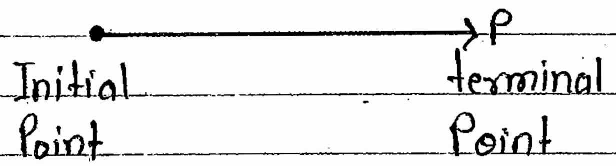
⁕ MATHEMATICAL PHYSICS ⁕

Vectors

Physical quantities having both magnitude and direction and which follow the vector law of addition are known as vectors.
e.g. velocity, acceleration, force, displacement etc.



$$\vec{OP} = \underbrace{|OP|}_{\text{mag.}} \; \underbrace{\hat{P}}_{\text{Dir}^n}$$

$\hat{OP}$ = Unit Vector along OP

Polar Vectors :-
The vector associated with a linear directional effect are called polar vectors. The examples of polar vectors are force, acceleration, linear velocity, ~~angular~~ linear momentum etc.

Axial Vectors :-
The vectors associated with rotation about an axis are called axial vectors. The examples of axial vectors are: torque, angular velocity, angular momentum etc.

Null Vector :-
Here magnitude is zero, hence initial and terminal point coincide.
Direction = orbitary direction
Displacement vector = Null vector

Reciprocal Vector :-

∵ We know -

$$\vec{a} = |\vec{a}|\,\hat{a}$$

So,

$$\boxed{\vec{a}^{\,-1} = \frac{1}{|\vec{a}|}\,\hat{a}}$$

Negative Vector :-

$$\boxed{-\vec{a} = |\vec{a}|\,(-\hat{a})}$$

Coplanar and Non-Coplanar Vectors :-

Three or more vectors are said to be coplanar when they are parallel to the same plane. A plane parallel to the system of coplanar vectors is said to be the plane of vectors. Three or more vectors which are not parallel to common plane are said to be non-coplanar.

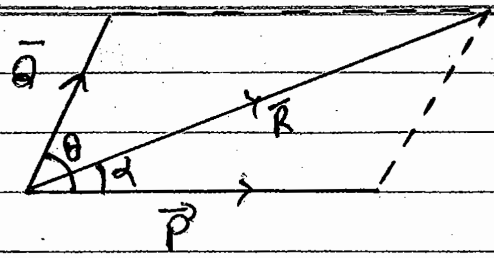
Unit Vectors :-

"A vector with unit scalar magnitude is defined as unit vector."

Let a vector be $x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{Unit Vector} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\boxed{R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}}$$



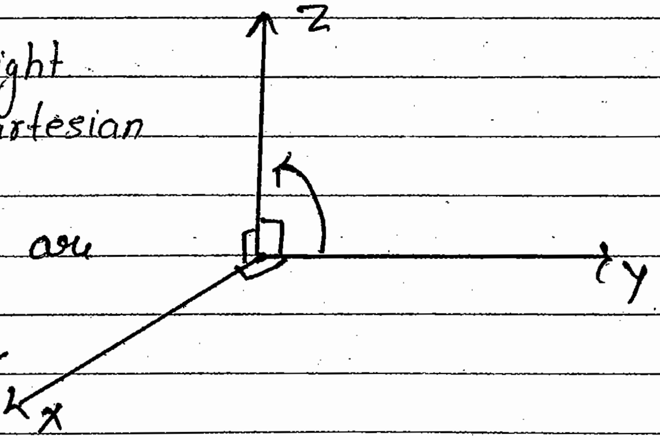
$$\boxed{\sin\alpha = \frac{Q\sin\theta}{R}}$$

* Co-Ordinate System :-

1. Right Handed Cartesian System :-

⇒ 3-D ~~Rectangul~~ Right handed Rectangular cartesian coordinate system.

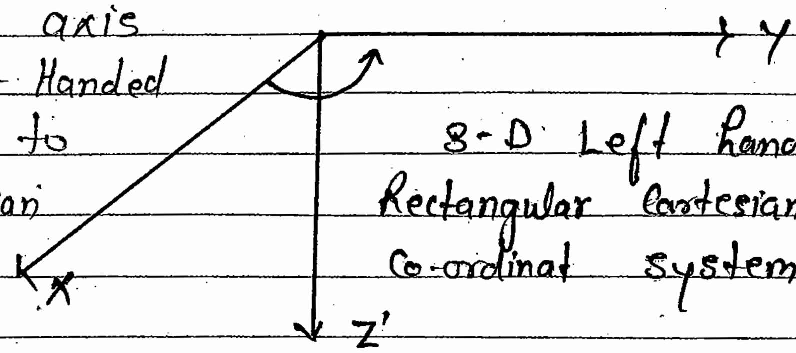
⇒ Here x, y, z axis are mutually perpendicular so it is rectangular



⇒ The orientation of x, y, z axis follow Right Handed Thumb Rule, so it is Right Handed.

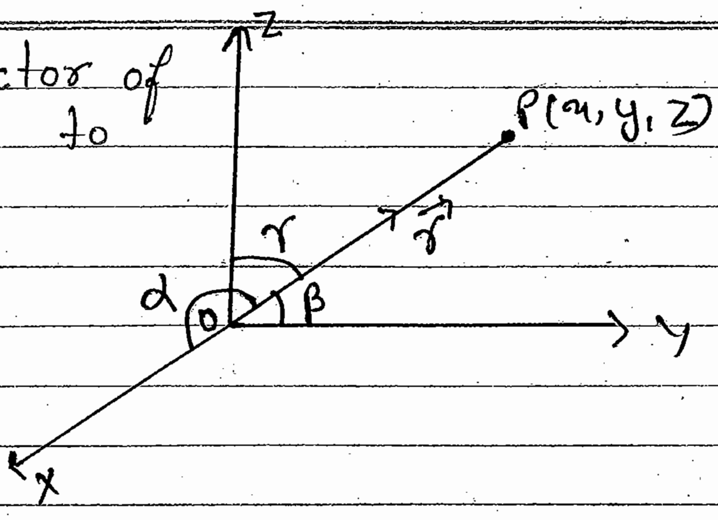
2. Left Handed Co-ordinate system :-

Invert any one axis to convert Right Handed Cartesian system to Left Handed cartesian system.



3-D Left handed Rectangular cartesian Co-ordinat system.

→ It follow Left Hand Thumb Rule.



$\vec{OP} = \vec{r}$ = Position Vector of point P with respect to origin O.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\boxed{r = \sqrt{x^2 + y^2 + z^2}}$$

Where $\hat{i}, \hat{j}, \hat{k}$ → Orthogonal basis

Note :-

$r = \sqrt{x^2+y^2+z^2}$, This formula is applicable only when $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular.

Unit Vector Along $\vec{r}$:-

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

So

$$\boxed{\hat{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}}$$

* Direction Cosines of $\vec{r}$:-

If α, β, γ is the angle, which $\vec{r}$ making with X, Y, Z axis, then -

$l = \cos\alpha = \frac{x}{r}$

$m = \cos\beta = y/r$

$n = \cos\gamma = z/r$

$$\boxed{l^2 + m^2 + n^2 = 1}$$

So $\boxed{\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}}$

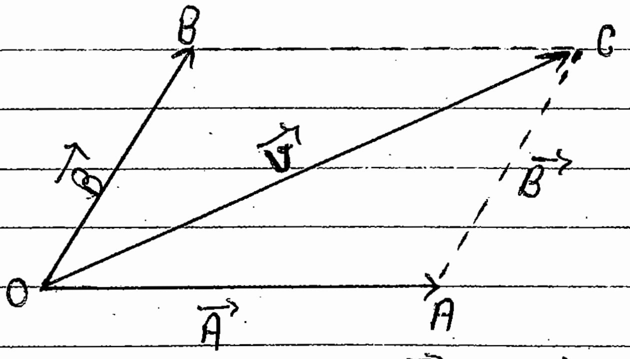
$$\boxed{\hat{r} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}}$$

Where l, m, n are the direction cosines.

* Addition of Vectors :-

The composition of two displacements of a point has all the characteristics of a summation.

Let us consider two vectors $\vec{A}$ and $\vec{B}$ acting on a point O as shown in figure (a). The resultant effect of the vectors $\vec{A}$ and $\vec{B}$ is same as that the their vector sum $\vec{V}$. The vector $\vec{V}$ is obtained by setting off the vector $\vec{B}$ at the end of $\vec{A}$ and drawing the vector $\vec{V}$ joining the begining of $\vec{A}$ to the end of $\vec{B}$.



So $\vec{V} = \vec{OC} = \vec{OA} + \vec{AC}$

$$\boxed{\vec{V} = \vec{A} + \vec{B}}$$

A similar result is obtained by starting with $\vec{B}$ and setting of the vector $\vec{A}$ on $\vec{B}$.

So,

$$\vec{V} = \vec{OC} = \vec{OB} + \vec{BC} = \vec{B} + \vec{A}$$

$$\boxed{\vec{A} + \vec{B} = \vec{B} + \vec{A}}$$

Which shows that the sum of two vectors is the diagonal of the parallelogram with the vectors as its sides and this sum obeys the law of commutation according to which the sum of two quantities is independent of the order of the quantities.

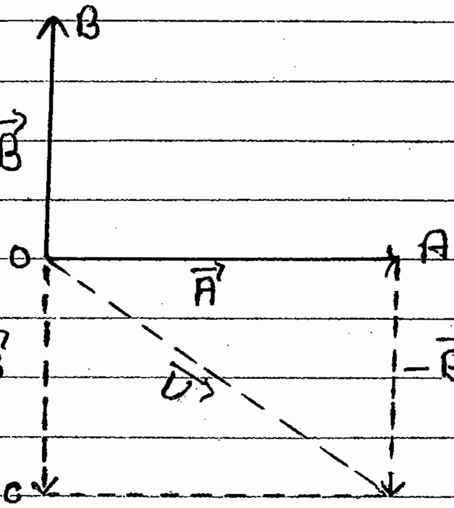
* Subtraction of Vectors:-

The sum of two vectors is zero when and only when they have the same lengths but opposite directions,

$$\vec{A} + \vec{B} = 0$$

When,

$$\vec{A} = -\vec{B}$$



Thus the negative of a vector is a vector of the same length but opposite direction $-\vec{B}$

So,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Which shows that subtracting a vector $\vec{B}$ from vector $\vec{A}$ is same as adding its negative.

* Product of Vectors:-

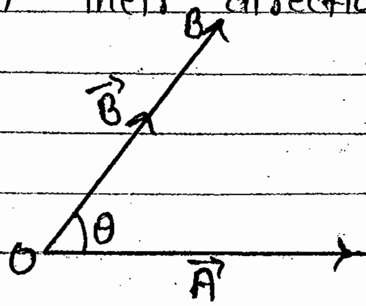
Two vectors are ~~multiplied~~ multiplied in two manners, i.e. scalarly and vectorily. These products are discussed seperately.

(a) The Scalar Product

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of vectors and the cosine of the angle between their directions,

Thus,

$$\vec{A}.\vec{B} = AB\cos\theta = OA.OB\cos\theta$$



Thus, scalar product of two vectors is the product of

the size of one vector with the component of the other in the direction of the first.
Moreover,

$$\vec{B}.\vec{A} = BA\cos(-\theta) = BA\cos\theta = A.B$$

Hence the scalar product of two vectors is commutative, i.e. independent of the order of vectors.

$$\because \quad \vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

Such that if -

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

Then

$$\vec{A}.\vec{B} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

Properties of $\hat{i}, \hat{j}, \hat{k}$:-

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} \ldots\ldots = 1$$
$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \ldots\ldots = 0$$

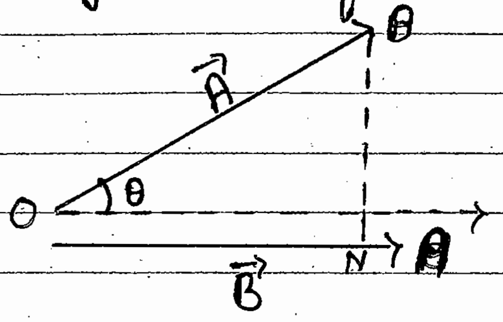
$$\langle \psi_1 | \psi_2 \rangle = 1 \longrightarrow \text{Normalization}$$

$$\langle \psi_1 | \psi_2 \rangle = 0 \longrightarrow \text{Orthogonal}$$

* Application of Dot Product :-

(i) $$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}$$

(2) Projection of $\vec{A}$ on $\vec{B}$



Complea of $\vec{A}$ along $\vec{B}$

$= \vec{A} \cdot \hat{B}$

$= |\vec{A}| |\hat{B}| \cos\theta$

$\cancel{= A \cos\theta}$

OR.

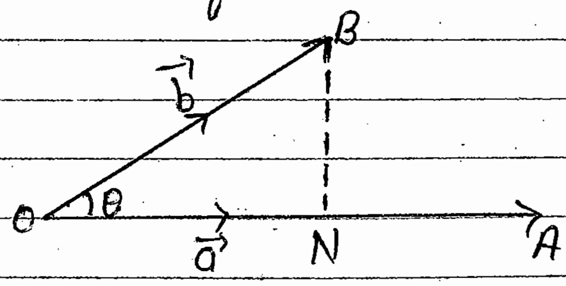
The scalar product of two vectors is the product of one vector and the length of the projection of the other in the direction of the first.

$\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

then,

$\vec{a} \cdot \vec{b} = (OA).(OB) \cos\theta$

$= (OA).(OB).\frac{(ON)}{(OB)}$

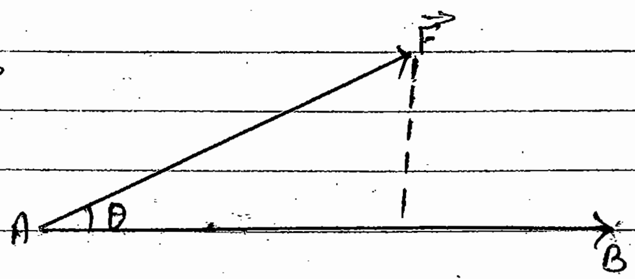


$= (OA).(ON)$

$= (\text{Length of } \vec{a})(\text{projection of } \vec{b} \text{ along } \vec{a})$.

(3) Work done as a Scalar Product :-

If a constant force $\vec{F}$ acting on a particle displaces it from A to B then,



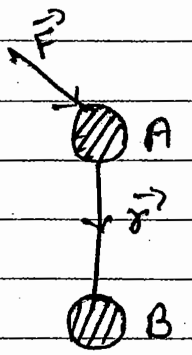
Workdone = (Component of $\vec{F}$ along AB). Displacement

$= F \cos\theta . AB$

$= \vec{F} \cdot \vec{AB}$

Workdone = Force . Displacement.

OR



Then $\boxed{W = \vec{F}.\vec{r}}$

(4) $d\phi = \vec{E}.d\vec{s}$ [Electric Flux]

(5) $d\phi = \vec{B}.d\vec{s}$ [Magnetic Flux]

CSIR June-2013

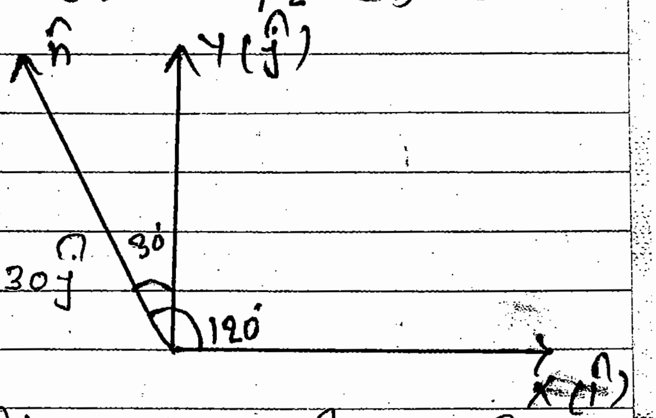
Ques A unit vector $\hat{n}$ on the xy-plane is an angle of 120° with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60°, find the relation between b and a.

(a) $b = \sqrt{3}a/2$ (b) $b = 2a/\sqrt{3}$ (c) $b = a/2$ (d) $b = a$

Sol^n

$\vec{u} = a\hat{i} + b\hat{n}$

$\vec{v} = a\hat{n} + b\hat{i}$



$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} = \cos 120\,\hat{i} + \cos 30\,\hat{j}$

$$\boxed{\hat{n} = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}}$$

then $\vec{u} = a\hat{i} + b\left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = \left(a - \frac{b}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}b\,\hat{j}$

$\vec{v} = a\left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) + b\hat{i} = \left(b - \frac{a}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}a\,\hat{j}$

$$\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$\left(a - \frac{b}{2}\right)\left(b - \frac{a}{2}\right) + \frac{3}{4}ab = \left\{\sqrt{\left(a - \frac{b}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2}\right\}\cdot\left\{\sqrt{\left(b - \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2}\right\} \times \cos 60^\circ$$

$$\left\{\left(a - \frac{b}{2}\right)\left(b - \frac{a}{2}\right) + \frac{3}{4}ab\right\}^2 = \left\{\left(a - \frac{b}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2\right\}\left\{\left(b - \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2\right\} \times \frac{1}{2}$$

$$\Rightarrow \cancel{(a-\frac{b}{2})}(b-\frac{a}{2}) + \cancel{(\frac{3}{4}ab)} + 2(a-\frac{b}{2})(b-\frac{a}{2})(\frac{3}{4}ab) = \Big[\cancel{(a-\frac{b}{2})}(b-\frac{a}{2}) + (a-\frac{b}{2})(\frac{\sqrt{3}}{2}a)^2$$
$$+ (\frac{\sqrt{3}}{2}b)^2(b-\frac{a}{2})^2 + \cancel{(\frac{3}{4}ab)^2}\Big]^{\frac{1}{2}}$$

$\Rightarrow 4$

<u>Ques</u> If the projection of the vector $\vec{A} = \hat{i} + m\hat{j} + \hat{k}$ on the vector $\vec{B} = 4\hat{i} - 4\hat{j} + 7\hat{k}$ is equal to 19/9 then the value of m is equal to?

(a) 1 (b) −1 (c) 2 (d) −2.

Soln: ∵ We know -

Projection of A along B = $\vec{A} \cdot \hat{B}$

And $\vec{A} = \hat{i} + m\hat{j} + \hat{k}$

$$\hat{B} = \frac{4\hat{i} - 4\hat{j} + \hat{k}}{\sqrt{16+16+49}}$$

$$(\hat{i} + m\hat{j} + \hat{k}) \cdot \frac{4\hat{i} - 4\hat{j} + 7\hat{k}}{\sqrt{16+16+49}} = \frac{19}{9}$$

$$\frac{4\hat{i}.\hat{i} - 4m\hat{j}.\hat{j} + 7\hat{k}.\hat{k}}{\cancel{\sqrt{81}}} = \frac{19}{\cancel{9}}$$

$$4\hat{i}.\hat{i} - 4m\hat{j}.\hat{j} + 7\hat{k}.\hat{k} = 19$$

$$4 - 4m + 7 = 19 \qquad \{\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1\}$$

$$11 - 4m = 19$$

$$-4m = 19 - 11 = 8$$

$$\boxed{m = -2}$$

(d) is correct.

Ques The value of constant 'm' for which $\vec{A} = m\hat{i} + \hat{j} + \sqrt{5}\hat{k}$ substends an angle $60°$ with $\vec{B} = 4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k}$ is equal to ?

Soln

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}| \cos 60°$$

$$(m\hat{i} + \hat{j} + \sqrt{5}\hat{k}).(4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k}) = \sqrt{m^2+1+5}\sqrt{16+25+5} \times \tfrac{1}{2}$$

$$\Rightarrow 4m\hat{i}.\hat{i} - \cancel{5\hat{j}.\hat{j}} + \cancel{5\hat{k}.\hat{k}} = \sqrt{m^2+6}\sqrt{46}\ \frac{1}{2}$$

$$8m = \sqrt{m^2+6}\sqrt{46} \qquad \{\hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1$$

On squaring both sides -

$$64m^2 = (m^2+6).46$$

$$= 46m^2 + 276$$

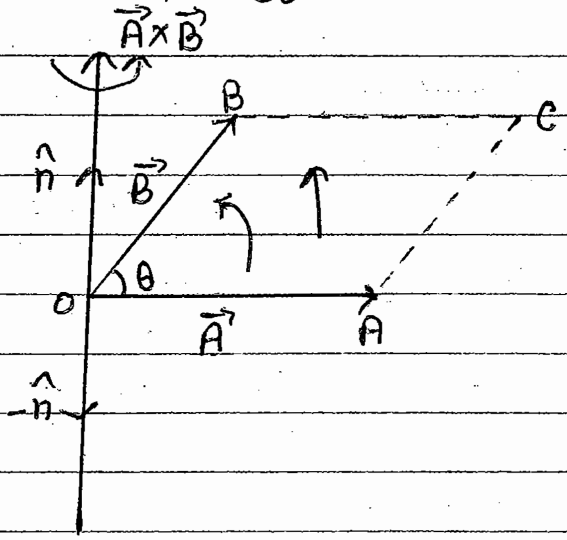
$$(64-46)m^2 = 276$$

$$m^2 = \frac{\cancel{276}\ 46}{\cancel{18}\ 3}$$

So $\boxed{m = \sqrt{\frac{46}{3}}}$

* The Vector Product or Cross Product :-

The vector product of two vectors is a vector quantity having magnitude and direction both.



(i) Its magnitude is $|\vec{A}||\vec{B}| \sin\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$

(ii) Its direction is perpendicular to both vectors $\vec{A}$ and $\vec{B}$.

(iii) It forms with a right handed system.

Let $\hat{n}$ be a unit vector perpendicular to the both vectors $\vec{A}$ and $\vec{B}$. Then,

$$\vec{V} = \vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta . \hat{n} = AB \sin\theta . \hat{n} \quad \text{— (I)}$$

The sense of this vector $\vec{V}$ along this perpendicular is defined by the right-hand screw rule according to which if the vector $\vec{A}$ is turned towards $\vec{B}$ through a small angle, then the right hand screw will proceed in the +ve direction of $\vec{V}$ by the rotation. Then,

$$\vec{B} \times \vec{A} = BA \sin(-\theta) . \hat{n} = -AB \sin\theta . \hat{n} \quad \text{— (II)}$$

Thus

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \text{—— (III)}$$

$\vec{A} \times \vec{B}$ has scalar magnitude equal to the area of the parallelogram OACB and is not commutative shown by equation (III).

* Useful Results :-

Since $\hat{i}, \hat{j}, \hat{k}$ are three mutually perpendicular unit vectors, then-

$$\cancel{\hat{i}\times\hat{i}} \quad \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$

$$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$$

$$\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$$

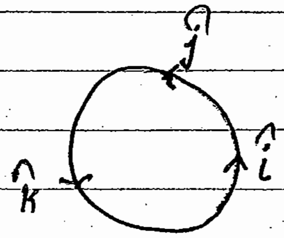
$$\hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$$

and,

$$\hat{j}\times\hat{i} = -\hat{i}\times\hat{j}$$

$$\hat{k}\times\hat{j} = -\hat{j}\times\hat{k}$$

$$\hat{i}\times\hat{k} = -\hat{k}\times\hat{i}$$



* Vector Product Expressed As a Determinant :-

If,

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

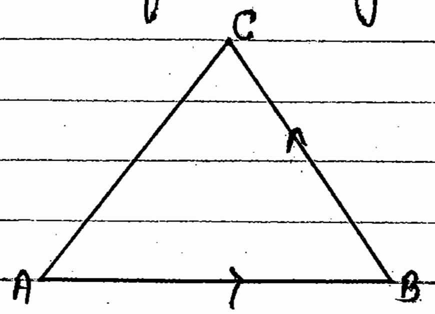
and $$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

Then

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

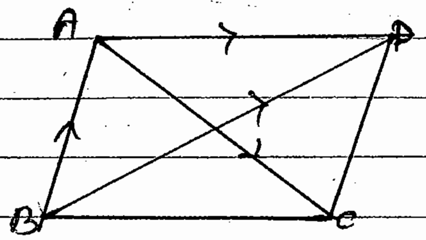
* Applications :-

1. Area of Triangle :-



$$\text{Area} = \frac{1}{2}\left|\vec{AB}\times\vec{BC}\right|$$

2. Area of Parallelogram :-



$$\text{Area} = \frac{2\times 1}{2} |\vec{AB} \times \vec{AD}|$$

$$= \frac{1}{2} |\vec{AC} \times \vec{BD}|$$

Ques Find the area of a parallelogram whose adjacent sides are $\hat{i} - 2\hat{j} + 3\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$

Sol^n Let $\vec{AB} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{AD} = 2\hat{i} + \hat{j} + 4\hat{k}$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix}$$

$$= \hat{i}(8-3) - \hat{j}(-4-6) + \hat{k}(1+4)$$

$$= 5\hat{i} + 10\hat{j} + 5\hat{k}$$

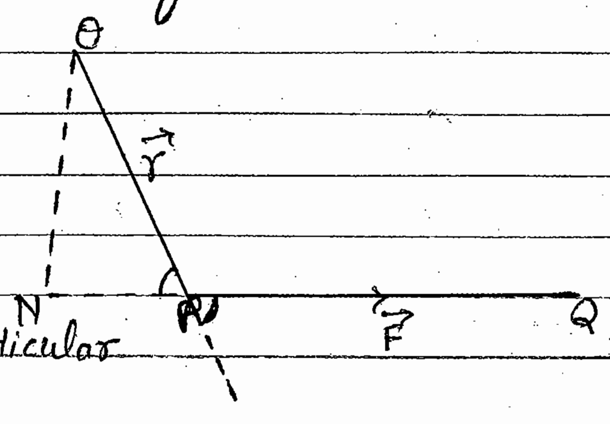
So

$$|\vec{AB} \times \vec{AD}| = \sqrt{(5)^2 + (10)^2 + (5)^2} = 5\sqrt{6}$$

So Area of Parallelogram =

3. Moment of a force OR Torque :-

Let a force $\vec{F}(\vec{PQ})$ act at a point P.



