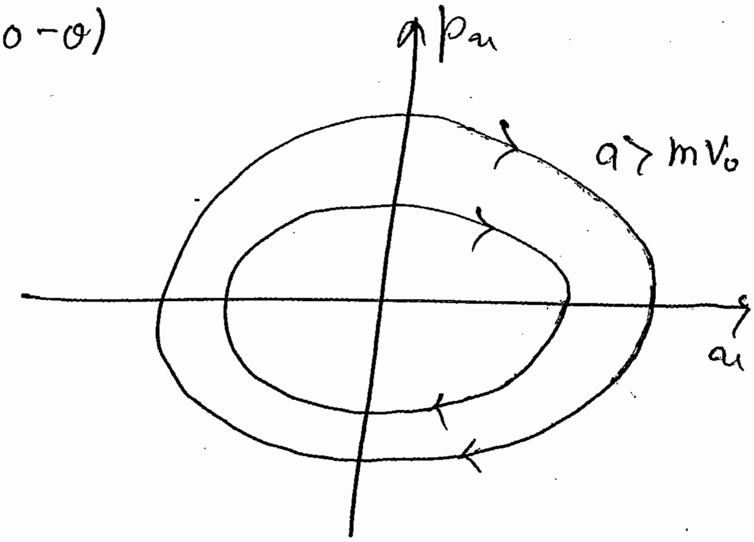
$V_x = -V_0 \cos(90-\theta)$

$\frac{p_x}{m} = -V_0 \sin\theta$

$\frac{p_x}{mV_0} = -\sin\omega t$

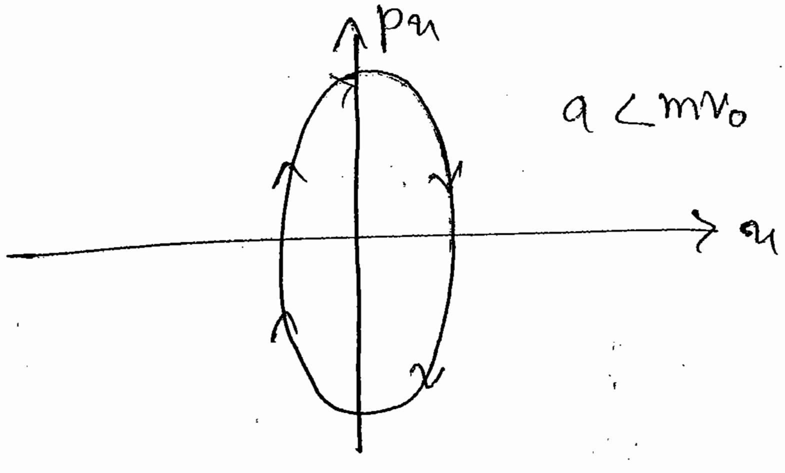
$\frac{x}{a} = \cos\omega t$

$\frac{p_x}{mV_0} = -\sin\omega t$



Square and add

$$\boxed{\frac{x^2}{a^2} + \frac{p_x^2}{(mV_0)^2} = 1}$$ eq^n of ellipse.



* Potential Based Questions:-

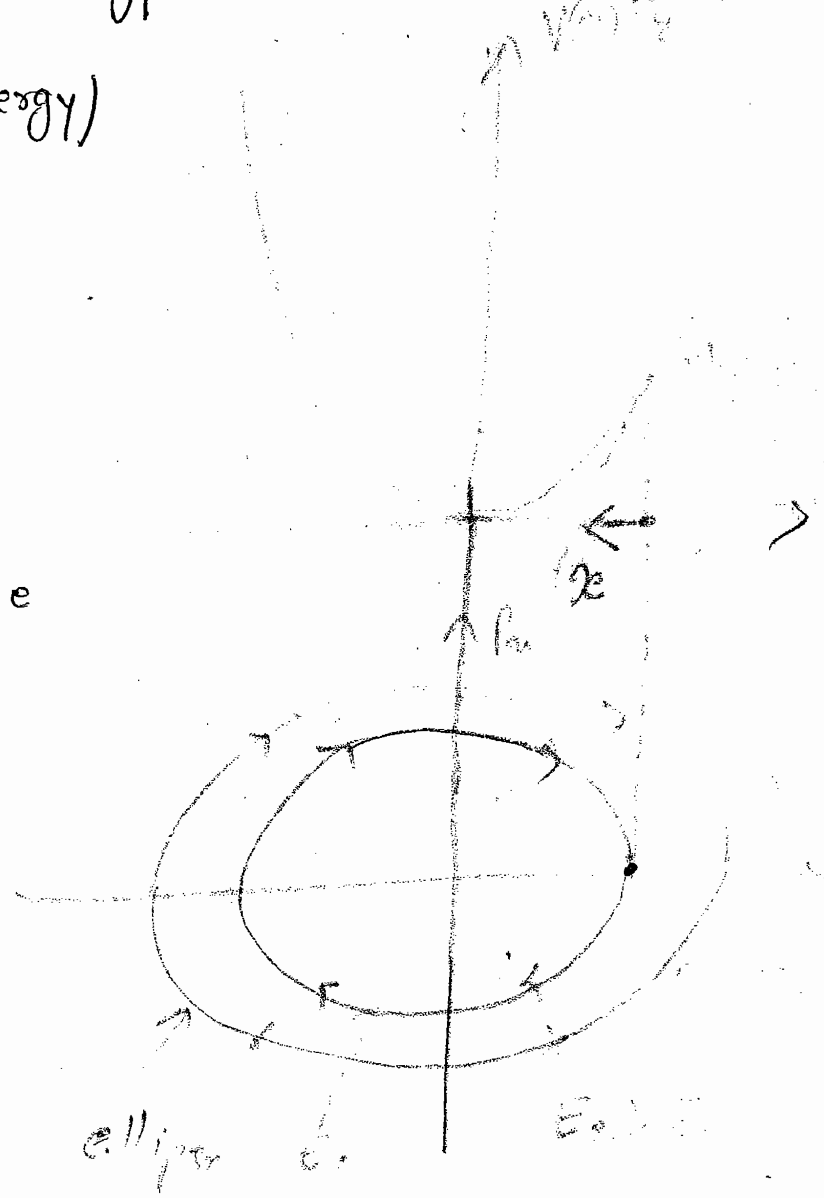
(1) Harmonic Oscillator:-

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = E \text{ (total Energy)}$$

If potential does not depends on velocity then H represents total energy.

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = E \text{ (energy)}$$

$$\frac{p_x^2}{2mE} + \frac{x^2}{\frac{2}{k}E} = 1$$



Here E does not change after
$p_x \to -p_x$
$x \to -x$

$\therefore$ Symmetric about x and p_x both.

Here infinite potential well then phase space lines are closed curve.

* If potential well is infinite then phase space lines <u>inside</u> the well are closed lines if

energy less than height of well. phase space lines are closed lines.