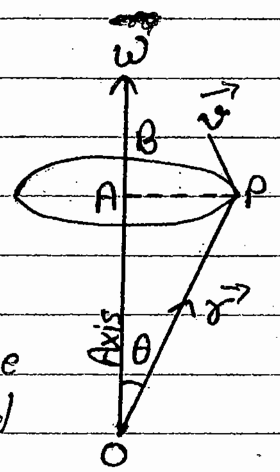
Moment of $\vec{F}$ about O =
Product of force $\vec{F}$ and perpendicular distance $(ON.\hat{n})$

$= (PQ)(ON)(\hat{n}) = (PQ)(OP)\sin\theta\,(\hat{n})$

$= \vec{OP} \times \vec{PQ}$

$\Rightarrow \quad \boxed{\tau = \vec{r} \times \vec{F}}$

4. Angular Velocity :-



Let a rigid body be rotating about the axis OA with the angular velocity $\vec{\omega}$ which is a vector and it's magnitude is ω radians per second and its direction is parallel to the axis of rotation OA.

Let P be any point on the body such that $\vec{OP} = \vec{r}$ and $\angle AOP = \theta$ and $AP \perp OA$.

Let the velocity of P be $\vec{v}$.

Let $\hat{n}$ be a unit vector perpendicular to $\vec{\omega}$ and $\vec{r}$.

$$\vec{\omega} \times \vec{r} = (r\omega \sin\theta)\,\hat{n} = (\omega AP)\,\hat{n}$$

$$= (\text{speed of } P)\,\hat{n}$$

$$= \text{Velocity of } P \perp \text{ to } \vec{\omega} \text{ and } \vec{r}$$

Hence, $\boxed{\vec{V} = \vec{\omega} \times \vec{r}}$

5. $\vec{B} = \vec{\nabla} \times \vec{A}$

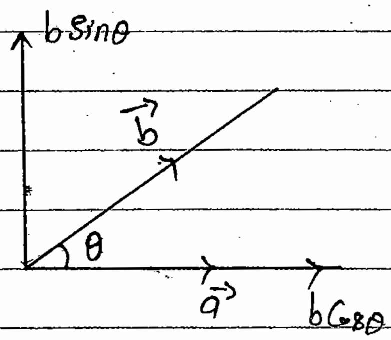
6. $\vec{S} = \vec{E} \times \vec{H}$ Poynting Vector.

C.S.I.R. June - 2011

Ques Let $\vec{a}$ and $\vec{b}$ be two distinct three-dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by -

(a) $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$ (b) $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{b^2}$ (c) $\frac{(\vec{a}\cdot\vec{b})\vec{b}}{b^2}$ (d) $\frac{(\vec{b}\cdot\vec{a})\vec{a}}{a^2}$

Sol^n Let two vectors $\vec{a}$ and $\vec{b}$ which are situated at an angle θ as shown in figure.



Now we check from options-
firstly we check option ⓐ

$$\frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$$

$$= \frac{\vec{a}\times[ba\sin\theta\,\hat{n}]}{a^2}$$

$$= \frac{ba\sin\theta\,(\vec{a}\times\hat{n})}{a^2}$$

$$= \frac{(b\cancel{a}\sin\theta\,(a.1\ \sin 90^\circ)}{\cancel{a^2}}$$

$= b\sin\theta$ which is perpendicular to $\vec{a}$

So option (a) is correct.

* Product of Three vectors :-

(A) Scalar Triple Product :-

$$\vec{a}\cdot[\vec{b}\times\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\vec{a}\cdot[\vec{b}\times\vec{c}] = \vec{b}\cdot[\vec{c}\times\vec{a}] = \vec{c}\cdot[\vec{a}\times\vec{b}]$$

Thus,

$$[\vec{A}\vec{B}\vec{C}] = \vec{A}\cdot(\vec{B}\times\vec{C})$$

$$= (A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= A_x[B_yC_z - C_yB_z] + A_y[C_xB_z - C_zB_x] + A_z[B_xC_y - B_yC_x]$$

or

$$[\vec{A}\vec{B}\vec{C}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = [\vec{C}\vec{A}\vec{B}] =$$

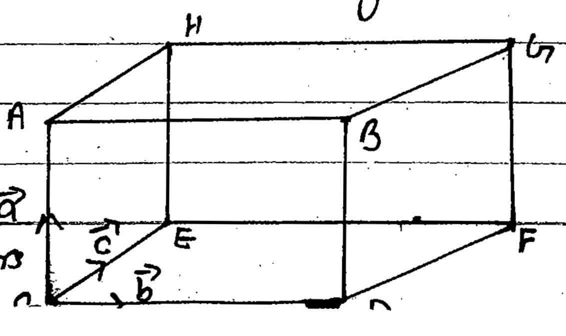
$$\begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix} = [\vec{B}\vec{C}\vec{A}]$$

Thus we have $[\vec{A}\vec{B}\vec{C}] = [\vec{C}\vec{A}\vec{B}] = [\vec{B}\vec{C}\vec{A}]$

or $\vec{A}\cdot[\vec{B}\times\vec{C}] = \vec{C}\cdot[\vec{A}\times\vec{B}] = \vec{B}\cdot[\vec{C}\times\vec{A}]$

Thus dot and cross may be inter-changed at will.

$\vec{A}\cdot(\vec{B}\times\vec{C})$ or the scaler triple product of three vectors is equal to the Volume of parallelopiped having vectors as concurrent edges



Let three vectors $\vec{a}, \vec{b}, \vec{c}$ are taken on a parallelo-piped having vectors $\vec{a}, b, \vec{c}$ as concurrent edges.

Then,

$$[\vec{a}\,\vec{b}\,\vec{c}\,] = \vec{a}.[\vec{b}\times\vec{c}\,]$$

So $$V = \left|\underset{\text{hight}}{\vec{a}}.[\underset{\text{base Area}}{\vec{b}\times\vec{c}}\,]\right| \qquad \left\{\text{Volume} = \text{hight} \times \text{Base Area.}\right\}$$

When the three vectors lie in a plane the volume of the parallelopiped is zero. hence, for three vectors to be coplanar, their scalar product vanishes.

[B] Vector Triple Product :-

$\vec{A}\times(\vec{B}\times\vec{C})$ is called vector triple product.

Since the vector $(\vec{B}\times\vec{C})$ is normal to the plane containing $\vec{B}$ and $\vec{C}$. Likewise, the vector $\vec{A}\times(\vec{B}\times\vec{C})$ is normal to the plane containing $\vec{A}$ and $(\vec{B}\times\vec{C})$, i.e. in the same plane as $\vec{B}$ and $\vec{C}$.

let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times[(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\times(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})]$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times[(b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}]$$

$$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\hat{i} + [a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\hat{j} + [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\hat{k}$$

$$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

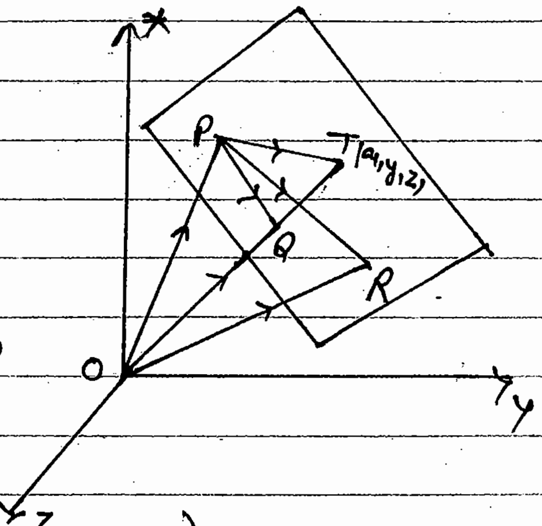
$$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}.$$ Ans

A-1

2.15 The equation of the plane that contains the points $P(2,-1,1)$, $Q(3,2,-1)$, $R(-1,3,2)$ is equal to?

(a) $11x+5y+13z=30$ (b) $11x+7y+9z=30$

(c) $4x+7y-9z=30$ (d) $4x+11y-3z=30$



Sol^n

$\vec{P} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{Q} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\vec{R} = -\hat{i} + 3\hat{j} + 2\hat{k}$

$$[\vec{P},\vec{Q},\vec{R}] = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} \neq 0$$

$\vec{OP}\cdot(\vec{OQ}\times\vec{OR}) \neq 0$ (not always $=0$)

So

$$\vec{PT}\cdot(\vec{PQ}\times\vec{PR}) = 0$$

$\vec{PT} = \vec{OT} - \vec{OP} = x\hat{i} + y\hat{j} + z\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= (x-2)\hat{i} + (1+y)\hat{j} + (z-1)\hat{k}$

$\vec{PQ} = \vec{OQ} - \vec{OP} = (3-2)\hat{i} + (2+1)\hat{j} + (-1-1)\hat{k}$

$= \hat{i} + 3\hat{j} - 2\hat{k}$

$\vec{PR} = \vec{OR} - \vec{OP} = (-1-2)\hat{i} + (3+1)\hat{j} + (2-1)\hat{k}$

$= -3\hat{i} + 4\hat{j} + \hat{k}$.

$P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$, $R(x_3,y_3,z_3)$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

$$= \hat{i}(3+8) - \hat{j}(1-6) + \hat{k}(4+9)$$

$$= 11\hat{i} + 5\hat{j} + 13\hat{k}$$

$$\{(x-2)\hat{i} + (1+y)\hat{j} + (z-1)\hat{k}\} . 11\hat{i} + 5\hat{j} + 13\hat{k} = 0$$

$$\Rightarrow 11(x-2) + 5(1+y) + (z-1)13 = 0$$

$$\Rightarrow 11x - 22 + 5 + 5y + 13z - 13 = 0$$

$$\Rightarrow 11x + 5y + 13z = 22 + 13 - 5$$

$$\Rightarrow 11x + 5y + 13z = 35 - 5$$

$$\Rightarrow \boxed{11x + 5y + 18z = 30}$$

Ans

Q.17 If $\vec{b} = \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$, then $\vec{b}$ can be simplified to ?

(a) 0 (b) $\vec{a}$ (c) $2\vec{a}$ (d) None of these.

Sol^n

$$b = \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j} + \hat{k} \times (\vec{a} \times \hat{k})$$

$$= \vec{a}(\hat{i}.\hat{i}) - \hat{i}(\hat{i}.\vec{a}) + \vec{a}(\hat{j}.\hat{j}) - \hat{j}(\hat{j}.\vec{a}) + \vec{a}(\hat{k}.\hat{k}) - \hat{k}(\hat{k}.\vec{a})$$

$$= 3\vec{a} - \vec{a}$$

$$= 2\vec{a}. \text{ Ans.}$$

Q.19 Three unit vectors $\vec{a}, \vec{b}, \vec{c}$ ($\vec{b}$ and $\vec{c}$ are not parallel) are such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c}$. The angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, respectively are ?

(a) $30°, 90°$ (b) $150°, 90°$ (c) $60°, 90°$ (d) $90°, 30°$

Sol^n

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,\underbrace{(\vec{a}.\vec{c})}_{\substack{\downarrow \\ \text{Coefficient} \\ \text{of } \vec{b}}} - \vec{c}\,\underbrace{(\vec{a}.\vec{b})}_{\substack{\downarrow \\ \text{Coefficient of } \vec{c}.}}$$

$\vec{a}.\vec{c} = 0 \quad \Rightarrow \quad \vec{a} \perp^{r} \vec{c} \; (90^\circ)$

$$-(\vec{a}.\vec{b}) = \frac{\sqrt{3}}{2}$$

$$ab \cos\theta = -\frac{\sqrt{3}}{2}$$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\cos\theta = \cos 150$$

$$\boxed{\theta = 150^\circ}$$

So $150^\circ, 90^\circ$ so option (b) is correct.

* SCALAR PRODUCT OF FOUR VECTORS:-

Prove the identity -

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$

Proof:- $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}).\vec{r} = \vec{a}.(\vec{b} \times \vec{r})$

dot and cross can be interchanged.

Put $\vec{c} \times \vec{d} = \vec{r}$

$= \vec{a}.[\vec{b} \times (\vec{c} \times \vec{d})] = \vec{a}.[(\vec{b}.\vec{d})\vec{c} - (\vec{b}.\vec{c})\vec{d}]$

$= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$

$$= \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$$

Proved

Q. If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{c} = \hat{i} + \hat{j} + \hat{k}$
find $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$

Sol^n

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = \hat{i}(-12+2) - \hat{j}(-8-1) + \hat{k}(4+3)$$

$$\vec{a} \times \vec{b} = -10\hat{i} + 9\hat{j} + 7\hat{k}$$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(3+1) - \hat{j}(2+1) + \hat{k}(2-3)$$

$$\vec{a} \times \vec{c} = 3\hat{i} - 3\hat{j} - \hat{k}$$

So $(\vec{a} \times b) \cdot (\vec{a} \times \vec{c}) = (-10\hat{i} + 9\hat{j} + 7\hat{k}) \cdot (4\hat{i} - 3\hat{j} - \hat{k})$

$$= -40 - 27 - 7$$
$$= -74 \quad \underline{\text{Ans}}$$

* VECTOR PRODUCT OF FOUR VECTORS :-

$$\boxed{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})}$$

Put $\vec{a} \times \vec{b} = \vec{r}$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{r} \times (\vec{c} \times \vec{d})$$
$$= (\vec{r} \cdot \vec{d})\vec{c} - (\vec{r} \cdot \vec{c})\vec{d}$$
$$= [(\vec{a} \times \vec{b}) \cdot \vec{d}]\vec{c} - [(\vec{a} \times \vec{b}) \cdot \vec{c}]\vec{d}$$
$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} \qquad (1)$$

$\therefore$ $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ lies in the plane of $\vec{c}$ and $\vec{d}$.

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{s} \qquad \text{put } \vec{c} \times \vec{d} = \vec{s}$$
$$= -\vec{s} \times (\vec{a} \times \vec{b})$$
$$= -(\vec{s} \cdot \vec{b})\vec{a} + (\vec{s} \cdot \vec{a})\vec{b}$$

$$= -[(\vec{c}\times\vec{d}).\vec{b}]\vec{a} + [(\vec{c}\times\vec{d}).\vec{a}]\vec{b}$$

$$= -[\vec{b}\vec{c}\vec{d}]\vec{a} + [\vec{a}\vec{c}\vec{d}]\vec{b}$$

$\therefore$ $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ lies in the plane of $\vec{a}$ and $\vec{b}$. — (II)

Geometrical Interpretation :-

From (1) and (2) we conclude that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is a vector parallel to the line of intersection of the plane containing $\vec{a},\vec{b}$ and plane containing $\vec{c},\vec{d}$.

Q. Show that -

$$(\vec{B}\times\vec{C})\times(\vec{A}\times\vec{D}) + (\vec{C}\times\vec{A})\times(\vec{B}\times\vec{D}) + (\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = -2(\vec{A}\vec{B}\vec{C})\vec{D}$$

Sol^n Proof :- $(\vec{B}\times\vec{C})\times(\vec{A}\times\vec{D}) + (\vec{C}\times\vec{A})\times(\vec{B}\times\vec{D}) + (\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})$

$$= [(\vec{B}\vec{C}\vec{D})\vec{A} - (\vec{B}\vec{C}\vec{A})\vec{D}] + [(\vec{C}\vec{A}\vec{D})\vec{B} - (\vec{C}\vec{A}\vec{B})\vec{D}]$$
$$+ [(-\vec{B}\vec{C}\vec{D})\vec{A} + (\vec{A}\vec{C}\vec{D})\vec{B}]$$

$$= (\vec{B}\vec{C}\vec{D})\vec{A} - (\vec{B}\vec{C}\vec{D})\vec{A} + (\vec{C}\vec{A}\vec{D})\vec{B} + (\vec{A}\vec{C}\vec{D})\vec{B}$$
$$-(\vec{B}\vec{C}\vec{A})\vec{D} - (\vec{C}\vec{A}\vec{B})\vec{D}$$

$$= -(\vec{A}\vec{C}\vec{D})\vec{B} + (\vec{A}\vec{C}\vec{D})\vec{B} - (\vec{A}\vec{B}\vec{C})\vec{D} - (\vec{A}\vec{B}\vec{C})\vec{D}$$

$$= -2(\vec{A}\vec{B}\vec{C})\vec{D}$$

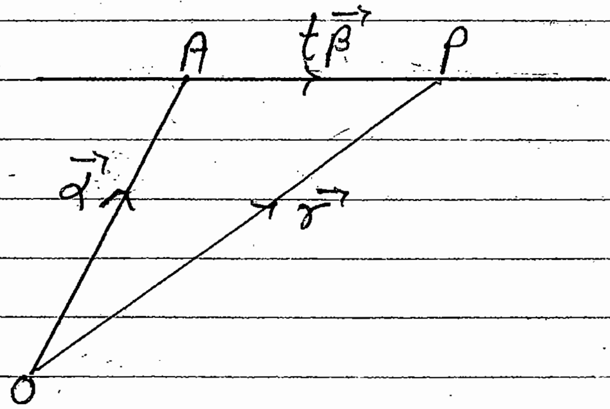
Proved

* **Vector Equation of the straight line passing through a given point and parallel to a given vector :-**

Given:- A be the given point whose position vector is α with respect to O and β be the given vector.

Solution:-

Let P be any point on the line AP, which is parallel to the vector $\vec{\beta}$.



$\vec{AP} = t\vec{\beta}$

where t is the scalar,

$\vec{OP} = \vec{OA} + \vec{AP}$

$$\boxed{\vec{r} = \vec{\alpha} + t\vec{\beta}}$$

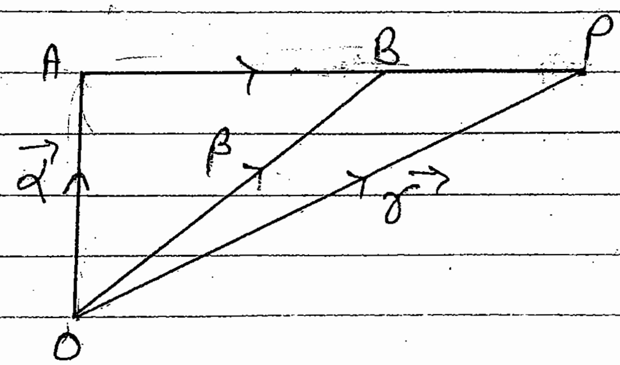
This is the vector equation of the straight line.

* **Vector Equation of the straight line passing through two given points :-**

Given:- A and B be the two given points whose position vectors are $\vec{\alpha}$ and $\vec{\beta}$ w.r.to O.

Solution:-

Let P be any point on the line AB,



$\vec{OP} = \vec{OA} + \vec{AP}$

$\vec{r} = \vec{\alpha} + t\vec{AB}$

$\vec{r} = \vec{\alpha} + t(\vec{\beta} - \vec{\alpha})$

$$\boxed{\vec{r} = (1-t)\vec{\alpha} + t\vec{\beta}}$$

This is the required equation of the straight line.

Their → उनके
There → वहाँ

* Condition of three points to be Collinear:-

Let A, B, C be three points whose position vectors are $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ with respect to O. Equation of the straight line through A and B is -

$$\vec{r} = (1-t)\vec{\alpha} + t\vec{\beta}$$

If $C(\vec{\gamma})$ point lies on the straight line AB,

$$\vec{\gamma} = (1-t)\vec{\alpha} + t\vec{\beta}$$

$$\Rightarrow (1-t)\alpha + t\vec{\beta} - \gamma = 0$$

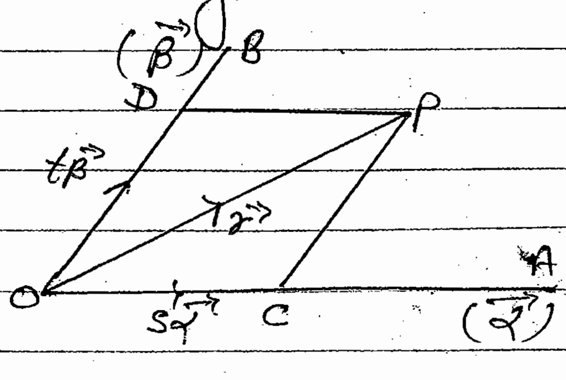
Sum of the coefficients of $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$

$$= (1-t) + t + (-1) = 0$$
$$= \cancel{1} - \cancel{t} + \cancel{t} - \cancel{1} = 0$$

This is the required condition.

So,
Condition of three points to be collinear = Sum of their coefficients.

* Vector Equation of the plane through the Origin and parallel to two given vectors $\vec{\alpha}$ and $\vec{\beta}$.



Let P be any point on the plane.

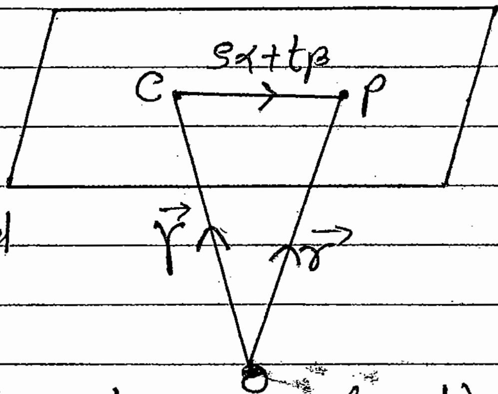
$$\vec{OA} = \vec{\alpha}, \quad \vec{OB} = \vec{\beta}$$

$$\vec{OP} = \vec{OC} + \vec{CP}$$
$$\vec{OP} = \vec{OC} + \vec{OD}$$
$$\boxed{\vec{r} = s\vec{\alpha} + t\vec{\beta}}$$

This is the required equation of the plane, where s and t are scalars.

* **Vector equation of the plane through a given point C and parallel to two given vectors $\vec{\alpha}$ and $\vec{\beta}$. :-**

Let P be any point on the plane.
$\vec{CP}$ lies on the plane parallel to $\vec{\alpha}$ and $\vec{\beta}$.



$\therefore \vec{CP} = s\vec{\alpha} + t\vec{\beta}$ (as above explained)
Let $\vec{\gamma}$ be the position vector of C.

$$\therefore \vec{OP} = \vec{OC} + \vec{CP}$$

$$\boxed{\vec{r} = \vec{\gamma} + s\vec{\alpha} + t\vec{\beta}}$$

represents the equation of the plane.

* **Vector Equation of the plane passing through three given points whose position vectors are α, β, γ :-**

Let $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ be the position vectors of the points A, B and C.
The plane ABC passes through A and parallel to AB and AC.

$$\vec{r} = \vec{\alpha} + s\vec{AB} + t\vec{AC} \quad \text{(as expained above)}$$
$$\vec{r} = \vec{\alpha} + s(\vec{\beta} - \vec{\alpha}) + t(\vec{\gamma} - \vec{\alpha})$$

$$\boxed{\vec{r} = (1-s-t)\vec{\alpha} + (s\vec{\beta} + t\vec{\gamma})}$$

This is the required equation of the plane.

Q. Show that $\vec{\alpha}-2\vec{\beta}+3\vec{\gamma}$, $-2\vec{\alpha}+3\vec{\beta}-\vec{\gamma}$, $4\vec{\alpha}-7\vec{\beta}+7\vec{\gamma}$ are collinear $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ being non-coplanar.

Sol^n The equation of a straight line through $\vec{\alpha}-2\vec{\beta}+3\vec{\gamma}$ and $-2\vec{\alpha}+3\vec{\beta}-\vec{\gamma}$ is – $[\because \vec{r}=(1-t)\vec{\alpha}+t\vec{\beta}]$

$$\vec{r}=(1-t)(\vec{\alpha}-2\vec{\beta}+3\vec{\gamma})+t(-2\vec{\alpha}+3\vec{\beta}-\vec{\gamma}) \quad \text{—} (1)$$

If $4\vec{\alpha}-7\vec{\beta}+7\vec{\gamma}$ lies on (1):-

$\Rightarrow 4\vec{\alpha}-7\vec{\beta}+7\vec{\gamma}=(1-t)(\vec{\alpha}-2\vec{\beta}+3\vec{\gamma})+t(-2\vec{\alpha}+3\vec{\beta}-\vec{\gamma})$

$\Rightarrow 4\vec{\alpha}-7\vec{\beta}+7\vec{\gamma}=(1-t)\alpha-2(1-t)\vec{\beta}+3(1-t)\vec{\gamma}-2t\vec{\alpha}+3t\vec{\beta}-t\vec{\gamma}$

$\Rightarrow 4\vec{\alpha}-7\vec{\beta}+7\vec{\gamma}=(1-3t)\vec{\alpha}+(-2+5t)\vec{\beta}+(3-4t)\vec{\gamma}$

Equating the coefficients of $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$, we get:

$$4=1-3t, \quad -7=-2+5t, \quad 7=3-4t$$

For $t=-1$, all the equations are ~~sta~~ satisfied.

Hence, the given vectors are collinear.

Proved

Q. What is the vector equation of the straight line through the points $\hat{i}-2\hat{j}+\hat{k}$ and $3\hat{k}-2\hat{j}$. Find where this line cuts the plane through the origin and the points $4\hat{j}$ and $2\hat{i}+\hat{k}$.

Sol^n:- The equation of the straight line through the points $\hat{i}-2\hat{j}+\hat{k}$ and $3\hat{k}-2\hat{j}$ is –

$[\vec{r}=\vec{\alpha}+t(\vec{\beta}-\vec{\alpha})]$

$$\vec{r}=\hat{i}-2\hat{j}+\hat{k}+t(3\hat{k}-2\hat{j}-\hat{i}+2\hat{j}-\hat{k})$$

$$\vec{r}=\hat{i}-2\hat{j}+\hat{k}+t(-\hat{i}+2\hat{k})$$

$$\vec{r}=(1-t)\hat{i}-2\hat{j}+(1+2t)\hat{k} \quad \text{—} (1)$$

The equation of the plane through the origin and the points $4\hat{j}$ and $2\hat{i}+\hat{k}$ is –

$$\vec{r}=0+s_1(4\hat{j})+t_1(2\hat{i}+\hat{k}) \qquad [\vec{r}=(1-s-t)\vec{\alpha}+s\vec{\beta}+t\vec{\gamma}]$$

$\vec{r} = 2t_1\hat{i} + 4s_1\hat{j} + t_1\hat{k}$ ——— (2)

The point of intersection of (1) and (2) is given by

$(1-t)\hat{i} - 2\hat{j} + (1+2t)\hat{k} = 2t_1\hat{i} + 4s_1\hat{j} + t_1\hat{k}$

Equating the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get-

$1-t = 2t_1, \quad -2 = 4s_1, \quad 1+2t = t_1$

On solving these equation, we get-

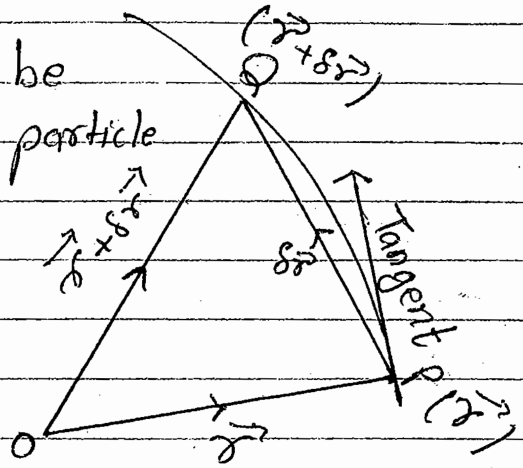
$t_1 = \frac{3}{5}, \quad t = -\frac{1}{5}, \quad s_1 = -\frac{1}{2}$

Hence, the point of intersection is $\left(1+\frac{1}{5}\right)\hat{i} - 2\hat{j} + \left(1-\frac{2}{5}\right)\hat{k}$ from (1) $\frac{6}{5}\hat{i} - 2\hat{j} + \frac{3}{5}\hat{k}$

Ans

{VECTOR CALCULUS}

* Differentiation of Vectors :-



Let O be the origin and P be the position of a moving particle at time t.

Let $\overrightarrow{OP} = \vec{r}$

Let Q be the position of the particle at the time $t+\delta t$ and the position vector of

$$\overrightarrow{OQ} = \vec{r} + \delta\vec{r}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (\vec{r}+\delta\vec{r}) - \vec{r} = \delta\vec{r}$$

$\frac{\delta\vec{r}}{\partial t}$ is a vector. As $\delta t \to 0$, Q tends to P and the chord becomes the tangent at P.

We define.

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t}, \quad \text{then} -$$

$\frac{d\vec{r}}{dt}$ is a vector in the direction of the tangent at P.

$\frac{d\vec{r}}{dt}$ is called the differential coefficient of $\vec{r}$ with respect to 't'.

Similarly $\frac{d^2\vec{r}}{dt^2}$ is the second order derivative of $\vec{r}$.

$\frac{d\vec{r}}{dt}$ gives the velocity of the particle at P, which is along the tangent to its path. Also $\frac{d^2\vec{r}}{dt^2}$ gives the acceleration of the particle at P.

$y = f(x) \rightarrow$ curve

$\frac{dy}{dx}$ = slope of tangent of the curve.

* **Vector Differentiation :-**

$\vec{r}$ is vector function of the variable t, then -

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\boxed{\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

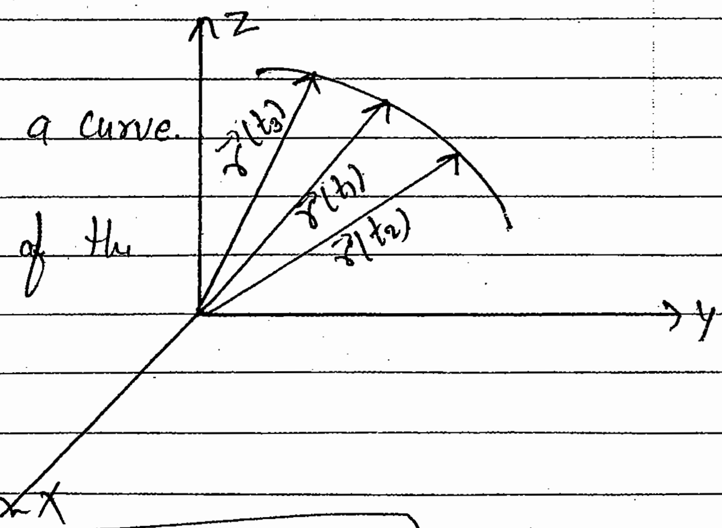
Significance of derivative of vector function:-

* $\vec{r}(t) \rightarrow$ displacement of the particle

$\frac{d\vec{r}(t)}{dt} \longrightarrow$ velocity

$\frac{d^2\vec{r}(t)}{dt} \longrightarrow$ Acceleration.

* Tip of $\vec{r}(t)$ traces a curve.



$\frac{d\vec{r}}{dt}$ = tangent vector of the curve.

Unit tangent vector or direction of the curve -

$$\boxed{\text{Unit Tangent Vector} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|}}$$

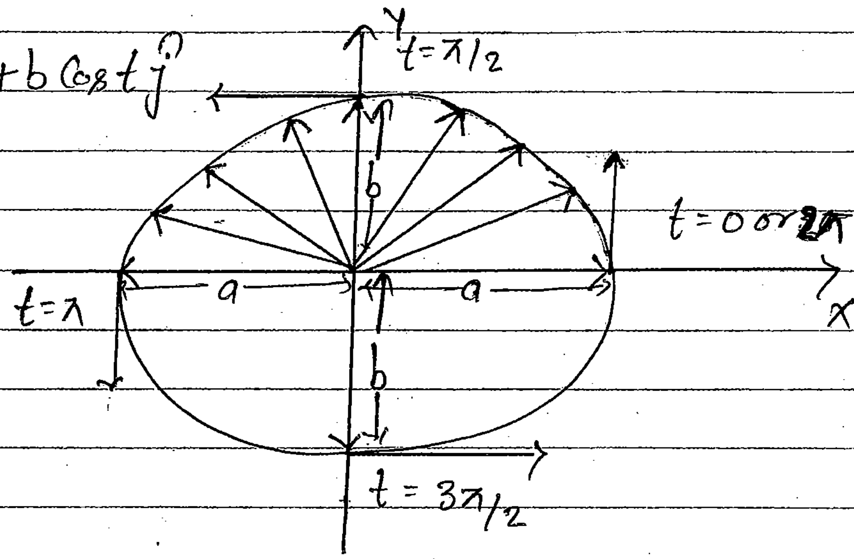
Anticlockwise or Counterclockwise is the positive direction of the curve.

* $\vec{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j}$

$\frac{d\vec{r}(t)}{dt} = -a\sin t\,\hat{i} + b\cos t\,\hat{j}$

at $t=0 = b\hat{j}$

at $t = \frac{\pi}{2} = -a\hat{i}$



7-1

Q.20 A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$. The components of its acceleration at $t=1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is equal to?

(a) $2/\sqrt{14}$ ✓(b) $-2/\sqrt{14}$ (c) $16/\sqrt{14}$ (d) $-16/\sqrt{14}$

Solⁿ $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$

at $t=1$ direcⁿ $= \hat{i} - 3\hat{j} + 2\hat{k}$.

$r(t) = x\hat{i} + y\hat{j} + z\hat{k}$

$r(t) = (2t^2)\hat{i} + (t^2 - 4t)\hat{j} + (3t-5)\hat{k}$

$\frac{d\vec{r}(t)}{dt} = 4t\hat{i} + (2t-4)\hat{j} + 3\hat{k}$

$\frac{d^2 r(t)}{dt^2} = 4\hat{i} + 2\hat{j}$

$\left[\frac{d^2 r(t)}{dt^2}\right]_{t=1} \cdot \hat{A}$

$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$

$(4\hat{i} + 2\hat{j})_{t=1} \cdot (\hat{i} - 3\hat{j} + 2\hat{k})$

$= 4\hat{i} - 6\hat{j}$

Q.9 If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then which of the following statements is true?

(a) $\vec{a}$ is parallel to $(\vec{b}-\vec{c})$ (b) $\vec{a}$ is perpendicular to $(\vec{b}-\vec{c})$

(c) ✓ either $\vec{a} = 0$ or $\vec{b} = \vec{c}$ (d) None of these.

Solⁿ

$$\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$$

$$\Rightarrow \vec{a}(\vec{b}-\vec{c}) = 0$$

$$\vec{a} \perp^{\text{or}} (\vec{b}-\vec{c})$$

and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$

$$\Rightarrow \vec{a}\times(\vec{b}-\vec{c})$$

$$\Rightarrow \vec{a} \parallel^{\text{lel}} (\vec{b}\cdot\vec{c})$$

$$\Rightarrow \vec{a} = 0 \quad \text{or} \quad \vec{b}-\vec{c} = 0$$

$$\Rightarrow \boxed{\vec{a} = 0,\ \vec{b} = \vec{c}}$$ Ans

Q.18 Which of the following relation is true?

(a) $(\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A}) = 0$

(b) ✓ $(\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A}) = [\vec{A}\cdot(\vec{B}\times\vec{C})]^2$

(c) $(\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A}) = [\vec{A}\times(\vec{B}\times\vec{C})]^2$

(d) $(\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A}) = [\vec{A}\times(\vec{B}\times\vec{C})]$

Solⁿ $(\vec{A}\times\vec{B})\cdot\underbrace{(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})} = 0$

$$\underbrace{(\vec{B}\times\vec{C})}_{\substack{M \\ a}}\times\underbrace{(\vec{C}\times\vec{A})}_{b\quad c}$$

$$\Rightarrow \vec{C}(\vec{M}\cdot\vec{A}) - \vec{A}(\vec{M}\cdot\vec{C})$$

$$\Rightarrow \vec{C}[(\vec{B}\times\vec{C})\cdot\vec{A}] - A[\underbrace{(\vec{B}\times\vec{C})\cdot\vec{C}}_{0}]$$

$\Rightarrow \vec{C}\,[\vec{A}\cdot(\vec{B}\times\vec{C})]$

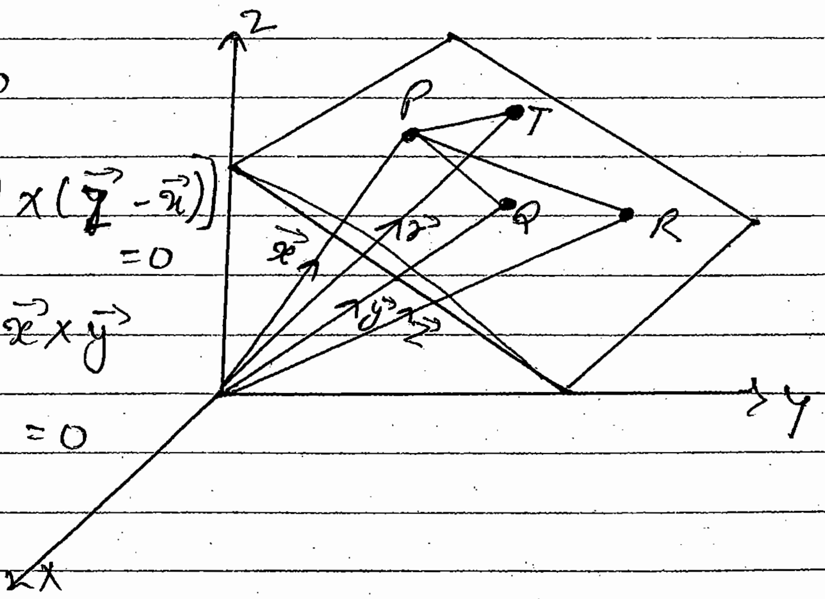
So $(\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A}) = (\vec{A}\times\vec{B})\cdot C[\vec{A}\cdot(\vec{B}\times\vec{C})]$

$\Rightarrow \quad = \vec{A}\cdot(\vec{B}\times\vec{C})\;\vec{A}\cdot(\vec{B}\times\vec{C})$

$= \left[\vec{A}\cdot(\vec{B}\times\vec{C})\right]^2$ <u>Ans</u>

A-2
Q.3 If the points P, Q, R are not lying on the same straight line having position vector $\vec{x}, \vec{y}, \vec{z}$ relative to a given origin, show that $\vec{x}\times\vec{y}+\vec{y}\times\vec{z}+\vec{z}\times\vec{x}$ is perpendicular to the plane containing P, Q, R.

Sol^n



$\vec{PT}\cdot(\vec{PQ}\times\vec{PR})=0$

$\Rightarrow (\vec{r}-\vec{x})\cdot[(\vec{y}-\vec{x})\times(\vec{z}-\vec{x})] = 0$

$\Rightarrow (\vec{r}-\vec{x})\cdot[\vec{y}\times\vec{z}+\vec{x}\times\vec{y} + \vec{z}\times\vec{x}+0] = 0$

$\Rightarrow (\vec{r}-\vec{x})\cdot\underbrace{[\vec{y}\times\vec{z}+\vec{x}\times\vec{y}+\vec{z}\times\vec{x}]}_{M} = 0$

So $\vec{M}$ is $\perp^{or}$ to $(\vec{r}-\vec{x})$.

So $\vec{M}$ is $\perp^{or}$ to the plane that contains P, Q, R.

Q.4 Find a set a vectors reciprocal to the following set $(2\hat{i}+3\hat{j}-\hat{k})$, $(\hat{i}-\hat{j}-2\hat{k})$, $(-\hat{i}+2\hat{j}+2\hat{k})$

Sol^n

$\vec{a}, \vec{b}, \vec{c} \longrightarrow$ Direct lattice vector

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$$

A-2

Q.6 If $\vec{A}$ has constant magnitude show that $\vec{A}$ and $\frac{d\vec{A}}{dt}$ are perpendicular to each other.

Sol^n

$$\vec{A}\cdot\vec{A} = |\vec{A}|^2$$

$$\Rightarrow \frac{d}{dt}(\vec{A}\cdot\vec{A}) = 0$$

$$\Rightarrow \vec{A}\cdot\frac{d\vec{A}}{dt} = 0$$

So $\vec{A}$ $\perp^{or}$ to $\frac{d\vec{A}}{dt}$.

A-2

Q.5 If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, show that

(i) $\vec{r}\times\frac{d\vec{r}}{dt} = \omega(\vec{a}\times\vec{b})$

(ii) $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$ [$\vec{a}, \vec{b}$ are constant vectors and 'ω' is constant].

Sol^n (i) $\vec{r} = \vec{a}\cos\omega t\,\hat{i} + \vec{b}\sin\omega t\,\hat{j}$

$$\frac{d\vec{r}}{dt} = -\vec{a}\omega\sin\omega t\,\hat{i} + \vec{b}\omega\cos\omega t\,\hat{j} \quad \text{—} (1)$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = (\vec{a}\times\vec{b})\omega\cos^2\omega t + (\vec{a}\times\vec{b})\omega\sin^2\omega t$$

$$= \omega(\vec{a}\times\vec{b})[\cos^2\omega t + \sin^2\omega t]$$

$$\boxed{\vec{r}\times\frac{d\vec{r}}{dt} = \omega[\vec{a}\times\vec{b}]}$$ Ans

(ii) From (1)

$$\frac{d^2\vec{r}}{dt^2} = -a\omega^2\cos\omega t\,\hat{i} - b\omega^2\sin\omega t$$

$$= -\omega^2(a\cos\omega t\,\hat{i} + b\sin\omega t)$$

$$\boxed{\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}}$$ Ans

Q.7 A particle moves so that its position vector is given by, $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ where 'ω' constant show that

(i) The velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.

(ii) The acceleration is directed towards the origin and has magnitude proportional to the distance from the origin.

(iii) $\vec{r} \times \vec{v}$ = Constant vector.

Sol (i)

$$\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -\omega \sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$$

$$\vec{v}\cdot\vec{r} = (-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j})\cdot(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$$

$$= \cancel{-\omega\sin\omega t\cos\omega t} + \cancel{\omega\sin\omega t\cos\omega t}$$

$$\boxed{\vec{v}\cdot\vec{r} = 0}$$

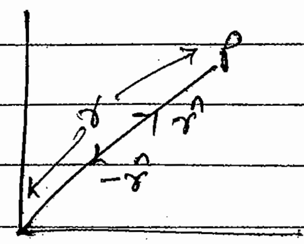
So velocity of the particle is perpendicular to $\vec{r}$.

(ii)

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\cos\omega t\,\hat{i} - \omega^2\sin\omega t\,\hat{j}$$

$$= -\omega^2\vec{r}$$

$$= -\underbrace{(\omega^2 r)}_{\text{mag.}}\ \underbrace{(\hat{r})}_{\text{direction}}$$



So mag $\propto r$ and acc^n is $(-\hat{r})$ direc^n

So Acc^n is towards to origin.

(viii) $\vec{r} = a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j}$

$\vec{V} = -\omega a\sin\omega t\,\hat{i} + b\omega\cos\omega t\,\hat{j}$

So
$$\vec{r}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\omega t & b\sin\omega t & 0 \\ -\omega a\sin\omega t & b\omega\cos\omega t & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left((ab\omega\cos^2\omega t) - (ab\omega\sin^2\omega t)\right)$$

So $\vec{r}\times\vec{V} = 0\hat{k}\left[-ab\omega(\cos^2\omega t + \sin^2\omega t)\right]$

$$\boxed{\vec{r}\times\vec{V} = -ab\omega\hat{k}}$$

$\because$ $\vec{r}\times\vec{V}$ is independent of t so it is a constant vector.

Q.8 If $\dfrac{d\vec{m}}{dt} = \vec{u}\times\vec{m}$ and $\dfrac{d\vec{n}}{dt} = \vec{u}\times\vec{n}$, then show that $\dfrac{d}{dt}(\vec{m}\times\vec{n}) = \vec{u}\times(\vec{m}\times\vec{n})$.

Sol^n
$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

$$\Rightarrow (\vec{b}\times\vec{c})\times\vec{a} = \vec{c}(\vec{a}\cdot\vec{b}) - \vec{b}(\vec{a}\cdot\vec{c})$$

$$\frac{d\vec{m}}{dt} = \vec{u}\times\vec{m}, \quad \frac{d\vec{n}}{dt} = \vec{u}\times\vec{n}$$

$$\frac{d}{dt}(\vec{m}\times\vec{n}) = \frac{d\vec{m}}{dt}\times\vec{n} + \vec{m}\times\frac{d\vec{n}}{dt}$$

$$= \underbrace{(\underset{b}{\vec{u}}\times\underset{c}{\vec{m}})\times\underset{a}{\vec{n}}} + \underbrace{\underset{a}{\vec{m}}\times(\underset{b}{\vec{u}}\times\underset{c}{\vec{n}})}$$

$$= \vec{m}(\vec{m}.\vec{u}) - \cancel{\vec{u}(\vec{m}.\vec{n})} + \cancel{\vec{u}(\vec{m}.\vec{n})}$$

$$- \vec{n}(\vec{m}.\vec{u})$$

$$= \underset{b}{\vec{m}}(\underset{a}{\vec{u}}.\underset{c}{\vec{n}}) - \underset{c}{\vec{n}}(\underset{a}{\vec{u}}.\underset{b}{\vec{m}})$$

$$= \vec{u}\times(\vec{m}\times\vec{n})$$

Ans

* FORMULAE OF DIFFERENTIOTION :-

(i) $\frac{d}{dt}(\vec{F}+\vec{G}) = \frac{d\vec{F}}{dt} + \frac{d\vec{G}}{dt}$

(ii) $\frac{d}{dt}(\vec{F}\phi) = \frac{d\vec{F}}{dt}\phi + \vec{F}\frac{d\phi}{dt}$

(iii) $\frac{d}{dt}(\vec{F}.\vec{G}) = \vec{F}.\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}.\vec{G}$

(iv) $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\times\vec{G}$

To order of the functions $\vec{F}, \vec{G}$ is not to be changed.

$$\text{Normal Vector} = \frac{d}{dt}\left[\frac{\text{Unit tangent Vector}}{\text{Magnitude of tangent Vector}}\right]$$

* Scalar And Vector Point Functions :-

(i) feeld :-

If a function is defined in any region of space, for every point of the region, then this region is known as field.

(ii) Scalar Point function :-

A function $\phi(x, y, z)$ is called a scalar point function if it associates a scalar with every point in space.
The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

(iii) Vector Point function :-

If a function $F(x, y, z)$ defines a vector at every point of a region, then $F(x, y, z)$ is called a vector point function. The velocity of a moving fluid, gravitational force are the example of vector point function.

$\vec{\nabla}$ is a vector operator
Vetor $\neq$ Vector operator.

* Gradiant of a Scalar Field or Scalar function :-

The vector differential operator Del is denoted by ∇. It is defined as -

$$\underset{\text{grad / del operator}}{\vec{\nabla}} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$\vec{\nabla}$ can not exist independentily.

$$\vec{\nabla}\cdot\vec{B} \neq \vec{B}\cdot\vec{\nabla} \longrightarrow \text{scalar operator}$$

$\rightarrow$ scalar quantity

Lets consider a scalar function ϕ then

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}\phi\,\hat{i} + \frac{\partial}{\partial y}\phi\,\hat{j} + \frac{\partial}{\partial z}\phi\,\hat{k}$$

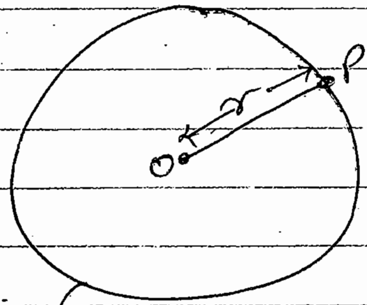
$$\vec{\nabla}\phi = \underbrace{\frac{\partial \phi}{\partial n}}_{\text{magnitude.}}\;\hat{n} \longrightarrow \text{direction}$$

$\Rightarrow$ Gradiant of a scalar function ϕ at any point $P(x,y,z)$ is a vector quantity whose magnitude is equal to the rate of change of ϕ with distance along to normal to the level surface and its direction is along normal to the level surface.

* Level Surface :-

It is the surface on which the value of the scalar function ϕ is same at all points.

ex-



$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$ (V is a scalar quantity)

↳ Equipotential Surface is an example of level surface.