Q. Particle moving under potential $V(x) = ax - bx^2$

Sol^n $H = \frac{p_x^2}{2m} + ax - bx^2 = E = \text{Constant}$

To draw $V(x)$ find zeros

$$V(x) = 0$$

$$ax - bx^2 = 0$$

$$x = 0, \quad x = \frac{a}{b}$$

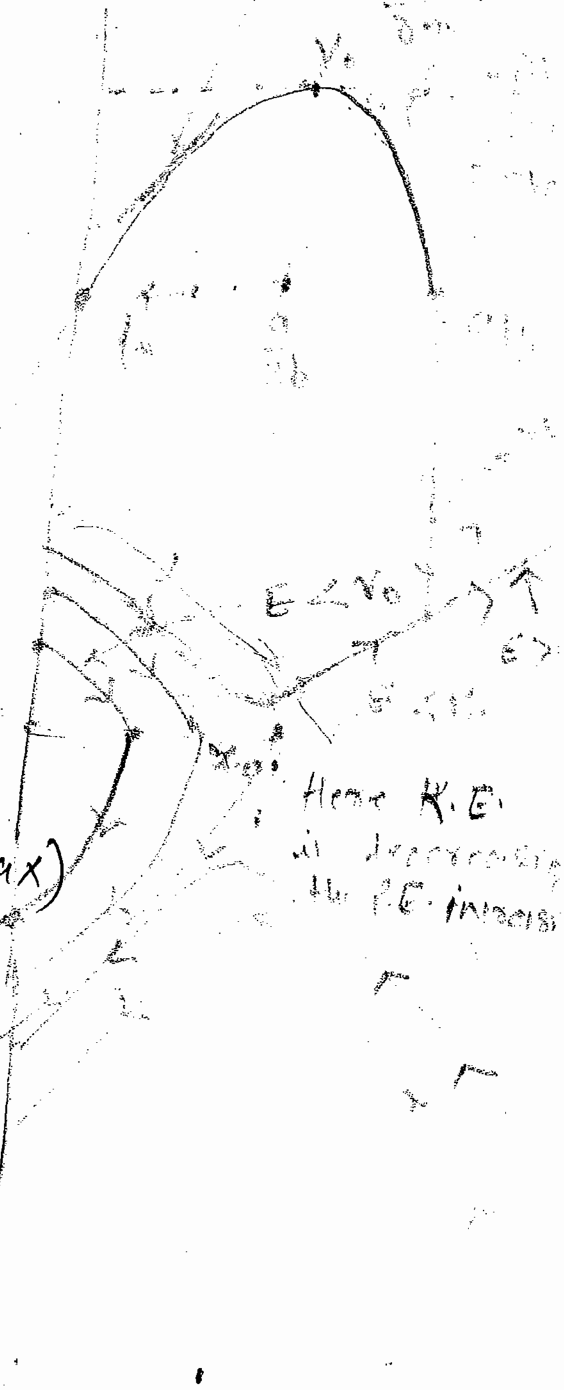
find extrimum points

$$\frac{dV}{dx} = 0$$

$$a - 2bx = 0$$

$$x = \frac{a}{2b}$$

$$\frac{d^2V}{dx^2} = -2b < 0 \quad (\text{max})$$



Q. Draw phase space trajectory for a particle moving under potential.

$$V(x) = ax - bx^3.$$

Solⁿ To draw $V(x)$:-

find zeros :-

$$V(x) = 0$$

$$ax - bx^3 = 0$$

$$x(a - bx^2) = 0$$

$$x = 0 \quad \text{or} \quad a - bx^2 = 0$$

$$x = \pm\sqrt{\frac{a}{b}}$$

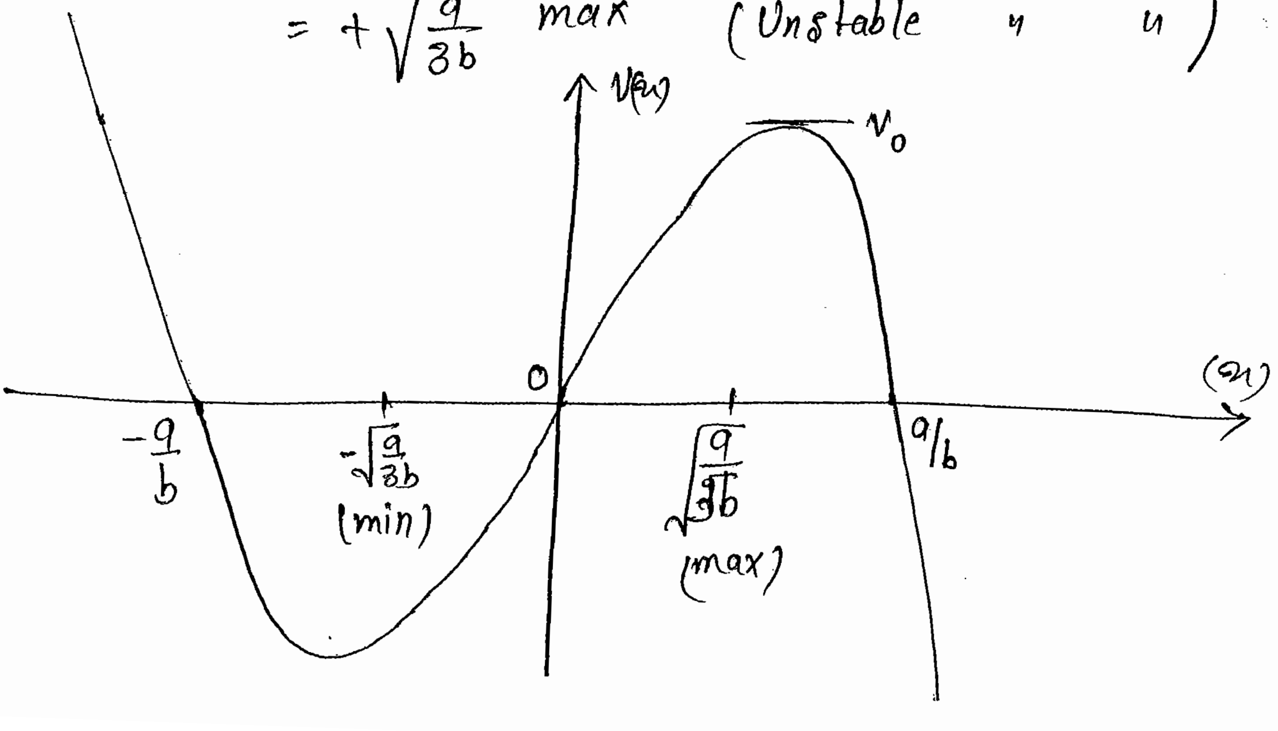
Extrimum points :-

$$\pm\sqrt{\frac{a}{3b}}$$

$$\frac{d^2V}{dx^2} = -6bx$$

$x = -\sqrt{\frac{a}{3b}}$ min (Stable equilibrium point)

$= +\sqrt{\frac{a}{3b}}$ max (Unstable " ")



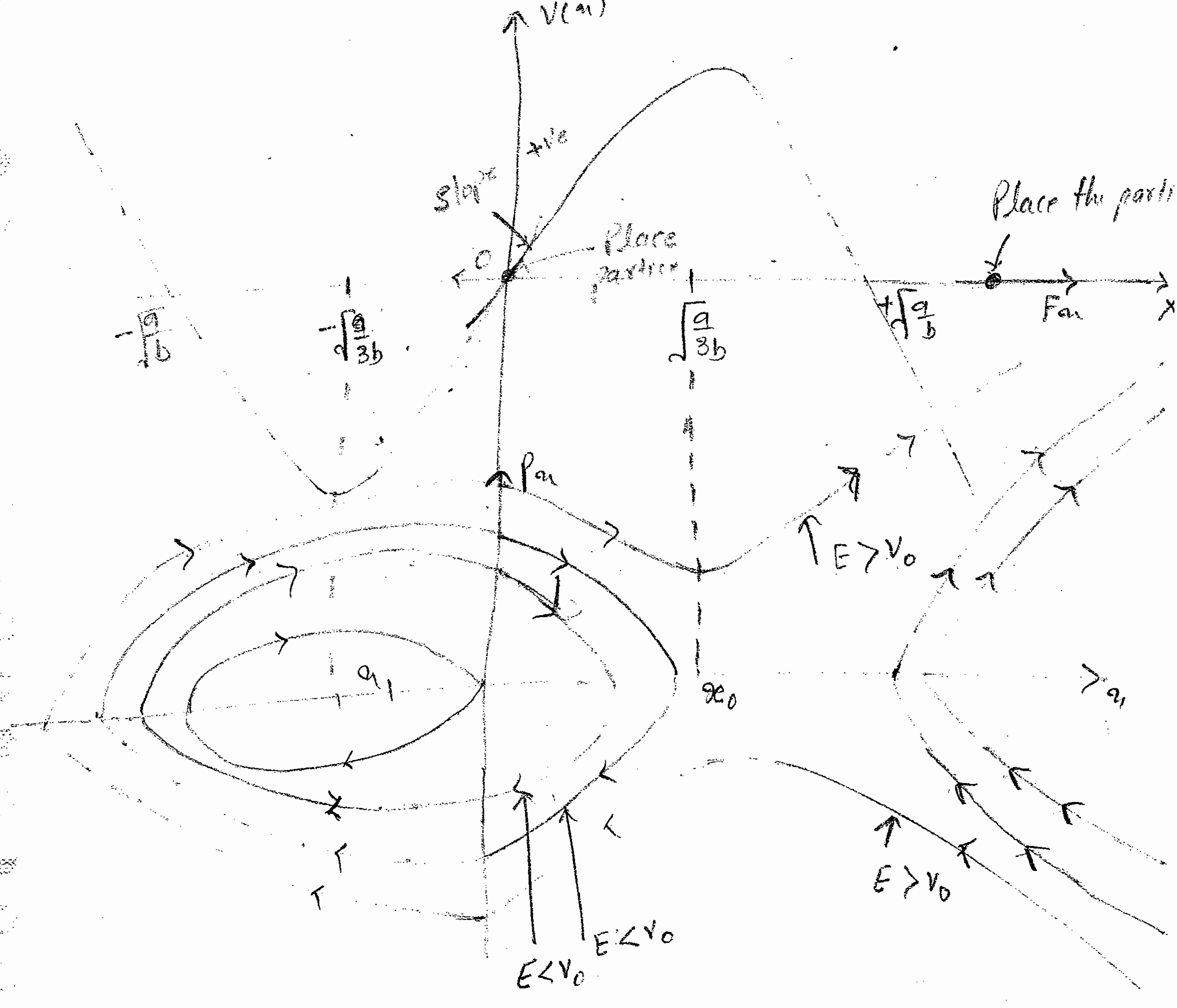
$\therefore \quad H = \dfrac{p_x^2}{2m} + ax - bx^3 = E$

Here $\quad p_x \rightarrow -p_x$

No change so symmetric about x axis

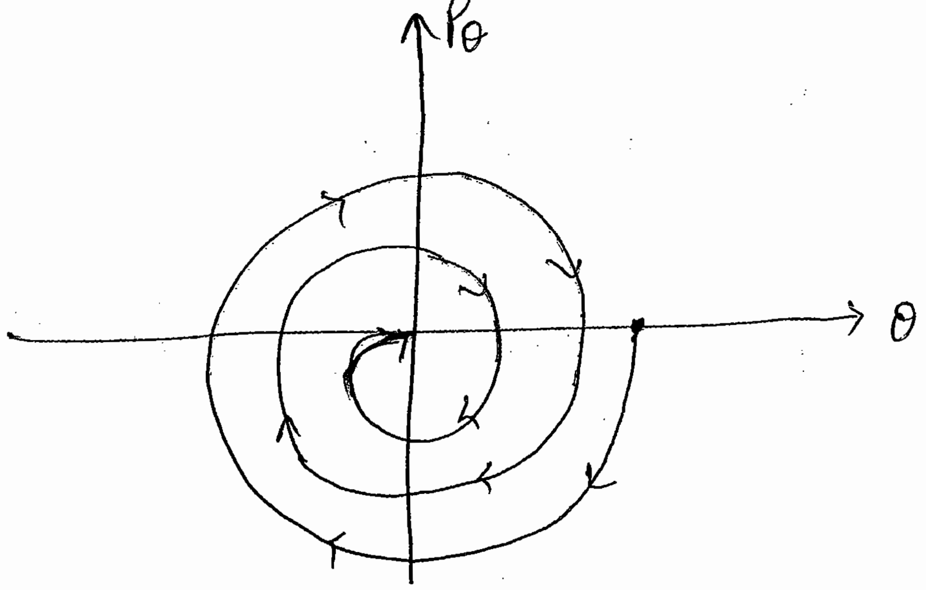
$x \rightarrow -x$

Change so does not symmetric about p_x-axis.