* General Equation :-

$$\phi = \text{Constant.}$$

⇒ Let equation of surface -

$$x^2 + y^2 + z^2 = r^2 \quad (\text{eq}^n \text{ of sphere})$$

define

$$\phi = x^2 + y^2 + z^2$$

$$\phi = \text{Constt.}$$

(2) $x^3 y + y z^3 + y z = 8$

Corresponding ϕ define -

$$\phi(x, y, z) = x^3 y + y z^3 + y z = \text{Const}$$

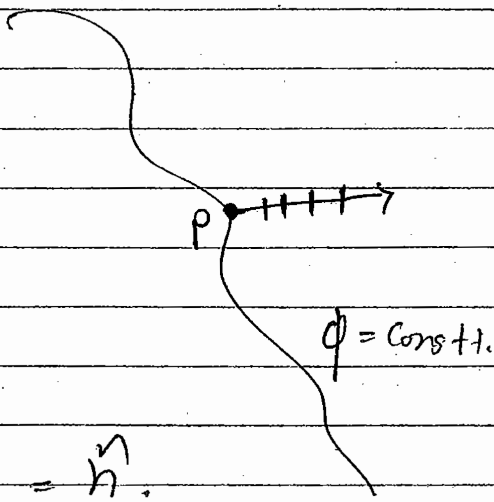
⇒ We have to define a scalar function ϕ which can be a function of (x, y, z).

$\delta\phi$ = Change in ϕ along δn, $\hat{n}$ = direc^n of normal surface.
δn = change in distance along normal to the level surface.

~~When~~ We then set ϕ = Constt. for level surface.

We want to define ϕ along P :-

Q. If equation of surface is -
$$\phi = \text{Constant}.$$
What is the normal vector?



Sol^n

Normal Vector : $\vec{\nabla}\phi$

Unit Normal Vector :- $\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \hat{n}$.

CSIR June - 2012
A-1
Q.98 The unit normal vector at the point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ on the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is?

(a) $\dfrac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$ ✓ (b) $\dfrac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$

(c) $\dfrac{b\hat{i} + c\hat{j} + a\hat{k}}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$ (d) $\dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

Sol^n
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
$$\hat{\phi} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$
$$\vec{\nabla}\hat{\phi} = \frac{2x}{a^2} + \frac{2y}{b^2} + \frac{2z}{c^2}$$

$$|\vec{\nabla}\phi| = \sqrt{\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2 + \left(\frac{2z}{c^2}\right)^2}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\frac{2x}{a^2}\hat{i} + \frac{2y}{b^2}\hat{j} + \frac{2z}{c^2}\hat{k}}{\sqrt{\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2 + \left(\frac{2z}{c^2}\right)^2}}$$

at, $\therefore$ $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$

$$\hat{n} = \frac{\frac{2a}{\sqrt{3}a^2}\hat{i} + \frac{2b}{\sqrt{3}b^2}\hat{j} + \frac{2c}{\sqrt{3}c^2}\hat{k}}{\sqrt{\left(\frac{2a}{\sqrt{3}a^2}\right)^2 + \left(\frac{2b}{\sqrt{3}b^2}\right)^2 + \left(\frac{2c}{\sqrt{3}c^2}\right)^2}}$$

$$= \frac{\frac{2}{\sqrt{3}a}\hat{i} + \frac{2}{\sqrt{3}b}\hat{j} + \frac{2}{\sqrt{3}c}\hat{k}}{\sqrt{\frac{4}{3a^2} + \frac{4}{3b^2} + \frac{4}{3c^2}}}$$

$$= \frac{\frac{2}{\sqrt{3}a}\hat{i} + \frac{2}{\sqrt{3}b}\hat{j} + \frac{2}{\sqrt{3}c}\hat{k}}{\frac{2}{\sqrt{3}}\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$= \frac{\cancel{\frac{2}{\sqrt{3}}}\left[\frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\cancel{abc}}\right]}{\cancel{\frac{2}{\sqrt{3}}}\sqrt{\frac{b^2c^2 + c^2a^2 + a^2b^2}{\cancel{a^2b^2c^2}}}}$$

$$\boxed{\hat{n} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}}$$

Ans

Q.24 The angle between the two surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z^2=3$ at the point $(2,-1,2)$ is?

(a) $\sin^{-1}\left(\frac{8}{3\sqrt{21}}\right)$ (b) $\cos^{-1}\left(\frac{8}{\sqrt{21}}\right)$ (c) $\sin^{-1}\left(\frac{8}{\sqrt{21}}\right)$

(d) $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

Sol^n

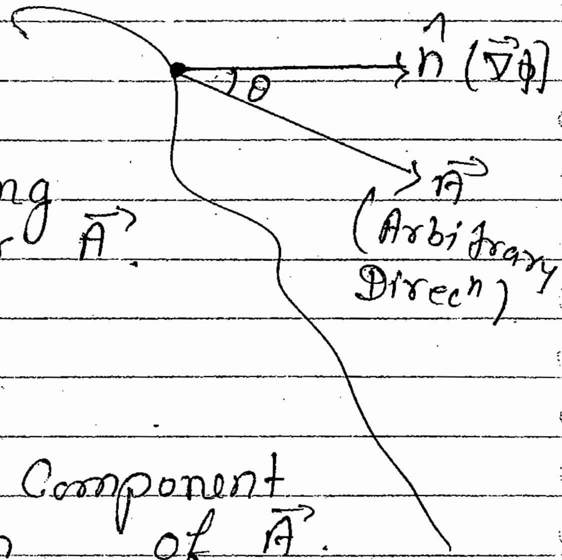
$$\phi_1 = x^2+y^2+z^2=9 \Rightarrow \vec{\nabla}\phi_1$$

$$\phi_2 = x^2+y^2-z^2 = 3 \Rightarrow \vec{\nabla}\phi_2$$

Imp.

* Directional Derivative of ϕ in the direction of $\vec{A}$:-

It is defined as the rate of change of $\hat{\phi}$ with ~~of~~ distance along the direction of vector $\vec{A}$.



$$\boxed{D.D = \underbrace{\vec{\nabla}\phi \cdot \hat{A}}}$$

Projection or Component of $\vec{\nabla}\phi$ in the direction of $\vec{A}$.

$$= |\vec{\nabla}\phi|\ |\hat{A}|\ \cos\theta$$

$$\boxed{D.D. = |\vec{\nabla}\phi| \cos\theta} \qquad \{\because |\hat{A}| = 1$$

θ = Angle between $\vec{A}$ and $\hat{n}$.

$$(D.D)_{max} = |\vec{\nabla}\phi|\ \cos\theta \xrightarrow{} max = 1$$

So $\boxed{(D.D.)_{max} = |\vec{\nabla}\phi|}$

The maximum value of the directional derivative occurs along to the normal to the level ~~surface~~ surface and value is $|\vec{\nabla}\phi|$.

=> In what direction the rate of change of ϕ is maximum => $\boxed{\hat{n} = \frac{\vec{\nabla}\hat{\phi}}{|\vec{\nabla}\hat{\phi}|}}$

What is the maximum rate of change of ϕ:-

$$= |\vec{\nabla}\phi|.$$

Q.28 The directions along which there is no change in the value of the function $f(x,y) = e^{(x^2+xy)}$ at the point $(3,-2)$ are equal to:-

(a) $0.6\hat{i} - 0.8\hat{j},\ 0.6\hat{i} + 0.8\hat{j}$ (b) $-0.6\hat{i} - 0.8\hat{j},\ 0.6\hat{i} + 0.8\hat{j}$

(c) $-0.6\hat{i} - 0.8\hat{j},\ 0.6\hat{i} - 0.8\hat{j}$ (d) $0.6\hat{i} + 0.8\hat{j},\ -0.6\hat{i} + 0.8\hat{j}$

Divergence → Vector lines spreading out

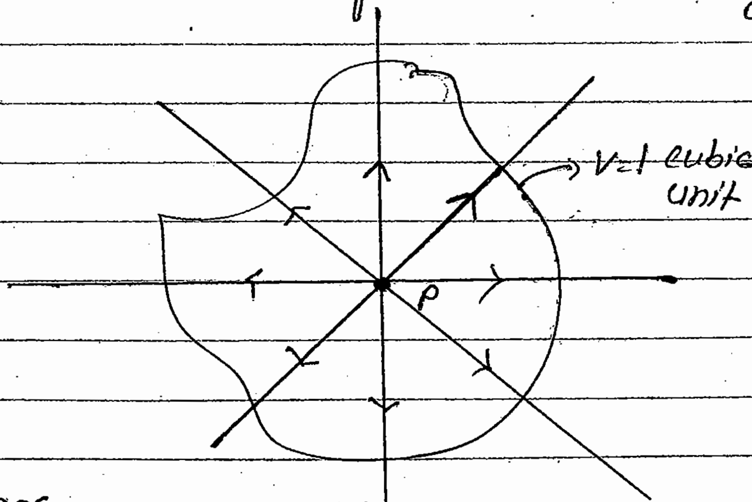
* Divergence of a vector field :-

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \text{scalar}$$

Divergence of a vector field $\vec{V}$ at any point of $P(x,y,z)$ is defined as the outwards flux of the vector field per unit volume inclose by an infinitisimly close volume inclose by an infinitisimly close surface sorrounding point of P.

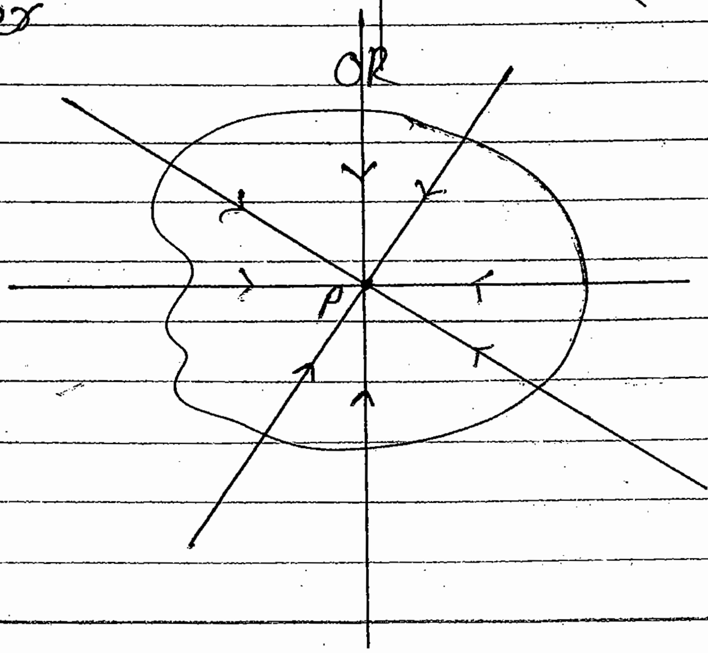
Outward (+ve) ϕ



Outward flux per unit volume.

OR

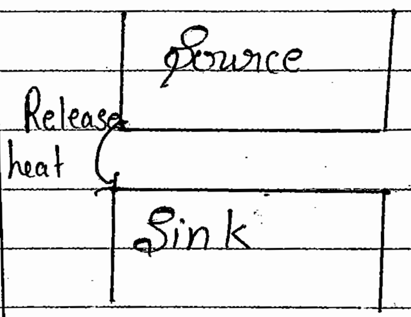
inward (−ve) ϕ.



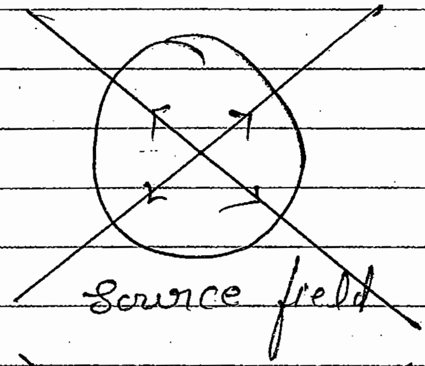
* $\vec{\nabla}.\vec{V} = +ve \Rightarrow$ Outward flux {Vector lines are going outwards}

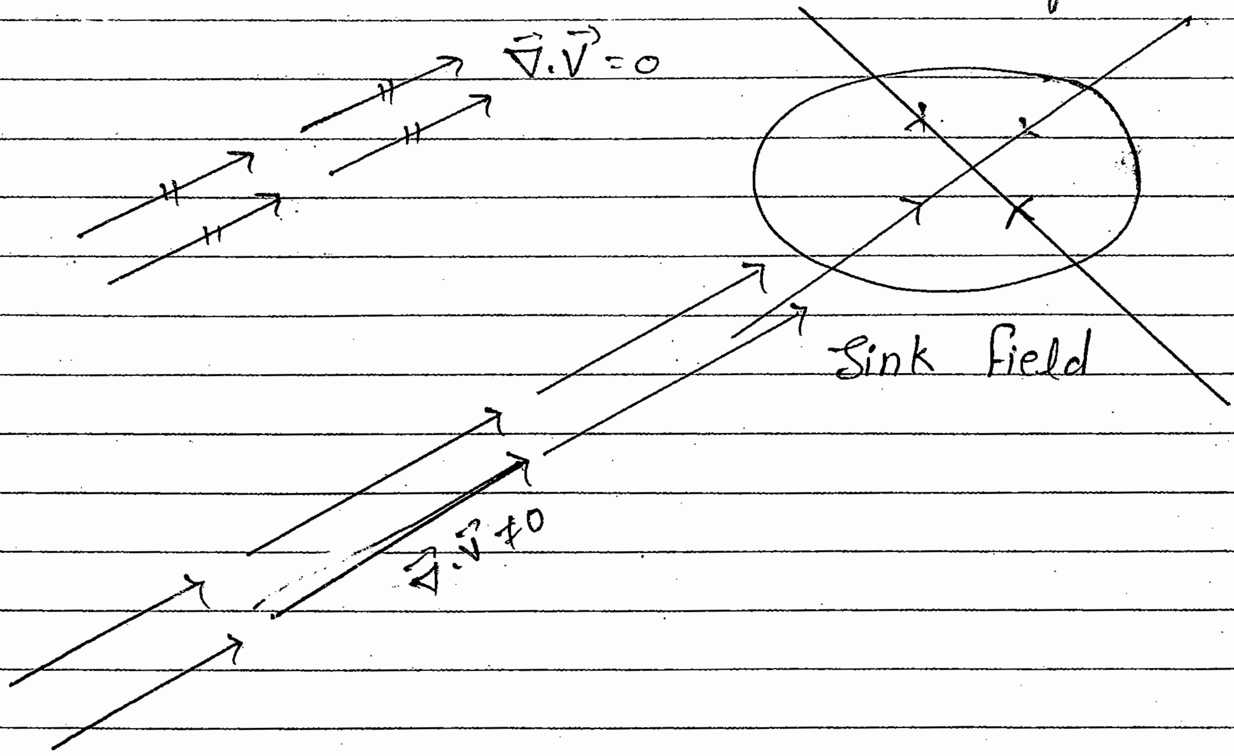
* $\vec{\nabla}.\vec{V} = -ve \Rightarrow$ Inward flux [Vector lines coming in or sink]

* $\vec{\nabla}.\vec{V} = 0 \Rightarrow$ Either no flux or net flux is zero. or Solenoidal field.



Same as -





* If the length or magnitude of the vector lines is changing along a direction along which the vector field is directed out then the vector field has a non zero divergence.

$$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k} \Rightarrow \vec{\nabla}\cdot\vec{V} = 0$$

Constant vector field

Independent of x, y, z

$$\Rightarrow |\vec{V}| = \sqrt{V_1^2 + V_2^2 + V_3^2}$$

Second Condition :-

$$\vec{V} = xy^2\hat{i}$$

$$|\vec{V}| = \sqrt{\quad}$$

$$\vec{\nabla}\cdot\vec{V} \neq 0$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{U}+\vec{V}) = \vec{\nabla}\vec{U} + \vec{\nabla}\cdot\vec{V}$$

$$\Rightarrow \vec{\nabla}\cdot(u\vec{V}) = (\vec{\nabla}u)\vec{V} + u(\vec{\nabla}\cdot\vec{V})$$

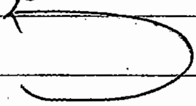
* Curl of a Vector field :-

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

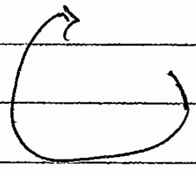
$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

⇒ The magnitude of curl of vector field is the measurement of rotation of the vector field. (Means rotation of vector lines corresponding to vector field)

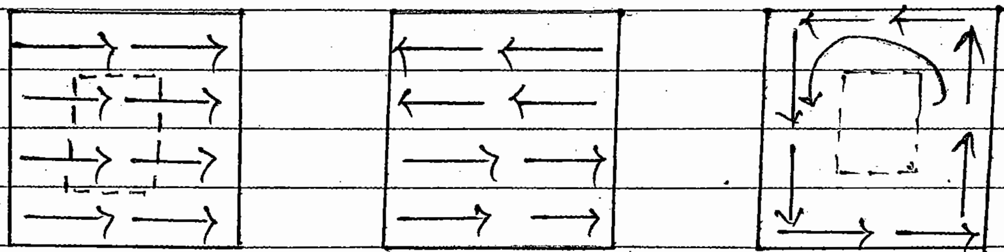
⇒ The direction of curl of a vector field is along the axis of rotation



Direc^n of curl = outward $\perp^{or}$ to the paper



Inward $\perp^{or}$ to the paper



$\vec{\nabla} \times \vec{V} = 0$ (becoz no rotation) | $\vec{\nabla} \times \vec{V} \neq 0$ | $\vec{\nabla} \times \vec{V} \neq 0$

⇒ $\vec{\nabla} \times (\vec{u} \times \vec{V}) = \vec{\nabla} \times \vec{u} + \vec{\nabla} \times \vec{V}$

⇒ $\vec{\nabla} \times (u\vec{V}) = (\vec{\nabla} u) \times \vec{V} + u(\vec{\nabla} \times \vec{V})$

⇒ $\vec{\nabla} \times \vec{V} = 0$ ⇒ Irrotational field or Conservative field.

⇒ $\boxed{\vec{V} \neq \vec{\nabla}\phi}$ → Calculate

$\vec{V}$: given; $\vec{\nabla}\phi$: gradiant of scalar

$\vec{\nabla} \times \vec{E} = 0$ → given

$\vec{E} = -\vec{\nabla} V$ (V → Calculate)

04/Aug/2013

1.2 1.13 Prove that,

(i) $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0 \quad (r \neq 0)$ (iii) $\vec{\nabla}\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right) = \frac{\vec{a}}{r^n} - \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}$

(ii) $\vec{\nabla}\cdot(r^n\vec{r}) = (3+n)r^n$

Sol^n (i) $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0$

$$\vec{\nabla}\cdot\left(\frac{\cancel{r}\hat{r}}{\cancel{r^3}}\right) = \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$$

$$= \frac{1}{r^2\cancel{\sin\theta}}\left[\frac{\partial}{\partial r}(\cancel{r^2}\cancel{\sin\theta})\left(\frac{\hat{r}\cdot\hat{r}}{\cancel{r^2}}\right)\right]$$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\hat{r}\cdot\hat{r}\right]$$

$$\boxed{\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0}$$

(ii) $\vec{\nabla}\cdot(r^n\vec{r}) = (3+n)r^n$

$$\vec{\nabla}\cdot(r^n\cdot r\hat{r}) = \vec{\nabla}\cdot(r^{n+1}\hat{r})$$

$$= \frac{1}{r^2\cancel{\sin\theta}}\frac{\partial}{\partial r}(r^2\cancel{\sin\theta})(r^{n+1})\,\hat{r}\cdot\hat{r}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+3})$$

$$= \frac{(n+3)}{r^2}r^{n+3-1} = \frac{(n+3)}{r^2}r^{n+2}$$

$$= (n+3)r^{n+2-2}$$

$$= (n+3)r^n \quad \underline{\text{Ans}}$$

(iii) $\nabla\left[\frac{\vec{a}\cdot\vec{r}}{r^n}\right] = \frac{\vec{a}}{r^n} - \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}$

L.H.S. :-

$$\vec{\nabla}\left[\frac{(\vec{a}\cdot\vec{r})}{r^n}\right] = \left[\vec{\nabla}(\vec{a}\cdot\vec{r})\right]\frac{1}{r^n} + (\vec{a}\cdot\vec{r})\,\vec{\nabla}\left(\frac{1}{r^n}\right)$$

$$= \frac{\vec{a}}{r^n} + (\vec{a}\cdot\vec{r})\left[-n r^{-n-1}\hat{r}\right]$$

$$= \frac{\vec{a}}{r^n} - \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}$$

R.H.S.

Q. If $\vec{a}$ is constant vector, then -

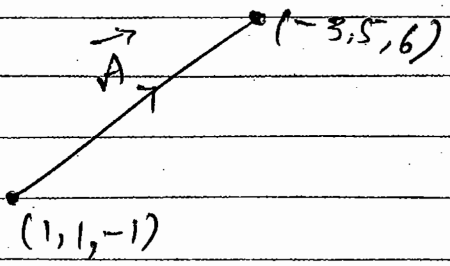
$$\vec{\nabla}(\vec{a}\cdot\vec{r}) = \vec{a}$$

$$\vec{\nabla}\times(\vec{a}\times\vec{r}) = 2\vec{a}$$

Q. Find the direction derivative of the scalar function $\phi = 4e^{(2x-y+z)}$ at the point $(1,1,-1)$ in a direction towards the point $(-3,5,6)$ what will be its maximum value ?

Sol^n

$$\phi = 4e^{(2x-y+z)}$$



$$\vec{A} = -4\hat{i} + 4\hat{j} + 7\hat{k}$$

$$\hat{A} = \frac{\vec{A}}{|A|} = \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{9}$$

$$\boxed{\hat{A} = -\frac{4}{9}\hat{i} + \frac{4}{9}\hat{j} + \frac{7}{9}\hat{k}}$$

$$\nabla\phi = 4\left[\frac{\partial}{\partial x} e^{(2x-y+z)}\hat{i} + \frac{\partial}{\partial y} e^{(2x-y+z)}\hat{j} + \frac{\partial}{\partial z} e^{(2x-y+z)}\hat{k}\right]$$

$$= 4\left[e^{(2x-y+z)} \cdot 2\hat{i} + e^{(2x-y+z)}(-1)\hat{j} + e^{(2x-y+z)}(1)\hat{k}\right]$$

$$\nabla\phi = \left[8e^{(2x-y+z)}\hat{i} - 4e^{(2x-y+z)}\hat{j} + 4e^{(2x-y+z)}\hat{k}\right]$$

$$\nabla\phi\Big|_{(1,1,-1)} = 8e^{(2-1-1)}\hat{i} - 4e^{(2-1-1)}\hat{j} + 4e^{(2-1-1)}\hat{k}$$

$$= 8e^{0}\hat{i} - 4e^{0}\hat{j} + 4e^{0}\hat{k}$$

$$\boxed{\nabla\phi\Big|_{(1,1,-1)} = 8\hat{i} - 4\hat{j} + 4\hat{k}}$$

So

$$\nabla\phi\Big|_{(1,1,-1)} \cdot \hat{A} = (8\hat{i} - 4\hat{j} + 4\hat{k}) \cdot \left(\frac{-4}{9}\hat{i} + \frac{4}{9}\hat{j} + \frac{7}{9}\hat{k}\right)$$

$$= \frac{8\times(-4)}{9} + \frac{(-4)(4)}{9} + \frac{(4)(7)}{9}$$

$$= \frac{-32}{9} - \frac{16}{9} + \frac{28}{9} = \frac{-48+28}{9}$$

$$= \frac{-20}{9} \text{ Ans}$$

$$\left(D.D\right)_{max} = \left|\vec{\nabla}\phi\right|_{(1,1,-1)}$$

$$= \sqrt{(8)^2 + (-4)^2 + (4)^2}$$

$$= \sqrt{64+16+16} = \sqrt{96}$$

$$= 4\sqrt{6}$$

Ans

A-2

Q16 Show that, $\vec{V} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$ is irrotational. Find the corresponding scalar potential 'ϕ'

Sol^n $\vec{V}$ = Given , $\vec{\nabla} \times \vec{V} = 0$

$$\vec{V} = \vec{\nabla}\phi$$

$$(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Comparing the Coefficient of $\hat{i}, \hat{j}$ & $\hat{k}$.

$$\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = 2xy - \frac{3}{2}y^2 - yz + f_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xz - yz - z^2 + f_3(x,y)$$

Here in above three equation gives the value of ϕ but it must be same among these 3 equations but it is not same. Here ϕ is the function of x, y, and z but in above 3 equations first eq^n is the func^n of x and yz treated as constant ~~similary~~ so missing terms are in $f_1(y,z)$ similary in second, third ~~and~~ eq^n missing terms are in $f_2(x,z)$ and $f_3(x,y)$. So net ϕ is –

$$\boxed{\phi(x,y,z) = \frac{x^2}{2} - \frac{3}{2}y^2 + 2xy + 4xz - yz + z^2 + c}$$

Ans

CSIR-2011 Dec

A-1

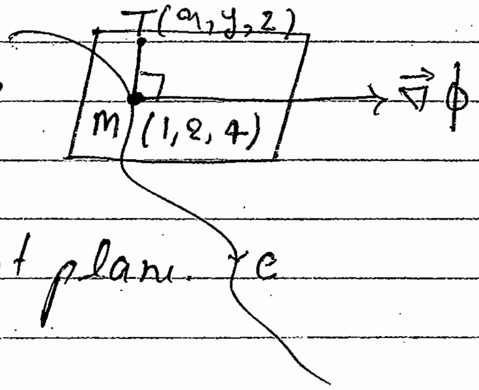
Q.46 The equation of the plane that is tangent to the surface $xyz = 8$ at the point $(1,2,4)$ is ?