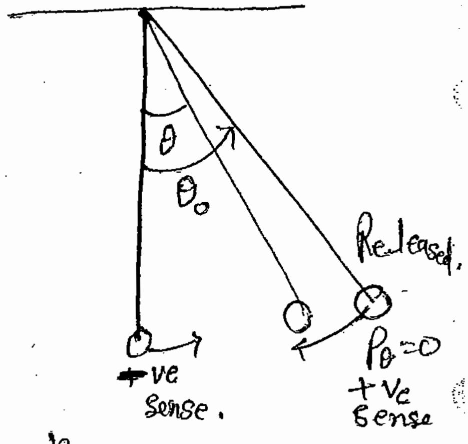


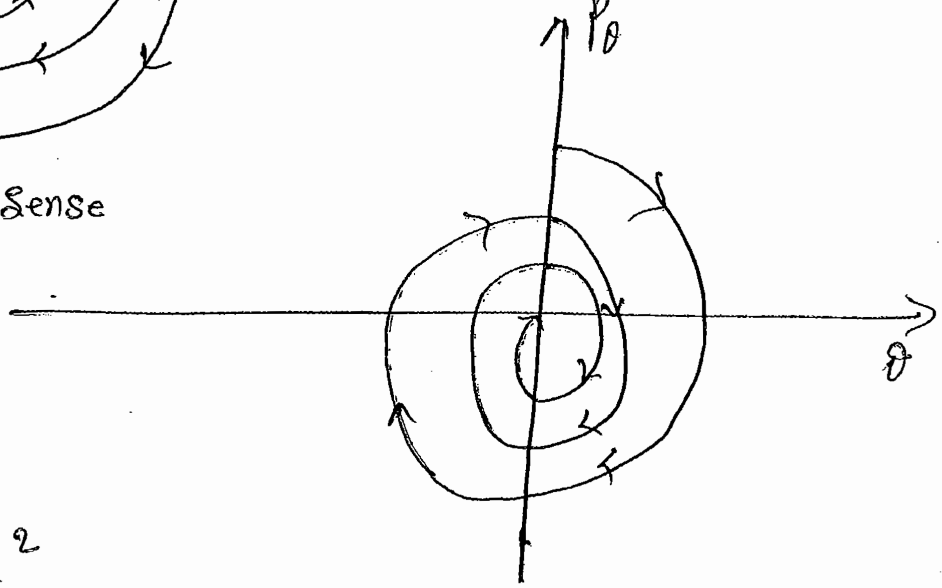
If a particle place a particle outside the potential well.

* Damped Oscillator (Pendulum) :-



for +ve sense

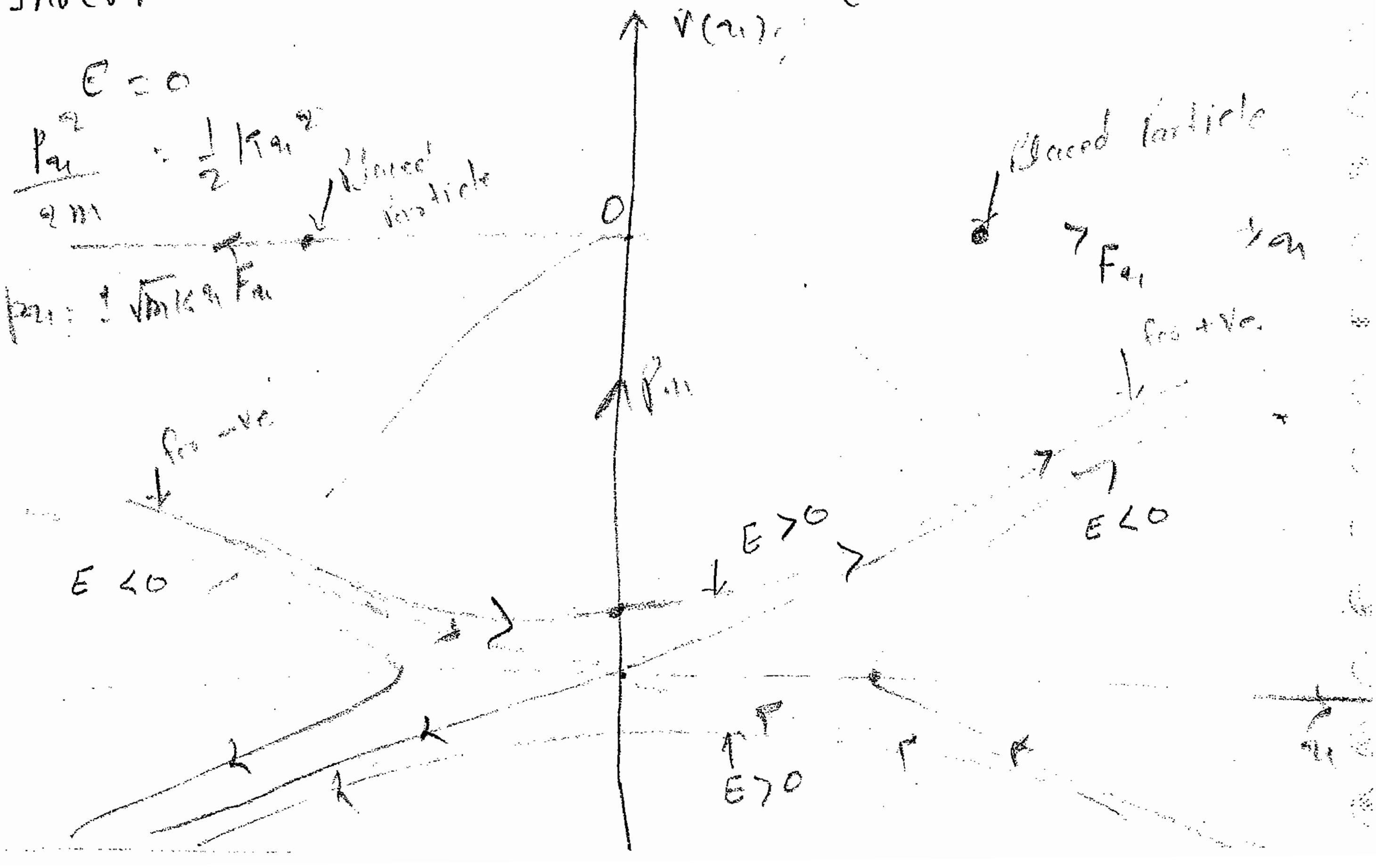




for -ve sense

$$H = \frac{P_x^2}{2m} - \frac{1}{2}kx^2$$

Inverted harmonic oscillator (No oscillation).



* Transformation :-

Point Transformation :- It is done in co-ordinate space.

$$(x, y, z) \longrightarrow (r, \theta, \phi)$$

It changes the form of Lagrangian but leave the form of Lagrangian's equation unchanged.

$$x, y, z \longrightarrow r, \theta, \phi$$

$$L \longrightarrow L'$$

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \text{P.E. term} \longrightarrow \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\sin^2\theta\dot{\phi}^2), - \text{P.E. term}.$$

Eq^n of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}}\right) - \frac{\partial L'}{\partial r} = 0$$

* Gauge Transformation of Lagrangian :-

If Lagrangian is changed in following manner

$$L \longrightarrow L' = L \pm \frac{dF}{dt}$$

, where F is function of q_i and t only, Then, Lagrangian equation of motion does not change.