(a) $x + 2y + 4z = 12$ ✓(b) $4x + 2y + z = 12$

(c) $x + 4y + 2z = 12$ (d) $x + y + z = 7$

Sol^n

$\because \left.\vec{\nabla}\phi\right|_{(1,2,4)}$ $\perp^{or}$ to the surface



So $\left.\vec{\nabla}\phi\right|_{(1,2,4)}$ $\perp^{or}$ to the tangent plane i.e

$$\left.\vec{\nabla}\phi\right|_{(1,2,4)} \cdot \overrightarrow{TM} = 0 \quad \text{---} \; (*)$$

Here $\overrightarrow{TM} = (x-1)\hat{i} + (y-2)\hat{j} + (z-4)\hat{k}$ — (a)

$$\because \nabla\phi = \frac{\partial}{\partial x}(xyz)\hat{i} + \frac{\partial}{\partial y}(xyz)\hat{j} + \frac{\partial}{\partial z}(xyz)\hat{k}$$

$$\nabla\phi = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\left.\nabla\phi\right|_{(1,2,4)} = 2\times 4\,\hat{i} + 1\times 4\,\hat{j} + 1\times 2\,\hat{k}$$

$$\left.\nabla\phi\right|_{(1,2,4)} = 8\hat{i} + 4\hat{j} + 2\hat{k} \quad \text{---} \; (b)$$

So from (*)

$$(8\hat{i} + 4\hat{j} + 2\hat{k}) \cdot \{(x-1)\hat{i} + (y-2)\hat{j} + (z-4)\hat{k}\} = 0$$

$$8(x-1) + 4(y-2) + 2(z-4) = 0$$

$$8x + 4y + 2z = 8 + 8 + 8 = 24$$

$$\boxed{4x + 2y + z = 12}$$ Ans

A-1

Q.34 For a constant vector $\vec{a}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Consider the following Statements:

(I) $\text{Curl}(\vec{a} \times \vec{r}) = 2[\text{grad}(\vec{a}\cdot\vec{r})]$

(II) $\text{div}[(\vec{a}\cdot\vec{r})\vec{r}] = 2(\vec{a}\cdot\vec{r})$, then -

(a) both statement are true ✓(b) Only statement I is true

(c) Only statement II is true (d) Both statements are false.

Sol^n

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}(a_2 z - y a_3) - \hat{j}(a_1 z - x a_3) + \hat{k}(a_1 y - a_2 x)$$

$$\text{Curl}(\vec{a} \times \vec{r}) = \vec{\nabla} \times [(\vec{a} \times \vec{r})]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_2 z - y a_3) & -(a_1 z - x a_3) & (a_1 y - a_2 x) \end{vmatrix}$$

$$= \hat{i}\left[\left(\frac{\partial}{\partial y}(a_1 y - a_2 x)\right) + \frac{\partial(a_1 z - x a_3)}{\partial z}\right] - \hat{j}\left[\frac{\partial}{\partial x}(a_1 y - a_2 x) - \frac{\partial}{\partial z}(a_2 z - y a_3)\right] + \hat{k}\left[\frac{-\partial(a_1 z - x a_3)}{\partial x} - \frac{\partial}{\partial y}(a_2 z - y a_3)\right]$$

$$= \hat{i}[a_1 + a_1] - \hat{j}[-a_2 - a_2] + \hat{k}[-[-a_3] - (-a_3)]$$

$$= 2a_1\hat{i} + 2a_2\hat{j} + 2a_3\hat{k}$$

$$\boxed{\text{Curl}(\vec{a} \times \vec{r}) = 2\vec{a}}$$

R.H.S. $= 2\,\text{grad}(\vec{a}\cdot\vec{r}) = 2\nabla(\vec{a}\cdot\vec{r})$

$\because\ \vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$

So $= 2\left[\nabla(a_1x + a_2y + a_3z)\right]$

$= 2\left[\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)\hat{i} + \frac{\partial}{\partial y}(a_1x+a_2y+a_3z)\hat{j} + \frac{\partial}{\partial z}(a_1x+a_2y+a_3z)\hat{k}\right]$

$= 2\left[a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right]$

$= 2\vec{a}$

$\therefore$ $\boxed{\text{L.H.S.} = \text{R.H.S.}}$ So statement I is true.

Second Statement - $\text{div}[(\vec{a}\cdot\vec{r})\vec{r}] = 2(\vec{a}\cdot\vec{r})$

$\Rightarrow \vec{\nabla}\cdot[(\vec{a}\cdot\vec{r})\vec{r}]$

$\Rightarrow \vec{\nabla}[(\vec{a}\cdot\vec{r})]\cdot\vec{r} + (\vec{a}\cdot\vec{r})(\vec{\nabla}\cdot\vec{r})$

$\Rightarrow \vec{a}\cdot\vec{r} + 3(\vec{a}\cdot\vec{r})$

$\Rightarrow 4(\vec{a}\cdot\vec{r})$

So second statement is false.

A-1

Q. 27 The surface $ax^2 + byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$. Which of the following relations is true for 'a' and 'b'?

(a) $a-b=4$ ✓(b) $2a-b=4$ (c) $a=3b$ (d) $3a+b=4$

Sol^n Say

$$\phi_1 = ax^2 - byz - (a-2)x = 0$$

$$\phi_2 = 4x^2y + z^3 - 4 = 0$$

$$\nabla\phi_1 = \frac{\partial}{\partial x}(ax^2 - byz - (a-2)x)\,\hat{i} + \frac{\partial}{\partial y}(ax^2 - byz - (a-2)x)\,\hat{j} + \frac{\partial}{\partial z}(ax^2 - byz - (a-2)x)\,\hat{k}$$

$$\nabla\phi_1 = (2ax - (a-2))\,\hat{i} + (-bz)\,\hat{j} + (-by)\,\hat{k}$$

$$(\nabla\phi_1)_{(1,-1,2)} = [2a\times 1 - (a-2)]\,\hat{i} - 2b\,\hat{j} + (-b\times -1)\,\hat{k}$$

$$\boxed{(\nabla\phi_1)_{(1,-1,2)} = (a+2)\,\hat{i} - 2b\,\hat{j} + (b)\,\hat{k}}$$

$$\nabla\phi_2 = \frac{\partial}{\partial x}(4x^2y + z^3 - 4)\,\hat{i} + \frac{\partial}{\partial y}(4x^2y + z^3 - 4)\,\hat{j} + \frac{\partial}{\partial z}(4x^2y + z^3 - 4)\,\hat{k}$$

$$\nabla\phi_2 = 8xy\,\hat{i} + 4x^2\,\hat{j} + 3z^2\,\hat{k}$$

$$(\nabla\phi_2)_{(1,-1,2)} = (8\times 1\times -1)\,\hat{i} + 4\times(1)^2\,\hat{j} + 3(2)^2\,\hat{k}$$

$$\boxed{(\nabla\phi_2)_{(1,-1,2)} = -8\,\hat{i} + 4\,\hat{j} + 12\,\hat{k}}$$

$$\nabla\phi_1 \cdot \nabla\phi_2 \Big|_{(1,-1,2)} = 0 = \left[(a+2)\,\hat{i} - 2b\,\hat{j} + (b)\,\hat{k}\right] \cdot \left[-8\,\hat{i} + 4\,\hat{j} + 12\,\hat{k}\right]$$

$$\Rightarrow -8(a+2) - 8b + 12(b) = 0$$

$$\Rightarrow -8a - 16 - 8b + 12b = 0$$

$$\Rightarrow -8a + 4b - 16 = 0 \Rightarrow 4(-2a+b) = 16$$

$$\Rightarrow \boxed{-2a+b=4}$$ Ans.

A-2

Q10 If $f = x+y+z$, $g = x^2+y^2+z^2$, $h = xy+yz+zx$ then show that $[\nabla f\ \nabla g\ \nabla h] = 0$

Sol^n

$\because [\vec{A}\vec{B}\vec{C}] = \vec{A} \cdot [\vec{B} \times \vec{C}]$

$\therefore [\vec{\nabla} f\ \vec{\nabla} g\ \vec{\nabla} h] = \vec{\nabla} f \cdot (\vec{\nabla} g \times \vec{\nabla} h)$

$\nabla f = \hat{i} + \hat{j} + \hat{k}$, $\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla h = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2x & 2y-2x & 2z-2x \\ y+z & x+\cancel{z}-y-\cancel{z} & x+\cancel{y}-\cancel{y}-z \end{vmatrix} \begin{matrix} C_2 - C_1 \to C_2 \\ C_3 - C_1 \to C_3 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2x & 2(y-x) & 2(z-x) \\ y+z & x-y & (x-z) \end{vmatrix} = 0$$

Expanding along first Row -

$\Rightarrow 1.[(x-z)\,2(y-x) - (x-y).2(z-x)] = 0$

$\Rightarrow \cancel{2xy} - \cancel{2yz} - \cancel{2x^2} + \cancel{2zx} - \cancel{2xz} + \cancel{2yz} + \cancel{2x^2} - \cancel{2xy} = 0$

$= 0$ Proved

A-2

Q17 Prove that, if a rigid body is rotating with a constant angular velocity, then the curl of linear velocity at any point gives twice its angular velocity.

Sol^n

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$$

$\because$ We know that $\vec{\nabla} \times (\vec{a} \times \vec{r}) = 2\vec{a}$ where $\vec{a}$ is a constant vector.

$$\therefore \quad \vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\omega} \times \vec{r}) = 2\vec{\omega}$$

becoz ω is constant vector   Proved

A-1

Q31 If $\vec{a}$ and $\vec{b}$ are irrotational in nature, then $\vec{a} \times \vec{b}$ will be -

(a) irrotational (b) solenoidal ✓ (c) rotational

(d) none of these

Sol^n

$$\because \quad \vec{\nabla} \times \vec{A} = 0 \quad \& \quad \vec{\nabla} \times \vec{B} = 0$$

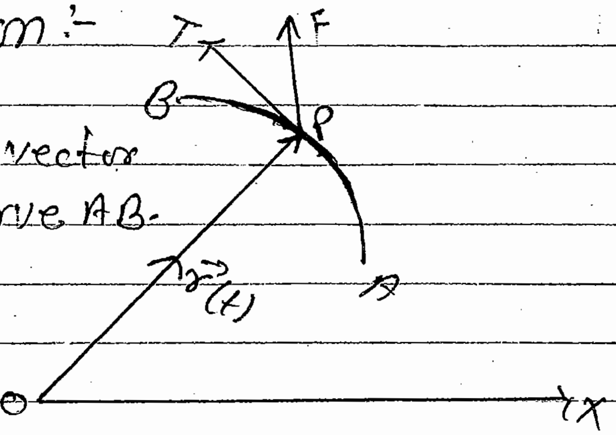
$$\text{So} \quad \underset{A}{\vec{\nabla}} \cdot (\underset{B}{\vec{A}} \times \underset{C}{\vec{B}}) = \underset{B}{\vec{B}} (\underset{A}{\vec{\nabla}} \times \underset{C}{\vec{A}}) - \underset{C}{\vec{A}} (\underset{A}{\vec{\nabla}} \times \underset{B}{\vec{B}})$$

$$= 0$$

So it is solenoidal.

* Integration Of Vectors :-

* Line Integration :-



Let $\vec{F}(x,y,z)$ be a vector function and a curve AB.

"Line integration of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

Component of $\vec{F}$ along a tangent PT at P

$= $ Dot product of $\vec{F}$ and unit vector along PT

$= \vec{F} \cdot \frac{d\vec{r}}{ds}$ $\left(\frac{d\vec{r}}{ds} \text{ is a unit vector along tangent PT} \right)$

Line integral $= \sum \vec{F} \cdot \frac{d\vec{r}}{ds}$ from A to B along the curve.

$$\boxed{\text{Line integral} = \int_c \left(F \cdot \frac{d\vec{r}}{ds} \right) ds = \int_c \vec{F} \cdot d\vec{r}}$$

by Sir

Line Integration :-

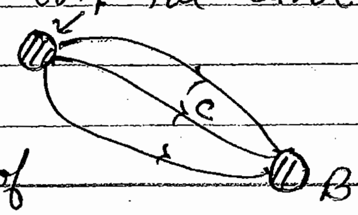
Integration of a vector along a curve. C.

Tip $\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$ trace a curve C.

Suppose $\vec{F}(t)$ is vector which define & continuous all the points on the curve C.

$$\boxed{F(t) = \int_A^B \vec{F} \cdot d\vec{r}}$$

← Mathematical representation of line integration.



It is the workdone by the force on the particle for the displacement from point A to B.

$$\boxed{W_{AB} = \int_A^B \vec{F}.d\vec{r}}$$

If $\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F} = \vec{\nabla}\phi$

If force is conservative -

$$W_{AB} = \int_A^B \vec{F}.d\vec{r} = \int_A^B \vec{\nabla}\phi . d\vec{r}$$

$$= \int_A^B d\phi$$

$\because \phi = \phi(x,y,z)$

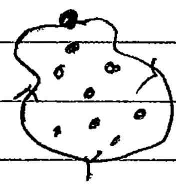
$\Rightarrow d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz.$

$$\boxed{W_{AB} = \phi_B - \phi_A}$$

Workdone by a conservative field is independent of path it only depends on the value of the scalar potential at the initial and final points.

$\Rightarrow$ Workdone by a conservative field along a closed path is zero.

$$\boxed{W = \oint \vec{F}.d\vec{r} = 0}$$



This condition is true only if field $\vec{F}$ is define and continuous at all points on curve and within the curve.

1.37 If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, then the value of the line integral $\int \vec{F}\cdot d\vec{r}$ along the curve C in the x-y plane by $y = x^3$, from the point (1, 1) to (2, 8) is equal to ?

(a) 35 ✓ (b) −35 (c) 47 (d) −47

Soln.

$$\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$$

$$C: y = x^3$$

(1, 1) to (2, 8) so $x = 1$ to 2

$$\int_C \vec{F}\cdot d\vec{r} = \int (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

$$\because y = x^3,\ dy = 3x^2 dx$$

$$= \int_1^2 (5x\cdot x^3 - 6x^2)\,dx + \int_1^2 (2x^3 - 4x)\cdot 3x^2\,dx$$

$$= \int_1^2 5x^4\,dx - \int_1^2 6x^2\,dx + \int_1^2 6x^5\,dx - \int_1^2 12x^3\,dx$$

$$= 5\left[\frac{x^5}{5}\right]_1^2 - 6\left[\frac{x^3}{3}\right]_1^2 + 6\left[\frac{x^6}{6}\right]_1^2 - 12\left[\frac{x^4}{4}\right]_1^2$$

$$= 5\left(\frac{2^5}{5} - \frac{1}{5}\right) - 6\left(\frac{2^3}{3} - \frac{1}{3}\right) + \left(\frac{2^6}{6} - \frac{1}{6}\right) - 12\left(\frac{2^4}{4} - \frac{1}{4}\right)$$

$$= \cancel{5}\left(\frac{32-1}{\cancel{5}}\right) - \overset{2}{\cancel{6}}\left(\frac{8-1}{\cancel{3}}\right) + \cancel{6}\left(\frac{64-1}{\cancel{6}}\right) - \overset{3}{\cancel{12}}\left(\frac{16-1}{\cancel{4}}\right)$$

$$= 31 - 14 + 63 - 45$$

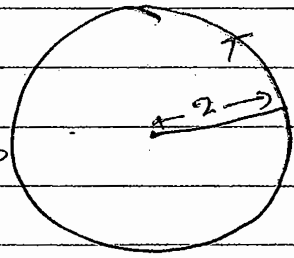
$$= 94 - 59 = 35 \quad \underline{\text{Ans}}$$

A-1

Q38 If a force field $\vec{A} = (y+2x)\hat{i} + (3x+2y)\hat{j}$ is applied to a particle, the the work done by the force field in traversing the particle around a circle C in x-y plane with center at the origin and radius 2 units is equal to (C is traverse in the counter-clockwise direction)

(a) 2π (b) 4π (c) 8π ✓ (d) 16π

Sol^n



$$\vec{A} = (y+2x)\hat{i} + (3x+2y)\hat{j}$$

$$\int \vec{A}\cdot d\vec{r} = \int \{(y+2x)\hat{i} + (3x+2y)\hat{j}\}\, d\vec{r}$$

$x = r\cos\theta$ $\quad dx = -2\sin\theta\, d\theta$ $\quad \{\because r = 2$ unit

$y = r\sin\theta$ $\quad dy = 2\cos\theta\, d\theta$

$$\Rightarrow \int A\,dr = \int (r\sin\theta + 2r\cos\theta)\hat{i}\,dx + \int (3r\cos\theta + 2r\sin\theta)\hat{j}\,dy$$

$$= -4\int_0^{2\pi} \sin\theta\sin\theta\, d\theta - 2\times4\int_0^{2\pi}\cos\theta\sin\theta\, d\theta$$

$$+ 2\times6\int_0^{2\pi}\cos\theta\cos\theta\, d\theta + 4\times2\int_0^{2\pi}\sin\theta\cos\theta\, d\theta$$

$$= -4\int_0^{2\pi}\sin^2\theta\, d\theta \cancel{- 4\int_0^{2\pi}\sin2\theta\, d\theta} + 12\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$\cancel{+ 4\int_0^{2\pi}\sin 2\theta\, d\theta}$$

$$= -2\int_0^{2\pi}(1-\cos2\theta)\,d\theta + 6\int_0^{2\pi}(1+\cos2\theta)\,d\theta$$

$$= -2\left(\theta - \frac{\sin2\theta}{2}\right)_0^{2\pi} + 6\left(\theta+\frac{\sin2\theta}{2}\right)_0^{2\pi}$$

$$= -4\pi + 12\pi$$

$$= 8\pi \text{ Ans}$$

$2\sin^2\theta = 1-\cos 2\theta$

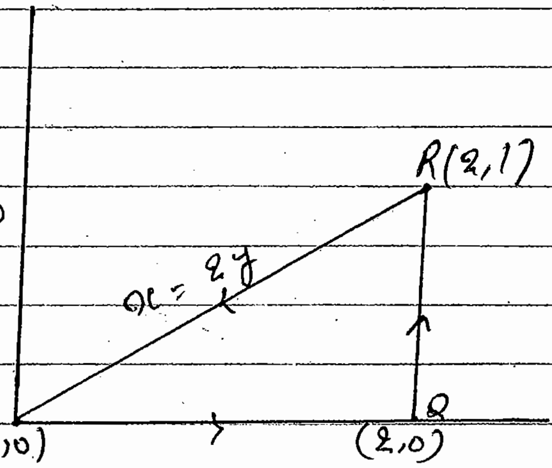
$2\cos^2\theta = 1+\cos 2\theta$

A-1

Q.39 When a force $\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ is applied on a particle, then it starts to move along a right angled triangle PQR having vertices P(0,0), Q(2,0), R(2,1) respectively. The amount of work done by the force is equal to -?

(a) $\frac{14}{3}$ (b) $-\frac{14}{3}$ ✓ (c) $\frac{22}{3}$ (d) $-\frac{22}{3}$

Sol



$\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{PQ} \vec{F}\cdot d\vec{r} + \int_{QR} \vec{F}\cdot d\vec{r} + \int_{RP} \vec{F}\cdot d\vec{r}$$

$\Rightarrow P \to Q$: $x: 0 \to 2$

$y: 0 \to 0 \quad dy = 0$

$$I_{PQ} = \int_{PQ} 2x\,dx = \int_0^2 2x\,dx = 4$$

$Q \to R$: $x: 2 \to 2$

$y: 0 \to 1$

$$\int_0^1 (2y-8)\,dy = 3\left(\frac{y^2}{2}\right)_0^1 - 8(y)_0^1$$

$$= -\frac{13}{2}$$

$R \to P$ x and y both change

$\therefore x = 2y$

If along a line if x, y both change then x and y are dependent. So we find relation between them.

$R \to P$ x and y both change

$\therefore x = 2y$

$= \int_1^0 (2(2y)+y^2)\,2dy + 3y\,dy - 4\times 2y\,dy$

$= 2\int_1^0 (4y+y^2)\,dy + \int_1^0 3y\,dy - 8\int_1^0 y\,dy$

$= 2\left(\frac{y^3}{3}\right) + \cancel{1}8\left(\frac{y^2}{2}\right)_1^0 + 3\left(\frac{y^2}{2}\right)_1^0 - 8\left(\frac{y^2}{2}\right)_1^0$

$= -\frac{4}{\cancel{}} - \frac{2}{3} - \frac{3}{2} + \cancel{4} = -\frac{4-9}{6} = \frac{-13}{6}$

So $P\to Q + Q \to R + R \to P$

$\Rightarrow \quad 4 - \frac{13}{2} - \frac{13}{6}$

$= -\frac{14}{3}$ Ans

2. $\vec{F} = (3x+2y)\hat{i} + (y-2z)\hat{j} + (z+4y)\hat{k}$

$\int \vec{A}\cdot d\vec{r}$

Along straight line from (x_1, y_1, z_1) to (x_2, y_2, z_2)

Sol^n $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$ (say)

$t \to 0$ to 1

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Bigg\}$$ Two point straight line eqn

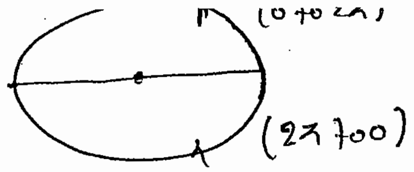
let $(-1, 2, 5)$ to $(-7, 3, -4)$

$x_1 \; y_1 \; z_1 \qquad x_2 \; y_2 \; z_2$

$$\frac{x+1}{-7+1} = \frac{y-2}{3-1} = \frac{z-5}{-4-5} = t$$

$$\frac{x+1}{6} = \frac{y-2}{2} = \frac{z-5}{-9} = t$$

$$\left.\begin{aligned} x &= -6t-1 \\ y &= t+2 \\ z &= -9t+5 \end{aligned}\right\}$$



A-2
Q.25 Show that, $\int_C \left[-\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}\right]\cdot d\vec{r} = 2\pi$ where C is the circle $x^2+y^2=1$ in the x-y plane described in counter clockwise direction.

Sol^n

$x^2+y^2=1$ $\quad r=1$

$x = r\cos\theta, \quad y = r\sin\theta, \quad \because r=1$

$x = \cos\theta, \quad dx = -\sin\theta$

$y = \sin\theta, \quad dy = \cos\theta$

$$= \int_C \frac{-y\,dx + x\,dy}{x^2+y^2 \to 1}$$

$$= \int -y\,dx + \int x\,dy$$

$$= \int_0^{2\pi} -\sin\theta(\sin\theta)\,d\theta + \cos\theta\cos\theta\,d\theta$$

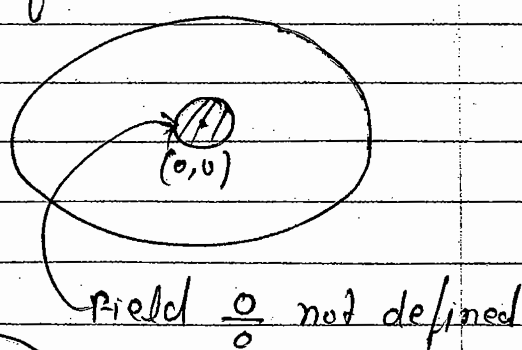
$$= \int_0^{2\pi} 1\,d\theta = [\theta]_0^{2\pi} = 2\pi \text{ Ans}$$

But if we calculate curl of $\vec{A}$

$\vec{\nabla}\times\vec{A} = 0$

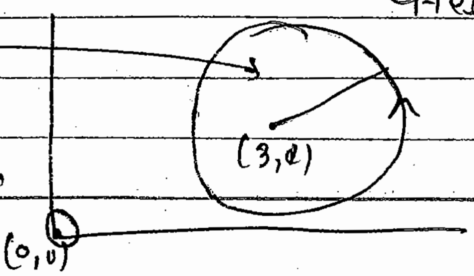
but $\oint \vec{A}\cdot d\vec{r} = 0$

but there is it is $= 2\pi$



* C: $(x-3)^2 + (y-4)^2 = 1$

$\boxed{\oint \vec{A}\cdot d\vec{r} = 0}$ becoz it is defined inside and outside of the curve.



A-2

Q.20 Given: $\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k}$ when $t = 2$ and $\vec{r}(t) = 4\hat{i} - 2\hat{j} + 3\hat{k}$ when $t = 3$. Show that, $\int_2^3 \vec{r} \cdot \frac{d\vec{r}}{dt}\, dt = 10$.