Example :-

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Thus Lagrangian equation of motion.

$$m\ddot{x} + kx = 0$$

Let $L' = L + \frac{dF}{dt}$ and let $F = \beta x^2$

So $L' = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2 + 2\beta x\dot{x}$

Equation of motion.

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x} + 2\beta x) + Kx - 2\beta\dot{x} = 0$$

$$m\ddot{x} + \cancel{2\beta\dot{x}} + Kx - \cancel{2\beta\dot{x}} = 0$$

$$\boxed{m\ddot{x} + Kx = 0}$$

In Gauge transformation if we add any term which is total derivative of position and time, the equation of motion remains unchanged.

Note :-

Guage transformation of electromagnetic potentials does not change equation of motion because Lagrangian changes by total time derivative of function of $(\vec{r}, t)$.

Consider a particle (non-relativistic) moving in e.m. field (A, ϕ), So Lagrangian is –

$$\boxed{L = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{V}}$$

Guage transformation of e.m. potential.

$$\vec{A}' = A + \vec{\nabla}\lambda(\vec{r}, t)$$

$$\phi' = \phi - \frac{\partial \lambda}{\partial t}$$

$$L' = \frac{1}{2}mv^2 - q\phi' + q\vec{A}'\cdot\vec{v}$$

$$L' = \frac{1}{2}mv^2 - q\phi + q\frac{\partial\lambda}{\partial t} + q\vec{A}\cdot\vec{v} + q(\vec{\nabla}\lambda)\cdot\vec{v}$$

$$L' = L + q\left[\frac{\partial\lambda}{\partial t} + (\vec{\nabla}\lambda\cdot\vec{v})\right]$$

$\because$ $f(x, y, z, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}) + \frac{\partial f}{\partial t}$$

$$\boxed{\frac{df}{dt} = \vec{\nabla}f\cdot\vec{v} - \frac{\partial f}{\partial t}}$$ $\rightarrow$ Total time derivative of any function.

So $$L' = L + q\frac{d\lambda}{dt}$$

$$\boxed{L' = L + \frac{d(q\lambda)}{dt}}$$

So equation of motion will not change.

B.A.2
Q.36

Sol^n $$L = \frac{1}{2}mv^2 + e\vec{A}\cdot\vec{v} - e\phi$$

$$= \frac{1}{2}mv_i^2 + eA_iv_i - e\phi$$

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + eA_i$$

$$\boxed{\vec{p} = m\vec{v} + e\vec{A}}$$

So first option is wroung.

Option (4) is correct. by previous relation.

A-II
Q.10

Sol^n

$L_1 = f(x, \dot{x})$, $\quad L_2 = f(x, \dot{x}) + A(x\dot{y} - y\dot{x})$

$L_2 = L_1 + A(x\dot{y} - y\dot{x})$

$L_3 = f(x, \dot{x}) + A(x\dot{x} + y\dot{y})$

$L_4 = f(x, \dot{x}) + A(x\dot{y} + y\dot{x})$

$L_3 = L_1 + A\frac{d}{dt}(x^2 + y^2)$

Here we can not write this term in the form of total time derivative so L_1 and L_2 can't have same eq^n of motion.

Here we can write total time derivative then L_3 and L_4 have same eq^n of motion.

* Canonical Transformation :- {CT} :-

It is a point transformation in phase space.

$$CT : (q_i, p_i) \longrightarrow (Q_i, P_i)$$

(q_i, p_i) ↓ Old co-ordinate & momentum

(Q_i, P_i) ↓ New Coordinate & momentum.

$$H \longrightarrow H'$$

↑ Old Hamiltonian ↑ New Hamiltonian.

* In canonical transformation form of Hamiltonian's eqn does not change.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{-\partial H}{\partial q_i} \longrightarrow \dot{Q}_i = \frac{\partial H'}{\partial P_i}, \quad \dot{P}_i = \frac{-\partial H'}{\partial Q_i}$$

* If form of Hamilton's equation does not change then form of Lagranges eqn of motion will also not change and this happens if Gauge condition is satisfied.

$$\boxed{L' = L - \frac{dF}{dt}(q_i, Q_i, t)}$$

$$\because L = p_i \dot{q}_i - H$$

So

$$\boxed{P_i \dot{Q}_i - H' = p_i \dot{q}_i - H - \frac{dF}{dt}(q_i, Q_i, t)}$$

This is condition for Canonical Transformation

Use $\frac{d}{dt}F(q_i, Q_i, t) = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$

$$P_i\frac{dQ}{dt} - H' = p_i\frac{dq_i}{dt} - H - \frac{\partial F}{\partial q_i}\frac{dq_i}{dt} - \frac{\partial F}{\partial Q_i}\frac{dQ_i}{dt} - \frac{\partial F}{\partial t}$$

$$P_i dQ - H'dt = p_i dq_i - H dt - \frac{\partial F}{\partial q_i}dq_i - \frac{\partial F}{\partial Q_i}dQ - \frac{\partial F}{\partial t}dt$$

Equating the coefficients of dq_i, dQ_i, dt from both side we get.

$$\left[p_i - \frac{\partial F}{\partial q_i}\right]dq_i = 0 \Rightarrow p_i - \frac{\partial F}{\partial q_i} = 0$$

$$\Rightarrow \boxed{p_i = \frac{\partial F}{\partial q_i}}$$

and $$\left[P_i + \frac{\partial F}{\partial Q_i}\right]dQ_i = 0 \Rightarrow P_i + \frac{\partial F}{\partial Q_i} = 0$$

$$\Rightarrow \boxed{P_i = -\frac{\partial F}{\partial Q_i}}$$

and $$\boxed{H' = H + \frac{\partial F}{\partial t}} \quad \text{—— (2)}$$

$$\left.\begin{aligned} P_i &= -\frac{\partial F}{\partial Q_i} \\ H' &= H + \frac{\partial F}{\partial t} \\ p_i &= \frac{\partial F}{\partial q_i}(q_i, Q_i, t) \end{aligned}\right\} \text{—— (1)}$$

If $q_i, p_i \longrightarrow Q_i$ is C.T. then we can write a function $F(q_i, Q_i, t)$ such that such that eqⁿ

(1) will be satisfies. Here $F(q_i, Q_i, t)$ is a generating function.

Now we can use eq^n (1) to show that in C.T. Poisson bracket

$$\boxed{\{Q_i, P_i\}_{q_i p_i} = 1 \quad \text{and} \quad \{q_i, p_i\}_{Q_i, P_i} = 1}$$

← for checking function is C.T. or not, we satisfies this poisson bracket if (1) bracket is satisfies then second also satisfies so we can check easily

* Jacobian of Transformation :-

Jacobian is defined for all type of function.