Sol^n

$$\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k} \quad at \Rightarrow t = 2$$

$$\vec{r}(t) = 4\hat{i} - 2\hat{j} + 3\hat{k} \quad at \Rightarrow t = 3$$

$$\int_2^3 \vec{r}\, \frac{d\vec{r}}{dt}\, dt \qquad \because \frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\frac{d\vec{r}}{dt}$$

$$= \frac{1}{2}\int_{t=2}^{t=3} \frac{d}{dt}(\vec{r}\cdot\vec{r})\, dt$$

$$= \frac{1}{2}(\vec{r}\cdot\vec{r})\Big|_{t=2}^{t=3}$$

CSIR Jun-2012

$$= 10 \text{ Ans}$$

A-1

Q.47 A vector perpendicular to any vector that lies on the plane defined by $x + y + z = 5$ is?

(a) $\hat{i} + \hat{j}$ (b) $\hat{j} + \hat{k}$ (c) $\hat{i} + \hat{j} + \hat{k}$ (d) $2\hat{i} + 3\hat{j} + 5\hat{k}$

Sol^n

$$\phi = x + y + z = 5$$

$$\boxed{\vec{\nabla}\hat{\phi} = \hat{i} + \hat{j} + \hat{k}}$$

Ans

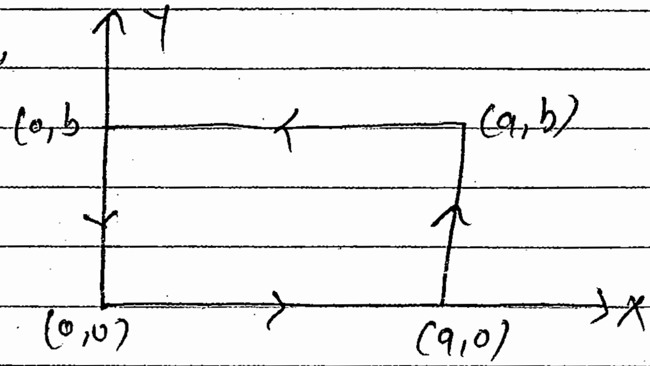
CSIR Dec-2013

Q.50 If $\vec{A} = yz\hat{i} + xz\hat{j} + xy\hat{k}$, then the integral $\oint_C \vec{A}\cdot d\vec{l}$ (where C is along the perimeter of a rectangular bounded by $x = 0$, $x = a$ and $y = 0$, $y = b$) is?

(a) $\frac{1}{2}(a^3 + b^3)$ (b) $\pi(ab^2 + a^2 b)$ (c) $\pi(a^3 + b^3)$ ✓(d) 0

Sol) $\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ $(\because \vec{\nabla}\times\vec{A} = 0)$

$\oint \vec{A}\cdot d\vec{l} = 0$ $\{\because$ field is defined



By Stokes theorem -

$\iint (\vec{\nabla}\times\vec{A})\, ds$

$= 0$ **Ans**

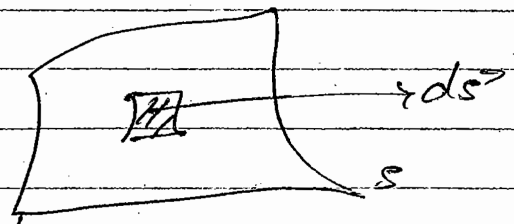
$\iint_S \vec{A}\cdot d\vec{s}$ = Flux of the vector field A through the surface S.

---

* **Surface Integration :-** It is a integration of a vector on a open or a close surface.

Any surface that incloses some volume is close surface, e.g. sphere, cube etc.

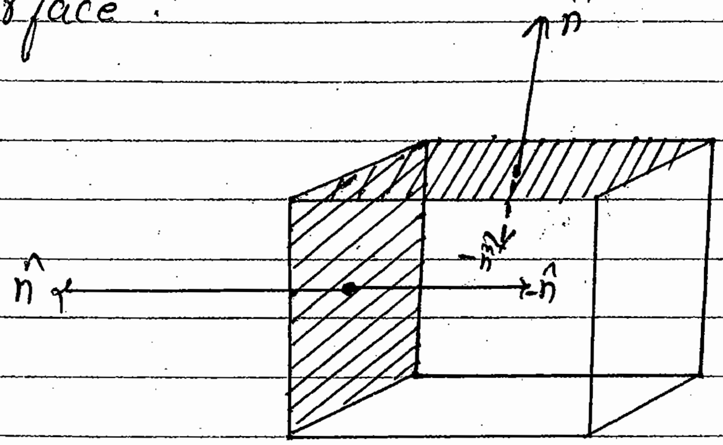
$$\iint_S \vec{A}\cdot d\vec{s} = \iint_S \vec{A}\cdot \hat{n}\, ds$$



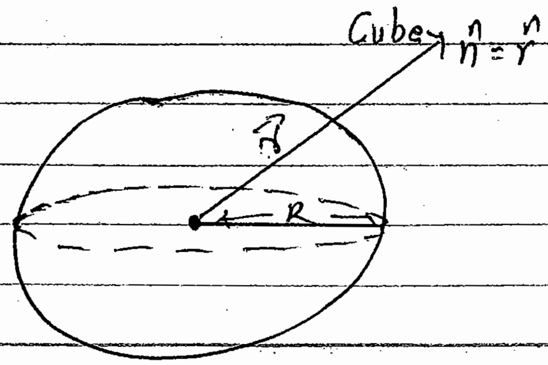
$\hat{n}$ is the direction of $\vec{A}$. And $d\vec{s}$ is magnitude of element.

$\hat{n}$ is a unit vector normal to the surface S indicating the +ve direc^n of surface.

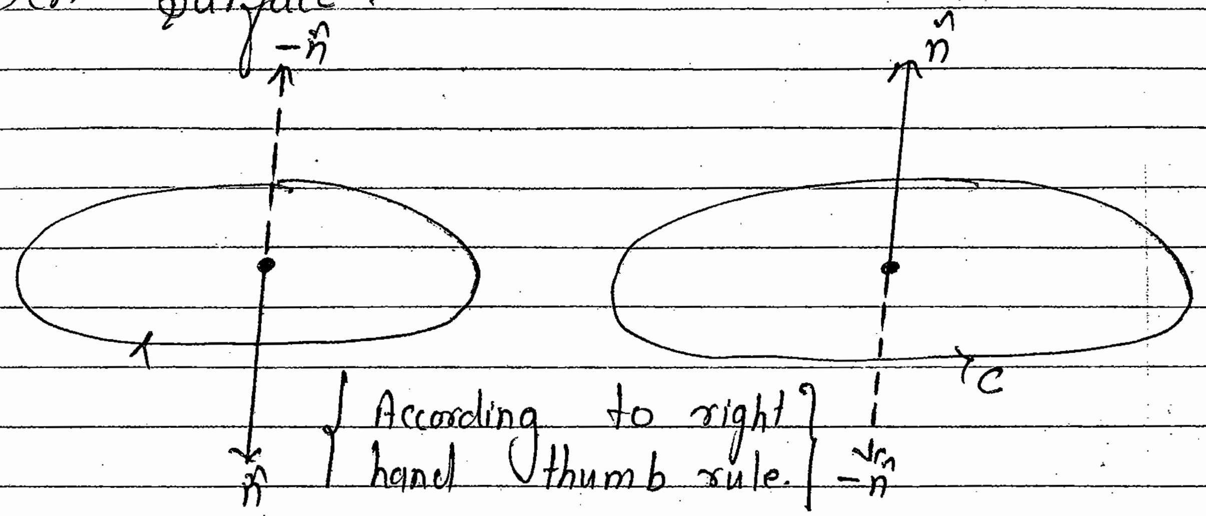
1. Close Surface :-

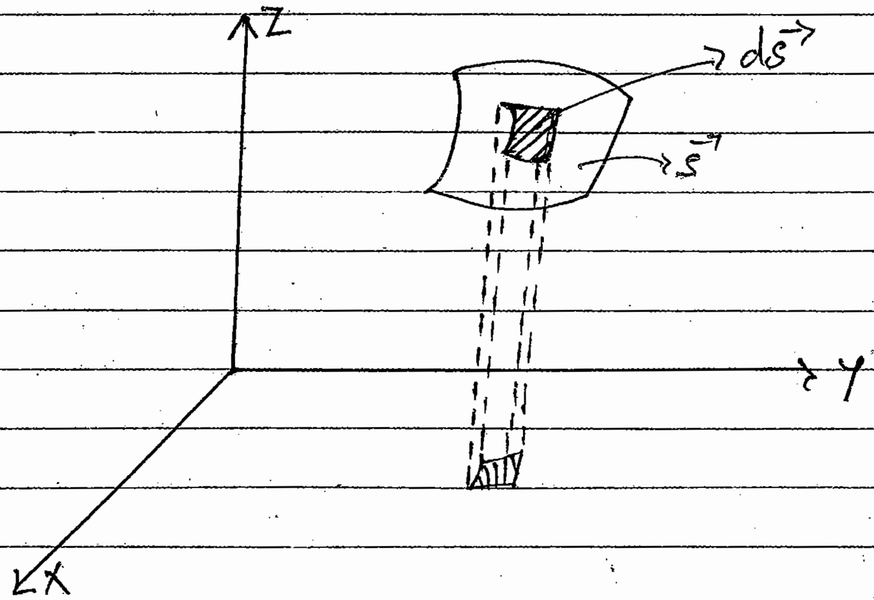


2. Sphere :-



2. Open Surface :-





If the surface of integration is not parallel to any one of the co-ordinate plane then we have to draw the projection on any one of co-ordinate plane.

~~Ete~~ Projection of $\vec{ds}$ in xy-plane = $dx\,dy$

$$\Rightarrow \quad ds.\hat{k} = dx\,dy$$

{ ∵ $\vec{A}$ $\hat{B}$
$\vec{ds}$ xy-plane's dir^n

$$\Rightarrow \quad \hat{n}\,ds.\,\hat{k} = dx\,dy$$

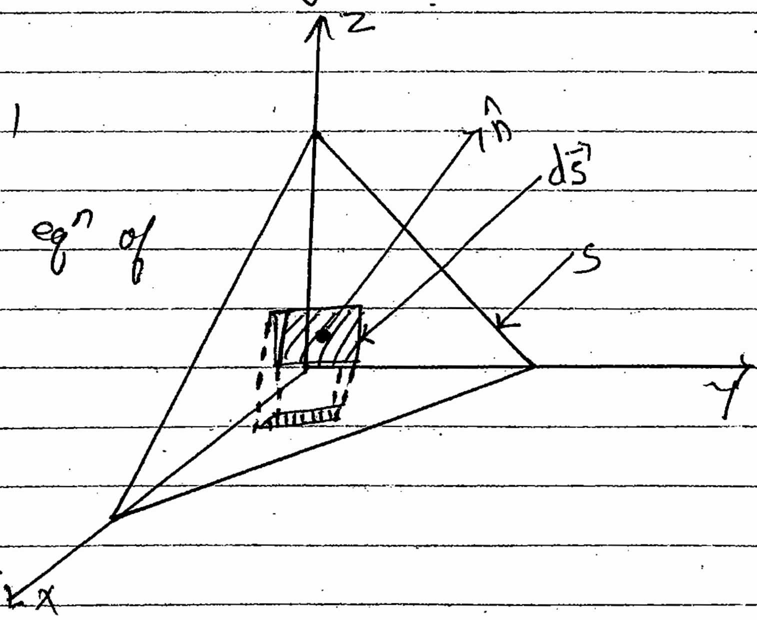
$$\Rightarrow \quad ds\,(\hat{n}.\hat{k}) = dx\,dy$$

$\Rightarrow \boxed{ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}}$

$$\boxed{\iint_S \vec{A}\cdot\hat{n}\,d\vec{s} = \iint \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}}$$

Here | | mod is becoz $\hat{n}\cdot\hat{k}$ is area and area is always +ve.

Ex $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ Calculate $\iint \vec{A}\cdot\hat{n}\,d\vec{s}$
S is surface of plane $2x + 3y + 6z = 12$ in the first octant.



Sol^n $\frac{x}{6} + \frac{y}{4} + \frac{z}{2} = 1$

$\because$ We know in 3-D eq^n of the plane is -

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\iint \vec{A}\cdot\hat{n}\,ds = \iint \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$S : 2x + 3y + 6z = 12$

$$\hat{n} = \frac{\vec{\nabla}S}{|\vec{\nabla}S|} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}$$

$$\hat{n}\cdot\hat{k} = \frac{6}{7}$$

$$\vec{A}\cdot\hat{n} = \frac{36z - 36 + 18y}{7}$$

$$\therefore \quad z = \frac{12 - 2x - 3y}{6}$$

$$\vec{A}\cdot\hat{n} = \frac{\frac{\cancel{36}^{\,6}(12-2x-3y)}{\cancel{6}} - 2x - 3y}{7}$$

$$\vec{A}\cdot\hat{n} = \frac{36-12x}{7}$$

~~If we draw the~~

$$\iint \vec{A}\cdot\hat{n}\,ds = \iint \frac{36-12x}{\cancel{7}} \cdot \frac{dx\,dy}{6/\cancel{7}}$$

$$= \iint (6-2x)\,dx\,dy$$

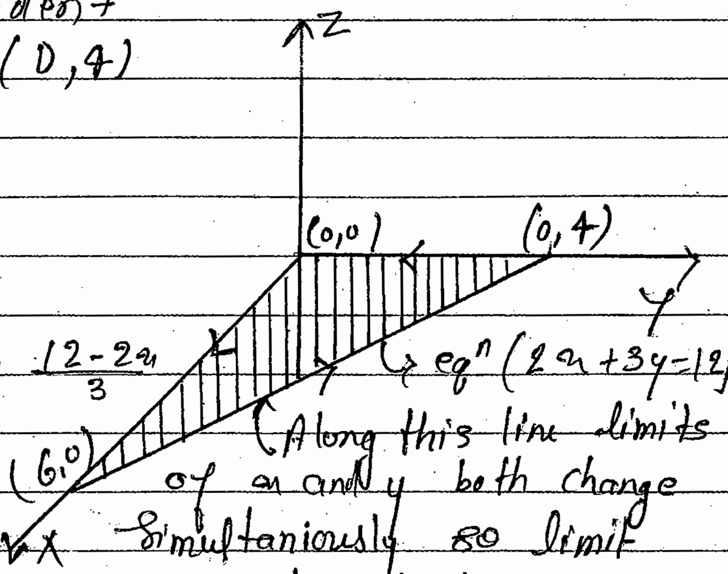
If we draw the total projection of the plane then it is like a triangle in X-Y plane.

$\because$ x & y are dependent

$\therefore$ Eqn of line $(6,0)$ to $(0,4)$ is :-

$$2x + 3y = 12$$

$$y = \frac{12-2x}{3}$$



(Along this line limits of x and y both change simultaniously so limit are dependent.

So y varies from 0 to $\frac{12-2x}{3}$

$\because$ y is already integrated.

So put $y=0$ in eqn

$$2x + 3\times 0 = 12$$

$$2x = 12$$

$$\boxed{x = 6}$$

So x varies from 0 to 6.

$$\Rightarrow \int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\,dx\,dy$$

$$\Rightarrow \int_0^6 \left[6y - 2xy\right]_0^{\frac{12-2x}{3}} dx$$

$$\Rightarrow \int_0^6 \left[\cancel{6}^{\,2}\frac{(12-2x)}{\cancel{3}} - 2x\frac{(12-2x)}{3}\right] dx$$

$$\Rightarrow \int_0^6 \left[24 - 4x - \frac{\cancel{24}^{\,8}x}{\cancel{3}} + \frac{4x^2}{3}\right] dx$$

$$\Rightarrow \int_0^6 \left(\frac{4x^2}{3} - 12x + 24\right) dx$$

$$\Rightarrow \frac{4}{3}\left(\frac{x^3}{3}\right)_0^6 - 12\left(\frac{x^2}{2}\right)_0^6 + 24(x)_0^6$$

$$\Rightarrow \frac{4}{\cancel{3}}\left(\frac{\cancel{216}^{\,\cancel{72}\ 24}}{\cancel{3}}\right) - 12\left(\frac{\cancel{36}^{\,18}}{\cancel{2}}\right) + 24(6)$$

$$\Rightarrow 96 - 216 + \cancel{264}\ 144$$

$$\Rightarrow 240 - 216$$

$$\Rightarrow = 24 \quad \underline{\text{Ans}}$$

Q. $\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$ $\quad \iint \vec{F}\cdot\hat{n}\,d\vec{s}$

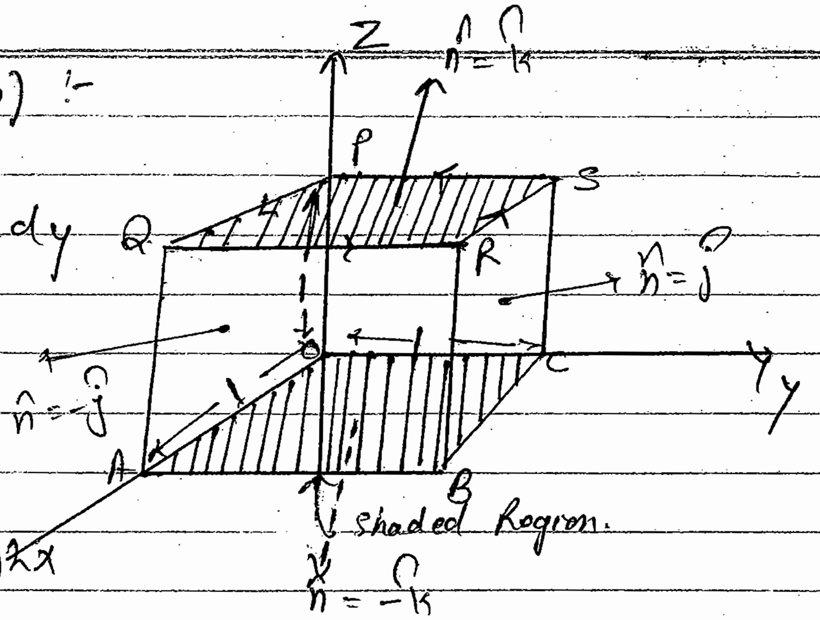
S: Surface of the cube bounded by eqn of plane $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$

Solⁿ I: PQRS Plane ($z=1$ plane):-

$$\hat{n} = \hat{k}, \quad ds = dx\,dy$$

$$\iint yz\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = \frac{1}{2}$$

II OABC Plane $(Z=0)$:-



$\hat{n} = -\hat{k}$, $d\vec{s} = dx\,dy$

$$\int_0^1\int_0^1 -yz\,dx\,dy = 0$$

III PQAO Plane ($y=0$ Plane) :-

$\hat{n} = -\hat{j}$, $d\vec{s} = dx\,dz$

$$\iint y^2\,dx\,dz = 0 \quad \{\because y=0\}$$

IV SRBC Plane ($y=1$ Plane) :-

$\hat{n} = \hat{j}$, $d\vec{s} = dy\,dz$

$$\iint y^2\,dy\,dz = \int_0^1\int_0^1 dx\,dz = [y]_0^1 \cdot [z]_0^1$$

$$= 1$$

V QRBA Plane ($x=1$ Plane) :-

$\hat{n} = \hat{i}$, $d\vec{s} = dz\,dy$

$$= \iint 4xz\,dz\,dy$$

$$= \int_0^1\int_0^1 4z\,dz\,dy$$

$$= 4\left(\frac{z^2}{2}\right)(y)_0^1 = 2$$

VI OPSC plane ($x=0$ Plane) :-

$\hat{n} = -\hat{i}$ $\qquad ds = dy\,dz$

A-1

Q.40 The flux of the vector field $\vec{V} = r^2\vec{r}$ over the region bounded by the surface S of a sphere of radius 'a' centered at the origin is equal to -

(a) πa^5 (b) $2\pi a^5$ (c) $4\pi a^5$ (d) $8\pi a^5$

Sol^n

$\vec{V} = r^2\vec{r}$

$\iint \vec{V}\cdot d\vec{s}$

$S: x^2+y^2+z^2 = a^2$

$\Rightarrow \iint r^2\vec{r}\cdot\hat{r}\,d\vec{s}$

$\Rightarrow \iint r^2\cdot r\,d\vec{s}$

$\Rightarrow r^3\iint d\vec{s} = 4\pi a^3[a^2]$

$= 4\pi a^5$ **Ans**

Q.43 The volume of the portion of the cyllinder $x^2+y^2=4$ in the first octant between the planes $z=0$ and $3x-z=0$ is ?

(a) 2 (b) 4

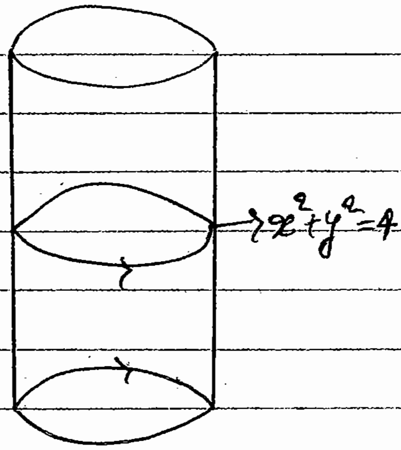
(c) 8 (d) 16.

Cylinder $x^2+y^2=4$

$z=0$

$3x-z=0$

$\iiint dx\,dy\,dz$ $\quad z=0$ to $3x$



$x^2+y^2=4$

$\Rightarrow y=\pm\sqrt{4-x^2}$

limit y: x to $\pm\sqrt{4-x^2}$

$x=\pm 2$

x: 0 to $+2$

$$V=\int_0^2\int_0^{\pm\sqrt{4-x^2}}\int_0^{3x} dx\,dy\,dz$$

$$V=\int_0^2\int_0^{\pm\sqrt{4-x^2}} 3x\,dx\,dy$$

$$V=\int_0^2\left[3xy\right]_0^{\pm\sqrt{4-x^2}}dx=\int_0^2\left[3x\sqrt{4-x^2}\right]dx$$

$$=3\int_0^2 x\sqrt{4-x^2}\,dx$$

$4-x^2=p^2$

$-\cancel{2}x\,dx=\cancel{2}p\,dp$

$$=-3\int_2^0 p^2\,dp=-3\left[\frac{p^3}{3}\right]_2^0$$

$$=-3\left[\frac{0}{3}-\frac{8}{3}\right]=8 \quad \underline{\text{Ans}}$$

Date :- 07/Aug/2014

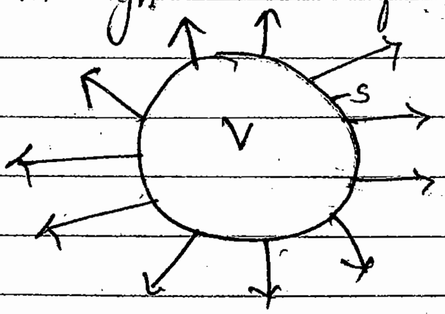
* Divergence Theorem :-

This theorem is applicable only for closed surfaces and this theorem converts volume integral or surface integral or vice-versa statement is -

If V is a volume bounded by a close surface S and $\vec{A}$ is a vector function of position with continuous derivative then -

$$\boxed{\oint\!\!\!\int_S \underbrace{\vec{A}\cdot d\vec{s}}_{\text{Total flux outward through the surface } s.} = \iiint_V \underbrace{(\vec{\nabla}\cdot\vec{A})}_{\text{Flux per unit volume.}}\, dv}$$

Total flux outward through the surface s. Flux per unit volume.



A-7

Q.41 If $\vec{A} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, then the value of the integral $\int \vec{A}\cdot\hat{n}\,ds$ (where S is the surface of a unit cube with two opposite ~~corn~~ corners at $(0,0,0)$ and $(1,1,1)$ respectively) is equal to -

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) 6 (d) $\frac{9}{2}$

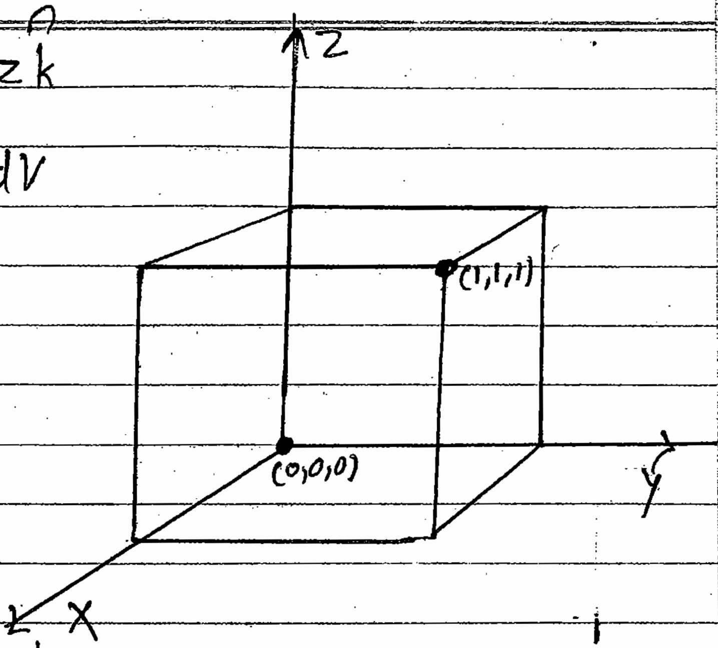
07/Aug/2014

$$\vec{A} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iiint (\vec{\nabla}\cdot\vec{A})\,dV$$

$$\vec{\nabla}\cdot\vec{A} = 4z - y$$

$$\iint_S \vec{A}\cdot\hat{n}\,d\vec{s} = \iiint (\vec{\nabla}\cdot\vec{A})\,dV$$



$$= \int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz$$

$$= \int_0^1 dx\,.\left(\int_0^1 4z\,dz - \int_0^1 dy\right)$$

$$= \frac{3}{2} \quad \underline{\text{Ans}}$$

A-2

Q.32 Consider the following vector field: $\vec{V} = xz^2\hat{i} - yz^2\hat{j} + z(a^2 - y^2)\hat{k}$

(a) Calculate the volume integral of the divergence of $\vec{V}$ over the region defined by $-a \le x \le a$, $-b \le y \le b$ and $0 \le z \le c$.

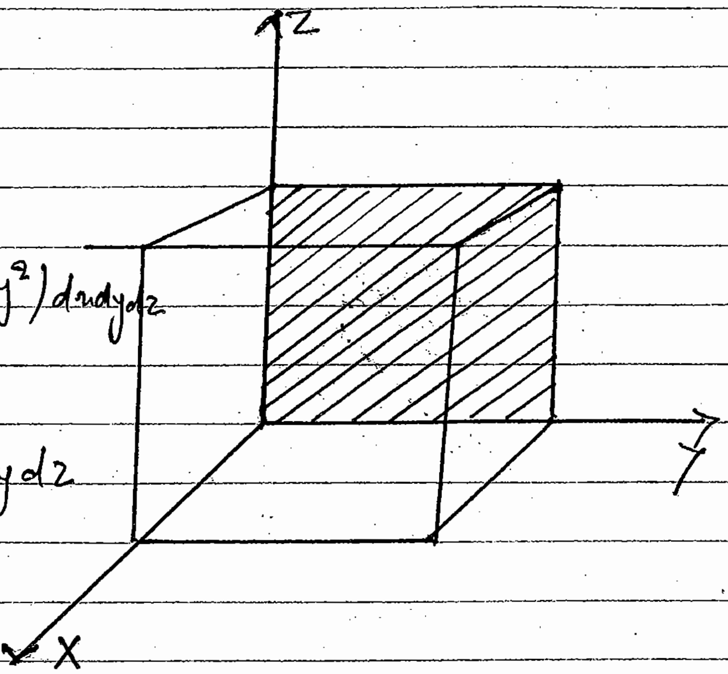
(b) Calculate the flux of $\vec{V}$ out of the region through the surface at $z = c$

(c) Hence deduce the net flux through the rest of the boundary of the region.

Sol^n $\vec{V} = xz^2\hat{i} - yz^2\hat{j} + z(a^2 - y^2)\hat{k}$

$$\vec{\nabla}\cdot\vec{V} = \cancel{z^2} - \cancel{z^2} + (x^2-y^2)$$

$$\boxed{\vec{\nabla}\cdot\vec{V} = x^2-y^2}$$



$$\int_{-a}^{a}\int_{-b}^{b}\int_{0}^{c} \nabla\cdot\vec{V}\,dv = \int_{-a}^{a}\int_{-b}^{b}\int_{0}^{c}(x^2-y^2)\,dx\,dy\,dz$$

$$= \int_{-a}^{a} x^2\,dx\,dy\,dz - \int_{-b}^{b} y^2\,dx\,dy\,dz$$

$$= \cancel{\frac{4}{3}abc(a^2-b^2)}$$

$$= 4\left[\frac{x^3}{3}\right]_0^a bc - 4\left[\frac{y^3}{3}\right]_0^b ca$$

$$= \frac{4}{3}\left[a^3bc - b^3ca\right]$$

$$= \frac{4}{3}abc\left[a^2-b^2\right]$$

Ans

(ii) $\iint \vec{V}\cdot\hat{n}\,ds$ $\qquad ds \Rightarrow dx\,dy$, $\hat{n} = \hat{k}$

$$= \iint z\,(x^2-y^2)\,dx\,dy$$

$$= c\iint (x^2-y^2)\,dx\,dy$$

$$= c\left[\iint x^2\,dx\,dy - \iint y^2\,dx\,dy\right]$$

$$= c\left[4\left(\frac{x^3}{3}\right)_0^a b - \frac{4}{3}\left(y^3\right)_0^b a\right]$$

$$= \frac{4}{3}abc\,(a^2-b^2)$$

Ans

(iii) So, the total flux through rest 5-faces is zero.
Because $\frac{4}{3}abc\,(a^2-b^2)$ is entering and $\frac{4}{3}abc\,(a^2-b^2)$ is leaving flux.

* Sphere :-

$$x^2+y^2+z^2=a^2$$

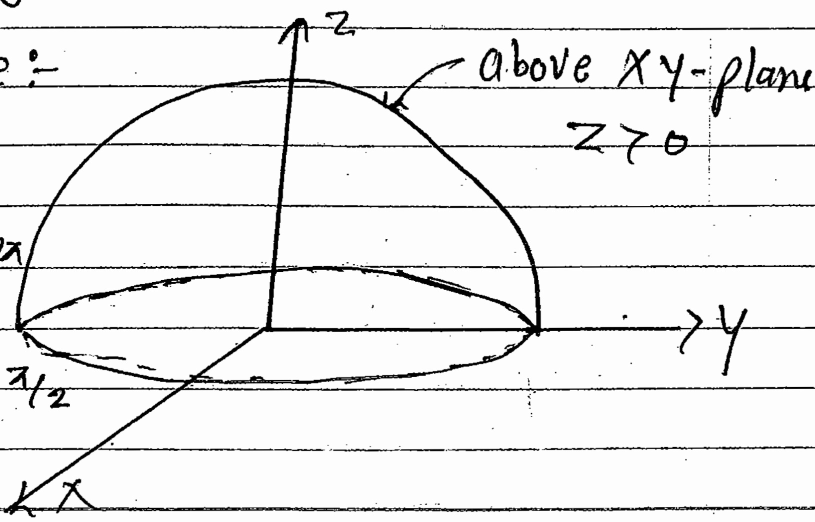
Volume $0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$

Surface: $r=a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$

* Hemisphere :-

$$x^2+y^2+z^2=a^2$$

Upper Hemisphere :-



Volume :- $0 \le r \le a$
$0 \le \theta \le \pi/2,\ 0 \le \phi \le 2\pi$

Surface :- $r=a,\ 0 \le \theta \le \pi/2$
$0 \le \phi \le 2\pi$

Lower Hemisphere :-

Volume :-

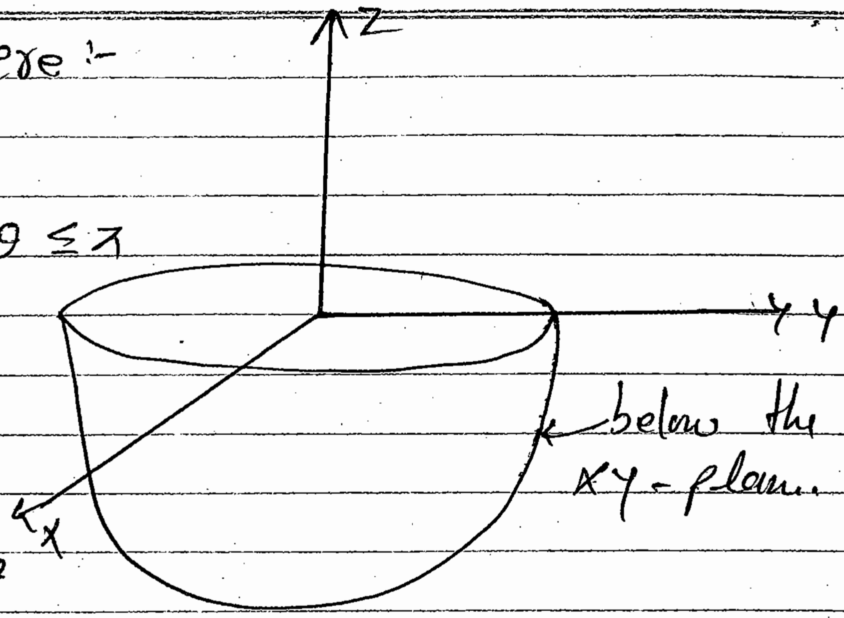
$0 \le r \le a$, $\frac{\pi}{2} \le \theta \le \pi$

$0 \le \phi \le 2\pi$

Surface :-

$r = a$, $0 \le \theta \le \pi/2$

$0 \le \phi \le 2\pi$



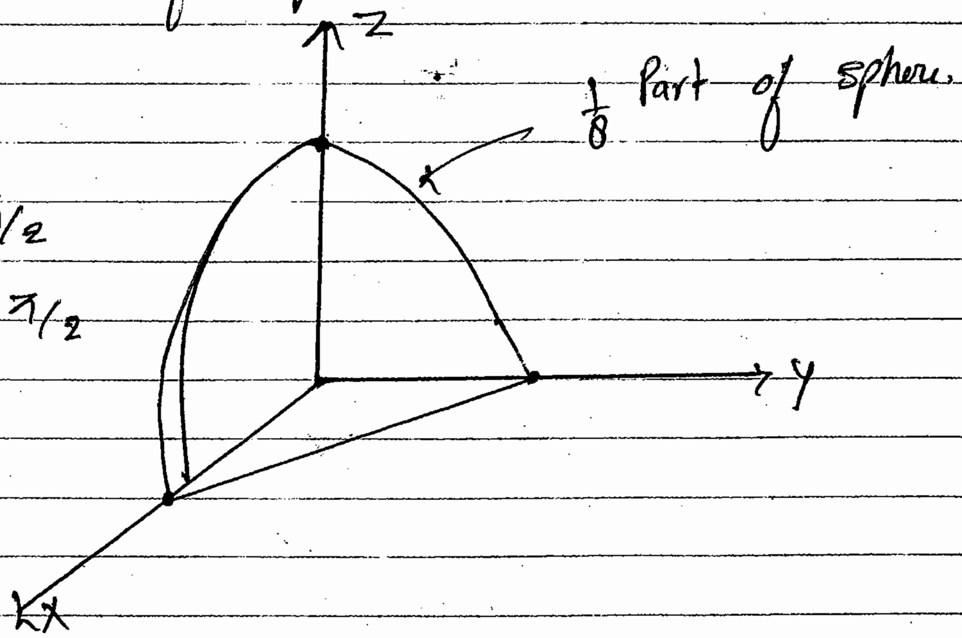
* First Octant Of Sphere :-

Volume :-

$0 \le r \le a$

$0 \le \theta \le \pi/2$

$0 \le \phi \le \pi/2$



A-2

Q.31 Using Gauss divergence theorem, evaluate $\iint_S (x^3 dy dz + y^3 dz dx + z^3 dx dy)$, where 'S' is the surface of the sphere $x^2+y^2+z^2=a^2$

Sol^n

$$\iint x^3 dy\,dz + y^3 dz\,dx + z^3 dx\,dy$$

$\because \ d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\because \ d\vec{s} = dx\,dy\,\hat{k} + dy\,dz\,\hat{i} + dx\,dz\,\hat{j}$

$$\iint \vec{A}\cdot d\vec{s} = \iint (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\cdot d\vec{s}$$

$$= \iint 3x^2 + 3y^2 + 3z^2$$

$$= \iint 3(x^2+y^2+z^2)\,dv$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$x^2+y^2+z^2 = r^2$

$$\iint \vec{A}\cdot d\vec{s} = \iint 3r^2\,dv$$

$$= 3\iiint r^2 \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\int_0^a r^4 dr \cdot \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= 3\left[\frac{a^5}{5} \cdot (\cos\theta)_0^\pi \quad 2\pi\right]$$

$$= \frac{3\times 4\pi\, a^5}{5} = \frac{12\pi a^5}{5} \quad \underline{\underline{\text{Ans}}}$$

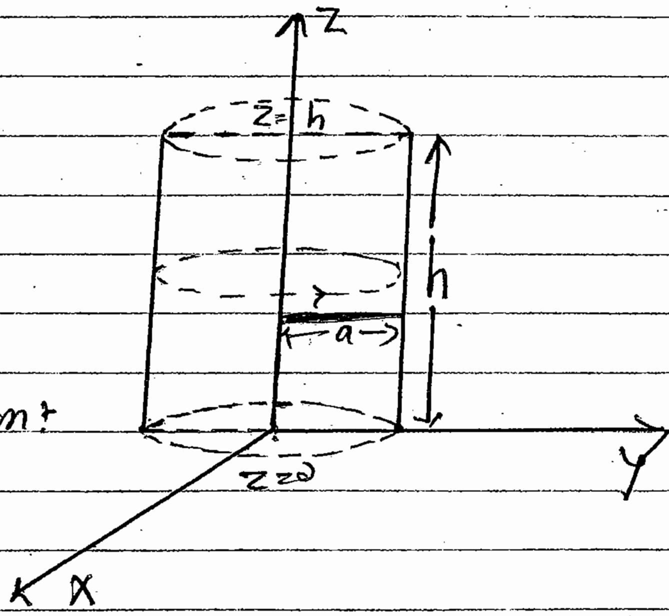
* Cyllindrical Co-ordinate :-

$$(\rho, \phi, z)$$

$$\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}\right\}$$

(i) $x^2+y^2 = a^2$

$z=0 \;,\; z=h$



Volume :-

$0 \le \rho \le a$

$0 \le \phi \le 2\pi$

$0 \le z \le h$

Curved Surface integration :-

ρ = Constant

$0 \le \phi \le 2\pi$

$0 \le z \le h$

Top Surface Integration -

$z = h$

$0 \le \rho \le a$

$0 \le \phi \le 2\pi$

Bottom Surface Integration -

$z=0$

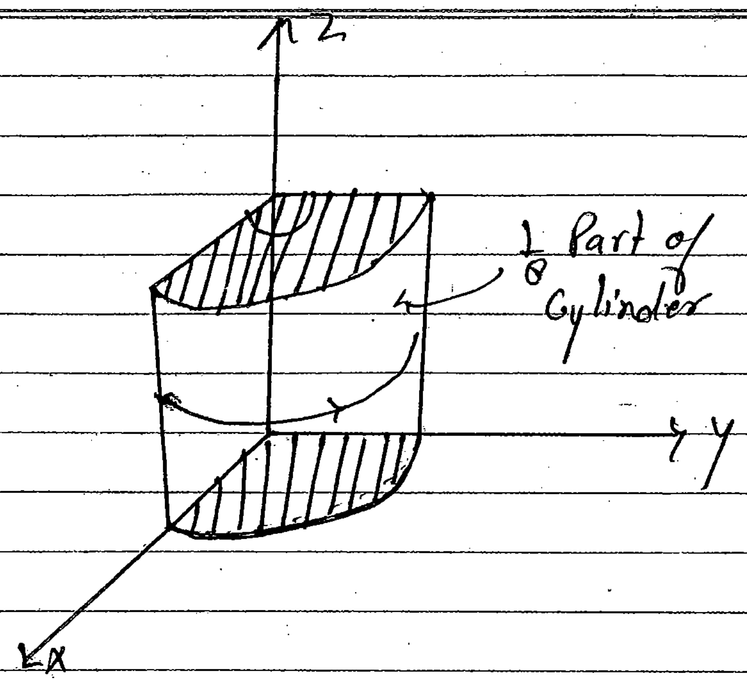
$0 \le \rho \le a \;,\; 0 \le \phi \le 2\pi$

first Ooctant :-

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq z \leq h$

$0 \leq \rho \leq a$



II $x^2+y^2=a^2$

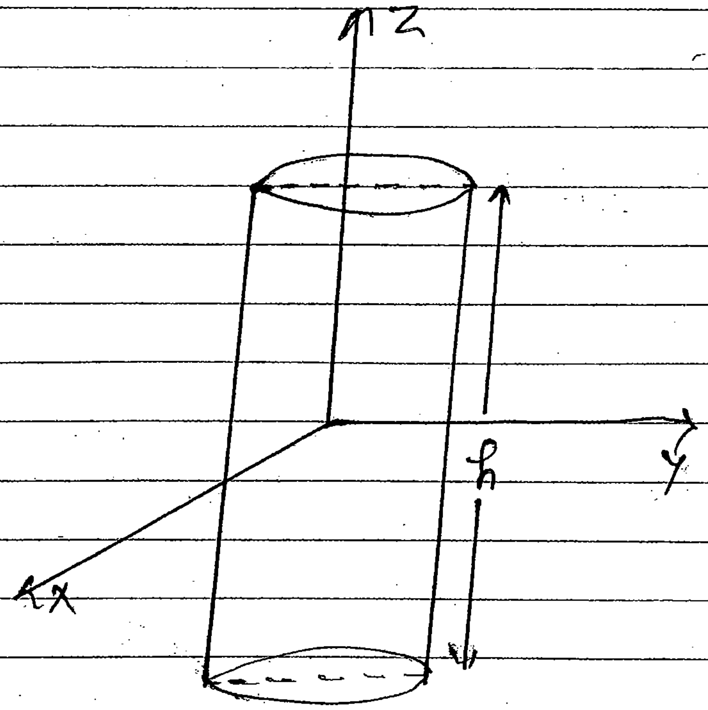
$z=-\frac{h}{2}$, $z=\frac{h}{2}$

Volume :-

$-\frac{h}{2} \leq z \leq +\frac{h}{2}$

$0 \leq \rho \leq a$

$0 \leq \phi \leq 2\pi$



* first Octant :- $\frac{1}{8}^{th}$ part of total cylinder

$0 \leq \rho \leq a$

$0 \leq \phi \leq \pi/2$

$0 \leq z \leq h/2$

Q-2

Q.33 Calculate the flux of the vector field $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ through the surface bounded by the region $x^2+y^2=4$, $z=0$, $z=3$

Sol^n

$$\vec{\nabla}\cdot\vec{F} = 4 - 4y + 2z$$

$$\iint (4-4y+2z)\,dV = \int_{\phi=0}^{2\pi}\int_{z=0}^{3}\int_{s=0}^{2} (4 - 4s\sin\phi + 2z)\, s\,ds\,d\phi\,dz$$

$$\Rightarrow \quad = 4\int_0^{2\pi}\iint s\,ds\,d\phi\,dz - 4\int_0^{3}\iint s\sin\phi\,ds\,d\phi\,dz$$

$$+ \iiint z s\,ds\,d\phi\,dz$$

$$= 4\left[\frac{s^2}{2}\right]_0^2 \left[\phi\right]_0^{2\pi} \left[z\right]_0^3 - 4\left[\frac{s^2}{2}\right]_0^2 \left[-\cos\phi\right]_0^{2\pi} \times 0$$

$$+ 2\left[\frac{s^2}{2}\right]_0^2 \left[\frac{z^2}{2}\right]_0^3 \left[\phi\right]_0^{2\pi}$$

$$= 4\times\frac{4}{2}\times 2\pi\times 3 - 0 + 2\times\frac{4}{2}\times\frac{9}{2}\times 2\pi$$

$$= 24\times 2\pi + 18\times 2\pi$$

$$= 48\pi + 36\pi$$

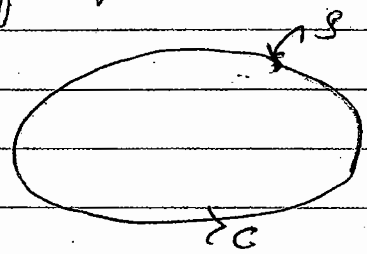
$$= 84\pi \quad \underline{\text{Ans}}$$

* Stokes Theorem :-

This theorem is applicable only for open surfaces and it converts surface integral to line integral or vice-versa.

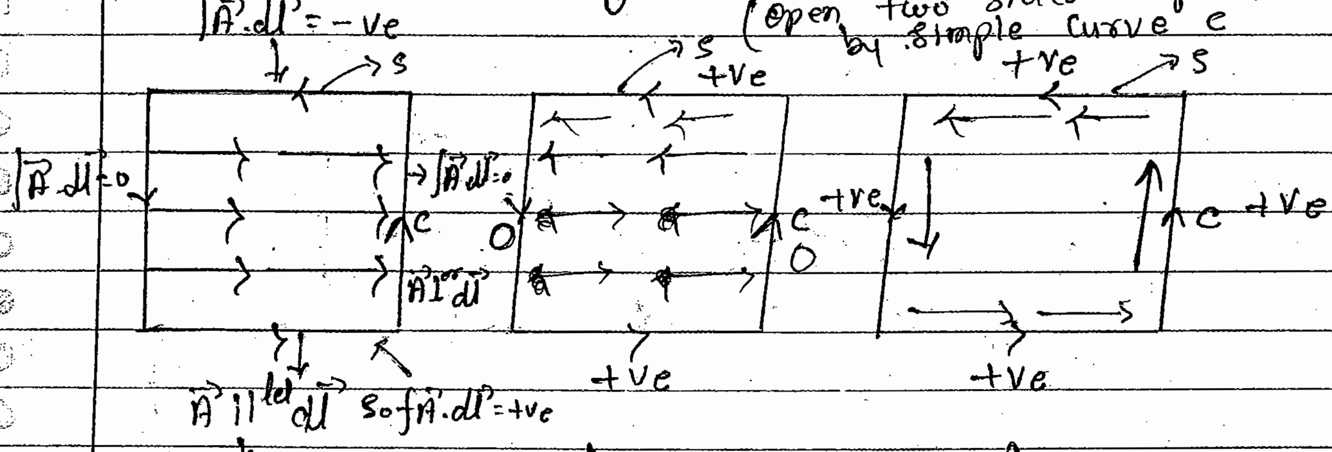
Suppose S is a open two sided surface bounded by a simple close curve C and $\vec{A}$ is a vector function of position with continuous derivatives.

$$\boxed{\oint_C \vec{A}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s}}$$



"Simple curve means the curve which is non-intersecting curve."

(open two sided surface, bounded by simple curve C)



$\oint \vec{A}\cdot d\vec{l} = 0$ $\qquad$ $\oint \vec{A}\cdot d\vec{l} = +ve < \oint \vec{A}\cdot d\vec{l} = +ve$

$$\boxed{\oint \vec{A}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{A})\, d\vec{s}}$$

$\because$ $\vec{A}\cdot d\vec{l}$ is related to $\vec{\nabla}\times\vec{A}$

So $\vec{A}\cdot d\vec{l}$ is called circulation of the vector field $\vec{A}$.

A-2

Q.35 Evaluate $\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$ where $\vec{F} = (3x-2y)\hat{i} + x^2 z\hat{j} + y^2(z+1)\hat{k}$ for a plane rectangular area with vertices at $(0,0)$, $(1,0)$, $(1,2)$, $(0,2)$ in the xy-plane.

Sol^n

$$\vec{F} = (3x-2y)\hat{i} + x^2 z\hat{j} + y^2(z+1)\hat{k}$$

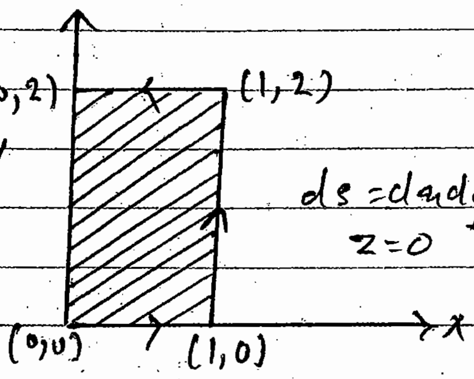
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-2y & x^2 z & y^2(z+1) \end{vmatrix}$$

$$= \hat{i}\{(z+1)2y - x^2\} + \hat{j}(0) + \hat{k}(2xz-2)$$

$$= \{2y(z+1) - x^2\}\hat{i} + (0)\hat{j} + 2(xz-1)\hat{k}$$

$$\vec{\nabla}\times\vec{F} = [2y(z+1) - x^2)\hat{i} + 2(xz+1)\hat{k}$$

$$\iint(\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint 2(x\cancel{z}+1)\,dx\,dy$$



$ds = dx\,dy$
$z=0$

$$= 2\int_0^2\int_0^1 dx\,dy$$

$$= 2[x]_0^2\,[y]_0^1 \qquad \hat{n} = \hat{k}$$

$$= 4 \quad \text{Ans}$$

Matrices

It is a square of rectangular array of numbers or functions.

e.g. :-

$$\begin{array}{c} \xrightarrow{\text{row}} \\ \text{column}\downarrow \begin{bmatrix} 1 & -2 & 3 \\ 3 & 7 & 5 \end{bmatrix} \end{array} \quad \& \quad \begin{bmatrix} x^2 & 2x \\ -7x & x^3-4 \end{bmatrix}$$

Order :-

If m number of rows and n number of column then,

$$\text{order} = m \times n$$

Representation :-

$$\underset{\text{Row Element}}{a_{i}}{}_{j} \leftarrow \text{Column index}$$

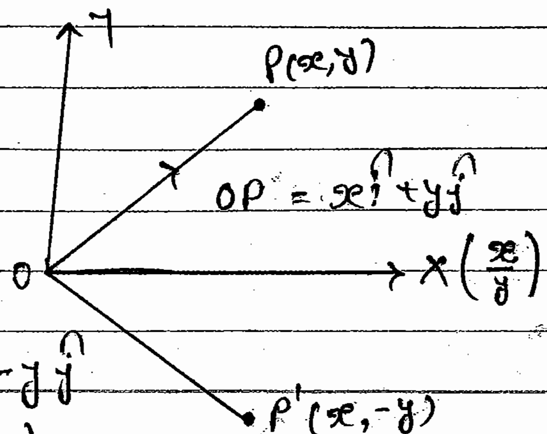
Where $1 \le i \le m$
$1 \le j \le n$

Product :-

If matrix A is $m\times n$ matrix and B is the matrix of order $p\times q$.