Here, in this case - Jacobian of Transformation is one. but it is valid only for canonical form.

$$\boxed{dp_i dq_i = dP_i dQ_i}$$

$$dp_1 dp_2 \text{----} dq_1 dq_2 \text{----} = dP_1 dP_2 \text{----} dQ_1 dQ_2 \text{----}$$

For Ex - $x, y, z \longrightarrow r, \theta, \phi$

$dx, dy dz \longrightarrow dr\, d\theta\, d\phi$

$dx\, dy\, dz \longrightarrow J dr\, d\theta\, d\phi$

When $J = r^2 \sin\theta$ is Jacobian of transformation

Q. Consider a transformation $x, p_x \longrightarrow x', p_x'$ such that

$x' = x\cos\theta - p_x \sin\theta$

$p_x' = x\sin\theta + p_x \cos\theta$

is this transformation a C.T?

P.B. $\{x', p_x'\}_{x, p_x} = 1$

$\Rightarrow (\cos\theta - 0)(0 + \cos\theta) - (-\sin\theta)\sin\theta = 1$

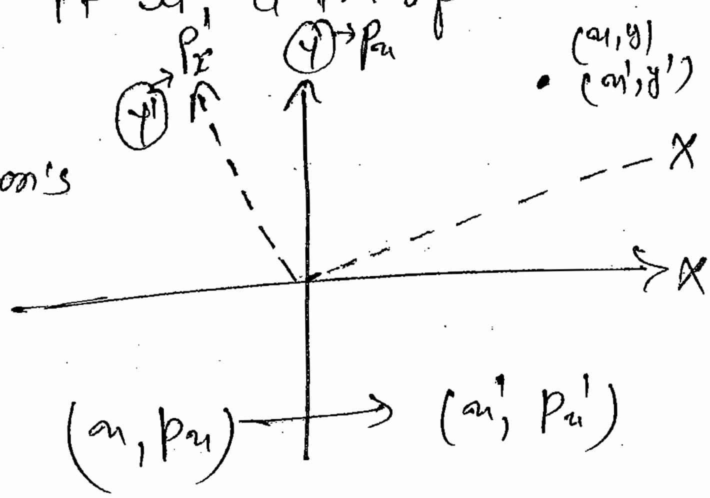
$\cos^2\theta + \sin^2\theta = 1$

$\boxed{1 = 1}$

So given transformation is canonical transformation

Meaning of C.T. is it is, a transformation in phase space.

That need the hamilton's eqn of motion remain's unchange.



$x' = x\cos\theta - p_x \sin\theta$

$p_x' = x\sin\theta + p_x\cos\theta$

Note :- Why we do this transformation.

$\cancel{H = \frac{p^2}{2m} + \frac{1}{2}kx^2 - bx}$

Say $V(x) = \frac{1}{2}kx^2 - bx$

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 - bx$$

$$H = \frac{p^2}{2m} + \frac{1}{2}k\left[\left(x^2 - \frac{2b}{k}x\right) + \left(\frac{b}{k}\right)^2 - \left(\frac{b}{k}\right)^2\right]$$

$$\Rightarrow H = \frac{p^2}{2m} + \frac{1}{2}k\left(x - \frac{b}{k}\right)^2 - \frac{b^2}{2k}$$

$$\left.\begin{aligned} x' &= \left(x - \frac{b}{K}\right) \\ P_x' &= p_x \end{aligned}\right\}$$ $\{$ C.T. is a general transformation $\}$

So $\boxed{E = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{b^2}{2K}}$

We can also solve this ~~transfor~~ problem by perturbati theory but by simplisity we can do this transforma to solve prob. easily. So this is a canonical transformation in phase space.

Q. If transformation

$$P = aq + bp$$
$$Q = cq + dp$$

is a C.T. then what is relation b/w constants a, b, c & d ?

Sol^n

$$\{Q, P\}_{q,p} = 1$$

$$cb - da = 1 \Rightarrow \boxed{bc - ad = 1} \quad \text{Ans}$$

Q. If transformation $Q = q^{\alpha}\sin\beta p$ and $P = q^{\alpha}\cos\beta p$ is a C.T. What is the relation of α and β.?

Sol^n

$$\{Q, P\}_{q,p} = 1$$

$$\{q^{\alpha}\sin\beta p, q^{\alpha}\cos\beta p\}_{q,p} = 1$$

$$-\alpha q^{\alpha-1}\sin\beta p \cdot \beta q^{\alpha}\sin\beta p - \beta q^{\alpha}\cos\beta p \cdot \alpha q^{\alpha-1}\cos\beta p = 1$$

$$-\alpha\beta q^{2\alpha-1} = 1 \cdot q^{0} \quad \text{—(1)}$$

"If we change q then Right side is not change so left side should also equating the power

$$2\alpha - 1 = 0$$

$$\boxed{\alpha = \frac{1}{2}}$$

from (1)

$$-\alpha \beta q = 1$$

$$-\frac{1}{2}\beta = 1$$

$$\boxed{\beta = -2}$$

* Generating function ⇒ It is a function of one new variable and one old variable which puts a condition on transformation in phase space to make it canonical.

If $F(q_i, Q_i)$ then it is related to co-ordinates as -

$$p_i = \frac{\partial F}{\partial q_i}(q_i, Q_i)$$

$$P_i = -\frac{\partial F(q_i, Q_i)}{\partial Q_i}$$

$$\boxed{\begin{matrix} q, Q \\ p, Q \\ p, P \\ q, P \end{matrix}}$$

* Legendre Transformation (Mathematics) :-

This is a most general transformation.

It changes one variable into other variable due to which form of the function changes.

If there is a function which depends on the n.

$$x \longrightarrow s$$

$$F(x) \longrightarrow G(s)$$

$$\boxed{G(s) = F(x) \pm xs}$$ ← This is Legendree transformation.

Here in mathematics + or − ve sign are taken by maxima or minima. we want $G(s)$ is maximum. So when $F(x)$ is −ve so xs is also −ve when $F(x)$ is +ve then xs is taken as positive.
But in Physics the consideration of +ve or −ve sign's concepts are different from mathematics. which is specified below.

* Generating functions :-

$$q_i \longrightarrow p_i$$

$$F_1(q_i, Q_i) \longrightarrow F_2(p_i, Q_i) \quad \{\text{Legendre Trans.}\}$$

$$F_2(p_i, Q_i) \longrightarrow F_1(q_i, Q_i) \pm q_i p_i$$

Diff. w.r. to q_i which is not present in F_2

$$0 = \frac{\partial F_1}{\partial q_i} \pm p_i$$

we will get previous relation $\left(p_i = \frac{\partial F}{\partial q_i}\right)$ if we use $(-)$ve sign.

i.e. ⇒ $$\boxed{F_2(p_i, Q_i) \longrightarrow F_1(q_i, Q_i) - q_i p_i}$$

Similarly, $\boxed{F_2(p_i, Q_i) = \underline{F_1(q_i Q_i)} - q_i p_i}$

If we put $F_1(q, Q) = F_2(P_i, Q_i) + q_i p_i$ in the relation.

$$P_i \dot{Q}_i - H' = p_i \dot{q}_i - H + \frac{d F_1}{dt}(q, Q)$$

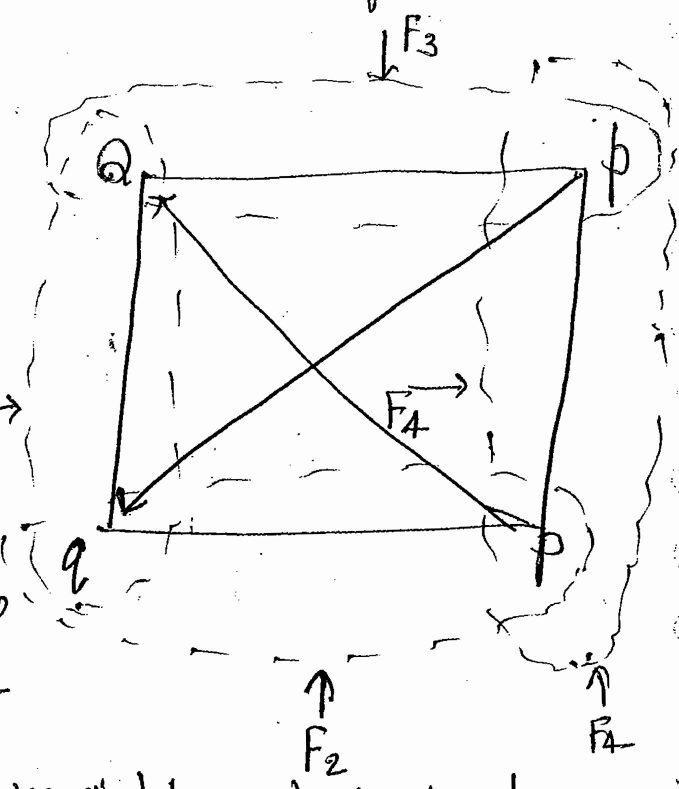
then we get relation for F_2.

Short Trick :- (No physical background)

It is a method to guess relation for different Generating function.

Steps :-

(I) Draw a rectangle $Q q P p$



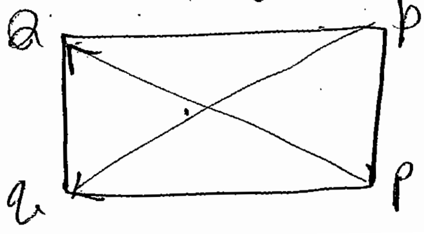
(II) Choose any two variable side wise.

(III) Relation for Generating function can be obtained as follows -

(a) Draw an arrow from the variable which is to be obtained towards diagonally opposite variable
If arrow points up use negative sign
If arrow points down use positive sign.

Ex- $F_1 = F_1(q, Q)$

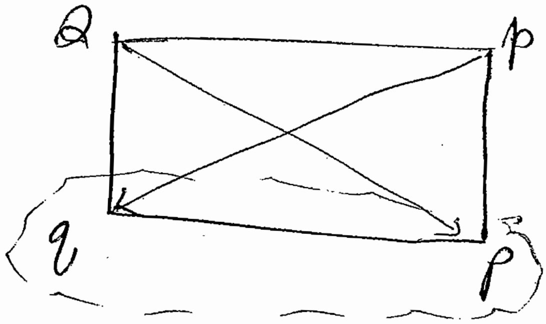
$p_i = \frac{\partial F_1}{\partial q_i}$, $P = -\frac{\partial F_1}{\partial Q_i}$



* Relation for $F_2(q,P)$:-

$$p = \frac{\partial F_2}{\partial q}$$

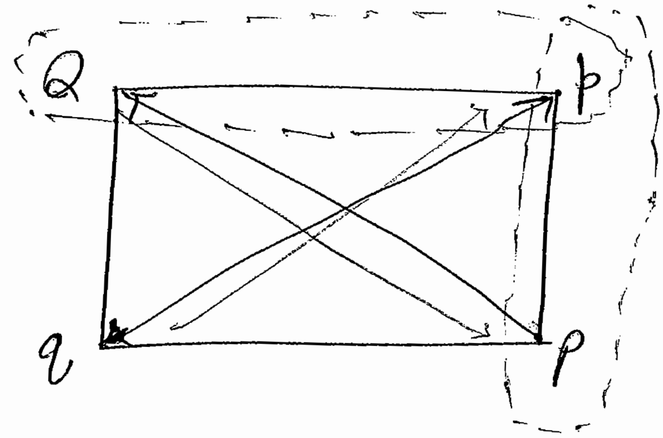
$$Q = \frac{\partial F_2}{\partial P}$$



(*) Relation for $F_3(p,Q)$:-

$$q = -\frac{\partial F_3}{\partial p}$$

$$P = -\frac{\partial F_3}{\partial Q}$$



$$Q = \frac{\partial F_4}{\partial P}$$

$$q = -\frac{\partial F_4}{\partial p}$$

---

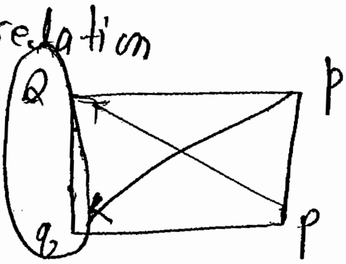
*Imp How to obtain generating function from a given transformation :-

① If we have to obtain $f(q,Q)$

② Then use its relation

$$p = \frac{\partial F}{\partial q}$$

$$P = -\frac{\partial f}{\partial Q}$$



③ Express L.H.S. of above relation in trems of chosen variables by using given transformation relation.

④ Integrate the relation for F.

⑤ Combine the two value of f obtain (write common ter once only).

BA-3

Q.96

Soln

$$(q, p) \longrightarrow (Q, P)$$

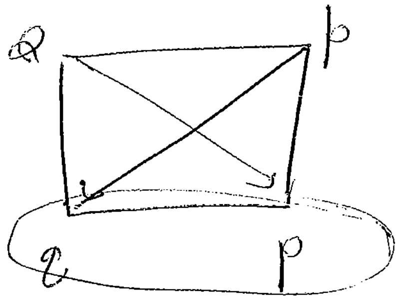
$$Q = q^2$$

$$P = \frac{p}{2q}$$

We have to find $F(q, P)$

$$Q = \frac{\partial F}{\partial P} \quad \text{— (i)}$$

$$p = \frac{\partial F}{\partial q} \quad \text{— (ii)}$$



from (i)

$$q^2 = \frac{\partial F}{\partial P}$$

$$F = q^2 P + G_1(q) \quad \text{— (iii)}$$

from (2)

$$p = \frac{\partial F}{\partial q}$$

$$2Pq = \frac{\partial F}{\partial q}$$

Integrate :-

$$F = q^2 P + G_2(P) \quad \text{— (iv)}$$

Using (iii) and (iv)

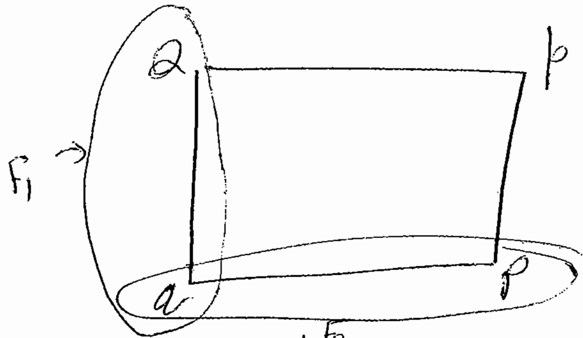
$$\boxed{F = q^2 P}$$

Ans

Q. 2

Solⁿ Identity Transformation -

$$Q = q$$
$$P = p$$



$f_1(q, Q)$

$p = \frac{\partial F(q,Q)}{\partial q}$

$P = -\frac{\partial F}{\partial Q}$

Here we can not F_2 express in terms of choosen variable.
So we can not solve it -

$$p = \frac{\partial F_2(q,P)}{\partial q}$$

$$Q = \frac{\partial F_2}{\partial P}$$

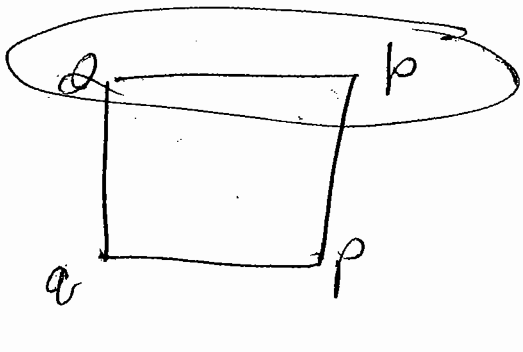
Q. Which of the following G.F. gives identity transformation

$F_1 = qP$, $F_2 = -Qp$

(a) Only F_1 (b) only F_2 (c) both (d) non

Solⁿ $P = \frac{-\partial F_2}{\partial Q_1} = p$

$q = -\frac{\partial F_2}{\partial p} = Q$



Ques 6

Soln $Q_i = p_i \tan t$

$P_i = q_i \tan t$

We have to find $F(q_i, Q_i)$

$$p_i = \frac{\partial F}{\partial q_i} \longrightarrow \frac{Q_i}{\tan t} = \frac{\partial F}{\partial q_i} \Rightarrow f = \frac{Q_i q_i}{\tan t} + C(Q)$$

$$P_i = -\frac{\partial F}{\partial Q_i} \longrightarrow \frac{\partial F}{\partial Q_i} = -q \tan t \Rightarrow f = -Q q_i \tan t + C(q_i)$$

$$\therefore F = \frac{Q_i q_i}{\tan t} + Q_i q_i \tan t$$

$$= Q_i q_i \left[\frac{\cos t}{\sin t} - \frac{\sin t}{\cos t}\right]$$

$$= 2 Q_i q_i \frac{[\cos 2t]}{2 \sin t \cos t}$$

* Rotation in phase space :-

$$(q, p) \longrightarrow (Q, P)$$

$$\begin{pmatrix} Q \\ P \end{pmatrix} = R \begin{pmatrix} q \\ p \end{pmatrix}$$

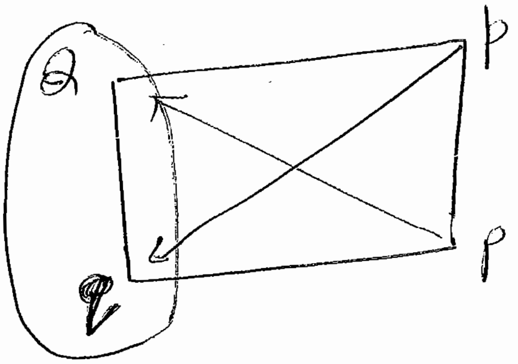
$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

$$Q = q\cos\theta - p\sin\theta$$
$$P = q\sin\theta + p\cos\theta$$
Transformation Relation

We have to find $F(q,Q)$:-

$$p = \frac{\partial F}{\partial q}$$

$$P = -\frac{\partial F}{\partial Q}$$



from (1) relation -

$$\frac{q\cos\theta - Q}{\sin\theta} = \frac{\partial F}{\partial q}$$

$$F = \frac{q^2}{2}\cot\theta - qQ\,\mathrm{cosec}\,\theta + c_1(Q)$$

from (2) relation -

$$P = \frac{-\partial F}{\partial Q}$$

$$-q\sin\theta - \cos\theta\left[\frac{q\cos\theta - \theta}{\sin\theta}\right] = \frac{-\partial F}{\partial Q}$$

$$\Rightarrow c_2(q) - qQ\sin\theta - \cot\theta\left[qQ\cos\theta - \frac{Q^2}{2}\right] = 0$$

$$\Rightarrow -qQ\sin\theta + \cot\theta\left[qQ\cot\theta - \frac{Q^2}{2}\right] + c_2(q).$$

$$\Rightarrow -qQ\left[\sin\theta + \frac{\cos^2\theta}{\sin\theta}\right] + \frac{Q^2}{2}\cot\theta + c_2(q).$$

$$\boxed{F = -qQ\,\mathrm{cosec}\,\theta + \frac{1}{2}(q^2 + Q^2)\cot\theta}$$

Ans.

H-12

Q.5

Soln Given $H = \frac{p^2}{2m} + mgq$

$$F(q, Q, t) = \frac{1}{3m^2 g}\left[2m^2 g(Q-q)\right]^{3/2} = \frac{1}{3m^2 g}(2m^2 g)^{3/2} \times (Q-q)^{3/2}$$

Here F is not depends on time explicitely.

$$H'(P, Q) = H(p, q) + \left(\frac{\partial F}{\partial t}\right)_{=0}$$ ['F' is not explicit function of t]

So $\boxed{H' = H}$

$$H = (P, Q) = \frac{p^2}{2m} + mgq$$

Express p & q in new variable

Relation of G.F. :-

$$p = \frac{\partial F}{\partial q} = \frac{-1}{3m^2 g}(2m^2 g)^{3/2}\frac{3}{2}(Q-q)^{1/2}(-1)$$

$$P = \frac{-\partial F}{\partial q} = +\frac{1}{3m^2 g}(2m^2 g)^{3/2}\frac{3}{2}(Q-q)(1)$$

So $\boxed{p = P}$

$$P = \frac{1}{\cancel{3}m^2 g}(2m^2 g)^{3/2}\frac{\cancel{3}}{2}(Q-q)^{1/2}$$

Squairing on both side :-

$$P^2 = \frac{1}{4a^2 g^2} \cdot 8m^6 g^3(Q-q).$$