$AB \Rightarrow n = p$
$BA \Rightarrow q = m$

$$\boxed{AB \neq BA}$$



$O\bar{P} = x\hat{i} - y\hat{j}$
$= \left(\frac{x}{-y}\right)$

- $$\begin{pmatrix} x \\ -y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{Reflection matrix / operator with respect to } x \text{ axis.}} \begin{pmatrix} x \\ y \end{pmatrix}$$

- $$\begin{pmatrix} -x \\ -y \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{Inversion Operator.}} \begin{pmatrix} x \\ y \end{pmatrix}$$

* Types of Matrix :-

1. Square Matrix :-

A matrix with same number of rows and columns is called square matrix. A square matrix A of order $n \times n$ is written as -

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The different powers of a matrix are defined only when it is a square matrix. Direct product of two matrices also has some meaning when and only when the matrices are square matrices.

A matrix which is not a square is known as rectangular matrix.

If in a rectangular matrix the number of columns is more than the number of rows, it is said horizontal matrix. Thus a matrix of order $m \times n$ is horizontal matrix when $n > m$. In case number of rows is more than the number of columns, the matrix is said to be vertical matrix.

2. Row Matrix :-

If a matrix has one row and n columns, it is said a row matrix. Thus a matrix of order $1 \times n$ is the row matrix or row vector. It is written as -

$$[X] = (a_{11} \; a_{12} \; \cdots \; a_{1n}) \text{ or } (a_1 \; a_2 \; \cdots \; a_n).$$

3. Column Matrix :-

A matrix with a single column and n rows is called the column matrix or column vector. Order of column matrix is $n \times 1$. It is written as -

$$[X] = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \text{ or } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

4. Upper Triangular Matrix :-

A square matrix of order $n \times n$ is called upper triangular matrix if its element below the principal diagonal is zero.

And it is represented by -

$$[A] = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

5. Lower Triangular Matrix :-

Matrix having the elements above the principal diagonal is zero.

e.g.

$$[B] = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}$$

6. Diagonal Matrix :-

A square matrix of order n, which is both upper triangular and lower triangular is called Diagonal Matrix.

e.g :-

$$[D] = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

It may also be denoted as -

$$\mathrm{Diag}\,(a_{11}, a_{22}, a_{33}, \cdots\cdots a_{nn})$$

7. Scalar Matrix:-

A diagonal matrix, all the diagonal elements of which are equal to a scalar number say λ, is called a scalar matrix. This matrix may be written as-

$$S = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \lambda \end{pmatrix} = diag(\lambda, \lambda \cdots \lambda)$$

Each element of this matrix is given as-

$$(S)_{ij} = a_{ij} = \lambda \delta_{ij}$$

Where, $\delta_{ij} = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases}$

8. Unit Matrix:-

A matrix with each diagonal element equal to unity is called a unit matrix. and is denoted by I. Obviously, it is a square matrix having unit elements in principal or leading diagonal and zero elements everywhere else. Therefore, $(i,j)^{th}$ element of unit matrix is given as-

$$(I)_{ij} = \delta_{ij} = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Periodic Matrix :-

A square matrix A for which $A^{k+1} = A$ is said to be periodic matrix and the least positive value of k ~~satif~~ stisfying this relation is called the period of this periodicity.

$$\boxed{A^{k+1} = A}$$

Where k = period of matrix

10. Idempotent Matrix :-

$$\boxed{A^2 = A}$$

Idempotent matrix is a periodic matrix of period 1.

$\Rightarrow A^2 - A = 0 \Rightarrow A(A - I) = 0$

$\Rightarrow |A||A - I| = 0$

Either $|A| = 0$ or $|A - I| = 0$

$\downarrow$ singular $\qquad \downarrow A = I$

Idempotent matrix will be either a singular matrix or a non singular unit matrix.

11 Nilpotent :- $\boxed{A^p = 0}$ Here A is a Nilpotent matrix of order/index p.

12 Involutary Matrix :-

$$\boxed{A^2 = I}$$

$AA^{-1} = I \Rightarrow A \cdot A = I$

$\boxed{A = A^{-1} \longrightarrow \text{Self inverse}}$

Properties of Matrix

1. Transpose :- Interchanging rows and columns.

e.g.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 7 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}$$

$\Rightarrow (A^T)^T = A$

$\Rightarrow (AB)^T = B^T A^T$

2. Conjugate of a Matrix :- The ~~matrix~~ elements of matrix may be real or complex. The matrix whose elements are complex conjugates of the corresponding elements of a matrix A is called conjugate of a matrix A. And it can be denoted by A^*

e.g. :-

$$A = \begin{bmatrix} 1+3i & 7i \\ 0 & -4i \end{bmatrix}$$

Then

$$A^* = \begin{bmatrix} 1-3i & -7i \\ 0 & 4i \end{bmatrix}$$

$$\left.\begin{aligned} &\Rightarrow (A^*)^* = A \\ &\Rightarrow (AB)^* = B^* A^* \end{aligned}\right\}$$

3. Transpose Conjugate of a Matrix :-

dagger

$$\boxed{A^{\dagger} = (A^*)^T = (A^T)^*}$$

$$A = \begin{bmatrix} 1+3i & 7i \\ 0 & -4i \end{bmatrix}$$

$$A^{\dagger} = \begin{bmatrix} 1-3i & 0 \\ -7i & 4i \end{bmatrix}$$

$$\left.\begin{aligned} &\Rightarrow (A^{\dagger})^{\dagger} = A \\ &\Rightarrow (AB)^{\dagger} = B^{\dagger} A^{\dagger} \end{aligned}\right\}$$

4.* Symmetric Matrix [Real] :-

$$\boxed{A^T = A} \rightarrow \boxed{a_{ji} = a_{ij}}$$

$$A = \begin{bmatrix} 1 & 6 & -7 \\ 6 & 3 & 0 \\ -7 & 0 & -4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 6 & -7 \\ 6 & 3 & 0 \\ -7 & 0 & -4 \end{bmatrix}$$

Ques What is the number of independent element $n \times n$ symetric matrix?

Sol^n:-

$$\boxed{\text{Number of independent elements} = \frac{1}{2} n(n+1)}$$

Proof:- No. of independent elements = No. of principal diagonal element + No. of upper half or lower half elements.

$$= n + \frac{n^2 - n}{2} = \frac{2n + n^2 - n}{2}$$

$$= \frac{1}{2} n(n+1)$$

Proved.

5.* Skew-Symmetric Matrix [Real] :-

$$\boxed{A^T = -A} \Rightarrow \boxed{a_{ji} = -a_{ij}}$$

The principal diagonal elements of a real skew symetric matrix are always zero.

Proof :-

Say for P.D.E. = $i = j \Rightarrow a_{ii} = -a_{ii}$

$\Rightarrow 2a_{ii} = 0$

$\Rightarrow \boxed{a_{ii} = 0}$

$A' = (A^T)$

Real $= (A^T)$

$\Rightarrow \boxed{a_{ii} = 0}$

Like :-

$$\begin{bmatrix} 0 & 6 & -7 \\ -6 & 0 & 4 \\ 7 & -4 & 0 \end{bmatrix}$$

The number of independent elements of a $n \times n$ real skew symmetric matrix is equal to $\frac{1}{2} n(n-1)$.

$$\boxed{\text{No. of independent element of skew symmetric matrix} = \frac{1}{2} n(n-1)}$$

Note :-

The complex form of symmetric matrix [Real] is Hermitian Matrix.

6. Hermitian Matrix :-

$$\boxed{A^{\dagger} = A} \Rightarrow \boxed{a_{ji}^{*} = a_{ij}}$$

The principal diagonal elements of a Hermitian matrix are always real.

$$\text{P.D.E.} = a_{ii}^{*} = a_{ii}$$

Assume,

$$a_{ii} = x + iy$$

then, $a_{ii}^{*} = a_{ii}$

$$\Rightarrow \cancel{x} + iy = \cancel{x} + iy$$

$$\Rightarrow 2iy = 0$$

$$\Rightarrow \boxed{y = 0}$$

So $\boxed{a_{ii} = x}$ ← Real.

e.g.

$$[A] = \begin{bmatrix} 2 & 6i & 3+4i \\ -6i & 4 & 0 \\ 3-4i & 0 & -7 \end{bmatrix}$$

"The minimum numbers of real elements of a most general $n \times n$ hermitian matrix will be n!"

7. Skew - Hermitian or Anti Hermitian Matrix :-

$$\boxed{A^{\dagger} = -A} \Rightarrow \boxed{a_{ji}^{*} = -a_{ij}}$$

"The principal diagonal element of a skew - hermitian matrix is either zero or purely imaginary."

P.D.E. $a_{ii}^{*} = -a_{ii}$

Let

$$a_{ii} = x + iy$$

So $x + \cancel{iy} = -x - \cancel{iy}$

$2x = 0$

Trace = Sum of the P.D.E.

$Tr(AB) = Tr(BA)$

$Tr(A+B) = Tr(A) + Tr(B)$

$\Rightarrow x = 0$

So,

$$\boxed{a_{ii} = iy}$$

if $y = 0 \longrightarrow a_{ii} = 0$

$y \neq 0 \longrightarrow a_{ii} = iy$ = Purily imaginary.

"The minimum number of real elements of the most general $n \times n$ skew hermitian matrix will be 'zero'."

e.g.

$$[A] = \begin{bmatrix} i & 7i & 3+4i \\ 7i & -i & -6i \\ -3+4i & -6i & 7i \end{bmatrix}$$

CSIR Dec - 2013

Ques If A, B, C are hermitian operators which of the following is wrong?

(I) $C = [A, B] = AB - BA$ [✓]

(ii) $C = AB + BA$

(iii) $C = ABA$

(iv) $C = A + B$

Soln $\because$ A, B, C are hermitian, So, $A^{\dagger} = A$, $B^{\dagger} = B$ & $C^{\dagger} = C$

Option (I) -

$$C^{\dagger} = (AB - BA)^{\dagger}$$
$$= (AB)^{\dagger} - (BA)^{\dagger}$$
$$= B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger}$$
$$= BA - AB$$
$$\boxed{C^{\dagger} = -C}$$

So this option is wrong.

Option (III)

$$C^{\dagger} = (ABA)^{\dagger}$$
$$= A^{\dagger}B^{\dagger}A^{\dagger}$$
$$= ABA$$
$$\boxed{C^{\dagger} = C}$$

(d) $\begin{pmatrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$

Ques 12:- The symmetric part of $P = \begin{pmatrix} a \\ b \end{pmatrix} (a-2 \quad b)$ is:?

16/July/2014

(a) $\begin{pmatrix} a^2-2 & ab-1 \\ ab-1 & b^2-2 \end{pmatrix}$ (b) $\begin{pmatrix} a(a-2) & b \\ b & b^2 \end{pmatrix}$ (c) $\begin{pmatrix} a(a-1) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$

Q.12 Solⁿ Any matrix can be written as sum of symmetric and Anti-symmetric matrix.

$$A = \underbrace{\frac{1}{2}[A + A^T]}_{\text{Symmetric}} + \underbrace{\frac{1}{2}[A - A^T]}_{\text{Anti-symmetric}}$$

$$P = \begin{pmatrix} a \\ b \end{pmatrix} (a-2 \quad b)$$

$$= \begin{pmatrix} a(a-2) & ab \\ b(a-2) & b^2 \end{pmatrix}$$ ~~Ans option (d) is correct.~~

So
$$\frac{1}{2}(P + P^T) = \frac{1}{2}\left[\begin{matrix} a(a-2) & ab \\ b(a-2) & b^2 \end{matrix}\right] + \left[\begin{matrix} a(a-2) & b(a-2) \\ ab & b^2 \end{matrix}\right]$$

$$= \frac{1}{2}\left[\begin{matrix} a(a-2) + a(a-2) & ab + b(a-2) \\ b(a-2) + ab & b^2 + b^2 \end{matrix}\right]$$

$$= \frac{1}{2}\left[\begin{matrix} 2a(a-2) & 2ab - 2b \\ 2ab - 2b & 2b^2 \end{matrix}\right] = \frac{\cancel{2}}{\cancel{2}}\left[\begin{matrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{matrix}\right]$$

$$\frac{1}{2}(P + P^{-1}) = \left[\begin{matrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{matrix}\right]$$ Ans option (d).

Ques 11 If A and B $n \times n$ square symmetric matrices, then AB will be symmetric in nature if.

(a) A and B both are triangular (b) A and B both are non singular

(c) A and B both Commute [✓] (d) A and B both anti-commute.

Solⁿ
$A^T = A$

$B^T = B$

$(AB)^T = AB$

$(AB)^T = B^T A^T = BA$ So A and B both Commute.

Ques 9) The product of the two matrices $\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$ and $\begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$ is zero. Which of the following statements is true?

(a) $(\theta-\phi)$ is odd multiple of $(\pi/2)$ (b) $(\theta-\phi)$ is even multiple of $(\pi/2)$

(c) $(\theta-\phi)$ is odd multiple of π (d) $(\theta-\phi)$ is even multiple of π

Sol^n Let $[A] = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$, $[B] = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

So

$$[AB] = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}\begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$\cancel{= \cos\theta\cos\phi \begin{vmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \cos\theta\cos\phi + \sin\theta\sin\phi \\ \sin\theta\cos\phi + \sin\theta\sin\phi & \sin\theta\sin\phi \end{vmatrix}}$$

$$= \begin{vmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\phi\cos\theta(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{vmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi\,(\cos(\theta-\phi)) & \sin\phi\cos\theta\,(\cos(\theta-\phi)) \\ \sin\theta\cos\phi\,(\cos(\theta-\phi)) & \sin\theta\sin\phi\,(\cos(\theta-\phi)) \end{bmatrix}$$

$$= \cos(\theta-\phi)\begin{pmatrix} \cos\theta\cos\phi & \sin\phi\cos\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{pmatrix} = 0$$

So either $\cos(\theta-\phi) = 0$ or $\begin{bmatrix} \cos\theta\cos\phi & \sin\phi\cos\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix} = 0$

$$\therefore \begin{bmatrix} \cos\theta\cos\phi & \sin\phi\cos\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

$$= \cos\theta\cos\phi\sin\theta\sin\phi - \cos\theta\cos\phi\sin\theta\sin\phi$$

$$= 0$$

So $\cos(\theta-\phi)$

AD $\cos(\theta-\phi) = 0$

$$\cos(\theta-\phi) = \cos\left((2n+1)\frac{\pi}{2}\right)$$

$$\theta-\phi = (2n+1)\frac{\pi}{2} \quad \text{where } n = 0, 1, 2, 3, \ldots$$

$\phi-\phi$ = add multiple of $\frac{\pi}{2}$.

Ques14 Suppose the matrix $B^{\dagger}AB$ is Hermitian in nature. Which of the following statement is true?

(a) A is a hermitian matrix (b) B is a hermitian matrix
(c) A is a skew hermitian matrix (d) B is a skew hermitian matrix

Sol^n Since $B^{\dagger}AB$ is hermitian

$\therefore$

$$(B^{\dagger}AB)^{\dagger} = B^{\dagger}AB \quad \text{—— (I)}$$

So,

$$(B^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}B \quad \text{—— (II)}$$

from (I) and (II)

$$A^{\dagger} = A$$

So A is hermitian so option (a) is correct.

* Orthogonal Matrix [Real]:-

(i) $AA^T = A^TA = I$

(ii) $A^T = A^{-1}$

(iii) $|A| = \pm 1$

(iv) Each row and each column of a orthogonal matrix is a normalised unit vector.

(v) Any two row or Any two column of a orthogonal matrix are orthogonal vectors of each other.

e.g. :-

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

$$\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

So $A^\dagger B = (A_1^*, A_2^*, A_3^*)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$

and $A^\dagger A = (A_1^*, A_2^*, A_3^*)\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$

$$\boxed{\vec{A}^\dagger \vec{B} = (A_1^* B_1 , A_2^* B_2 , A_3^* B_3)}$$

$$\boxed{\vec{A}^\dagger \vec{A} = A_1^* A_1 + A_2^* A_2 + A_3^* A_3 = |A|}$$

$$\boxed{|\vec{A}| = \sqrt{\vec{A}^\dagger \vec{A}}}$$

Normalisation Condition:-

$$\boxed{\vec{A}^\dagger \vec{A} = 1}$$

Orthogonality Condition :-

$$\boxed{\vec{A}^{\dagger}\vec{B} = 0}$$

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$\boxed{A_1^{\dagger} A = 1}$ Normalisation Condition satisfied

If

$$A_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$A_1^{\dagger} A_1 = 1 \qquad A_2^{\dagger} A_2 = 1$$

but

$\boxed{A_1^{\dagger} A_2 = 0}$ orthogonality Condition satisfied.

- Complex form of orthogonal matrix is unitary matrix.

* Unitary Matrix :-

(i) $AA^{\dagger} = A^{\dagger}A = I$

(ii) $A^{\dagger} = A^{-1}$

(iii) Determinant of a unitary matrix will be of unit modulus.

(iv) Each row and each column of a unitary matrix is normalised unit vector.

(v) Any two row and any two column of a unitary matrix is orthogonal to each other.

Ques If A and B are orthogonal matrices then AB is orthogonal or not?

Sol^n

$$AA^T = A^TA = I$$
$$BB^T = B^TB = I$$

Thus,

$$(AB)^T (AB) = I.$$
$$B^TA^TAB = B^TIB = B^TB$$

So,

The product of two orthogonal matrices is always orthogonal.

* If,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then,

1. Minor :-

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{Minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Minor of } a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

2. Co-factor :-

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

$C_{11} = M_{11}$, $C_{23} = -M_{23}$

3. Determenant in terms of Co-factor :-

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\boxed{|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}}$$

* Adjoint of Matrix :-

Adj(A) = Transpose of Co-factor matrix of A.

$$Adj(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$\boxed{A.(Adj\,A) = (Adj\,A)\,A = |A|\, I_n}$$

Ques Suppose A is a Square matrix of order n and its determinant is M. Now calculate determinant of adjoint matrix.

Sol^n

$A = n \times n$
$|A| = m$

$$|Adj\ A| = |A|\ I_n = \begin{pmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & & |A| \end{pmatrix}$$

$$\Rightarrow |A||Adj\ A| = |A|^n$$

$$\boxed{|Adj\ A| = |A|^{n-1}}$$

- Also find determinat of inverse matrix -

$$AA^{-1} = I$$

Taking determinant on both sides -

$$|A||A^{-1}| = |I|$$

$$|A||A^{-1}| = 1$$

$$\boxed{|A^{-1}| = \frac{1}{|A|}}$$

Ques If I is an identity matrix of order $n \times n$ and $adj\,(2I) = 2^k I$, then k is equal to ?

(a) 1 (b) 2 (c) $n-1$ ✓ (d) n

Soln

$$Adj\,(2I) = 2^k I \longrightarrow k = ?$$

$$(2I)\,Adj\,(2I) = |2I|\,I_n = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 2 \end{pmatrix}$$

$$2\,Adj(2I) = 2^n I_n$$

$$Adj(2I) = 2^{n-1} I_n \quad \text{So } \boxed{k = n-1}$$

Ans

Ques If 'A' is a 3x3 matrix with determinant 2, then the determinant of adj.[adj.[adj.(A^{-1})]] is equal to ?

(a) $\frac{1}{512}$ (b) $\frac{1}{1024}$ (c) $\frac{1}{128}$ (d) $\frac{1}{256}$ ✓

Sol^n $A = 3 \times 3$ matrix and $|A| = 2$

$\because$ $|A| = 2$ So $|A^{-1}| = \frac{1}{2}$

So, Method-I :-

$$\text{adj.}(\text{adj.}(\text{adj.}[A^{-1}))) = ??$$

$$|A^{-1}| = 1/2$$

$$\left(\frac{1}{2}\right)^{3-1} = \frac{1}{4}$$

$$\left(\frac{1}{4}\right)^{3-1} = \frac{1}{16}$$

$$\left(\frac{1}{16}\right)^{3-1} = \frac{1}{256}$$

Method - II :-

$$|A| = 2$$

$$|A^{-1}| = \frac{1}{2}$$

$$\underset{M}{\underline{\text{adj}\,[A^{-1}]}} = \left(\frac{1}{2}\right)^{3-1} = \frac{1}{4}$$

$$\underset{N}{\underline{\text{adj}\,(M)}} = \left(\frac{1}{4}\right)^{3-1} = \frac{1}{16}$$

$$\text{adj}\,(N) = \left(\frac{1}{16}\right)^{3-1} = \frac{1}{256} \quad \underline{\text{Ans}}$$

v. Imp.

* Eigen Values & Eigen Vectors :-

The values of λ for which we will get a non zero solution of X are known as eigen values of matrix A or operator A.

Corresponding to each value of λ we will get a non zero solution of X and that is known as eigen vector of matrix or operator A. corresponding to that λ.

Eigen Value eq^n. $\rightarrow$ $\hat{A}\vec{X} = \lambda X$ $\rightarrow$ Scalar (real or complex)

$$\Rightarrow \boxed{(A - \lambda I)X = 0}$$

Similarly,

$|\psi\rangle \rightarrow$ State Vector

$$\hat{H}\psi = E\psi$$

Energy eigen state

Here ψ is an eigen function of $\hat{H}$ correspon-ding to eigenvalue E.

ψ = State of a particle
E = Energy of the particle.

Same as,

Position Vector

$$\hat{x}\psi = \lambda\psi$$

λ $\rightarrow$ Position of the particle

Similarly -

momentum operator

$$\hat{p}_x\psi = \lambda\psi$$

λ $\rightarrow$ momentum of the particle

So $\boxed{A}\,X = \boxed{\lambda}\,X \rightarrow$ state of the particle

Act of Measurement. Result of the measurement

$$(A-\lambda I)X = 0$$

Since we know that it is a homogeneous equation of type $AX=0$ and it has trivial solution (at $X=0$) and Non-trivial solution (at $X\neq 0$)

For Non-trivial solution $|A| = 0$ for homogeneous equation.

So $|A-\lambda I| = 0 \rightarrow$ Characteristic equation
OR
Eigen Value equation.

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} \cos\theta-\lambda & \sin\theta \\ -\sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\Rightarrow \boxed{\lambda = \cos\theta \pm i\sin\theta}$$

Eigen Vector :-

$$\begin{bmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-i\sin\theta\, x_1 + \sin\theta\, x_2 = 0$$

$$x_2 = i x_1$$

Say $x_1 = K$, $x_2 = iK$

So $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ iK \end{pmatrix}$

$$X^\dagger X = 1$$

$$\Rightarrow (K \;\; -iK) \begin{pmatrix} K \\ iK \end{pmatrix} = 1$$

$\Rightarrow \boxed{K = \pm \frac{1}{\sqrt{2}}}$ ← Normalisation ~~Condition~~ Constant

So, $\boxed{X = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}}$

* Properties Regarding eigen values and eigen vectors :-

(i) Any square matrix A and its transpose have same eigen values.

(ii) Sum of eigen values of matrix A is equal to trace of matrix A.

(iii) Product of the eigen value of matrix A is equal to determinant of the matrix.

(iv) If A has eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots$ then -

(a) KA, where k is scalar quantity is -

$$KA \rightarrow K\lambda_1, K\lambda_2, \ldots$$

(b) $A^m \longrightarrow \lambda_1^m, \lambda_2^m, \ldots$

(c) $A^{-1} = \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots$

(v) The eigen values of a real symmetric matrix and hermitian matrix are always real.

(vi) Eigen values of a real skew symmetric matrix and skew hermitian matrix are either zero or purely emaginary.

(vii) Eigen values of a real orthogonal matrix and unitary matrix are of unit modulus.

(viii) Eigen values of a diagonal matrix, upper triangular matrix, lower triangular matrix are just the principal diagonal elements.

$A^2 = I$ — involutary

- eigen valu is +1 or -1

(ix) Any two eigen vectors corresponding to distinct eigen values of a hermitian matrix and unitary matrix are orthogonal to each other.

like → $\lambda_1 \neq \lambda_2$ where $\lambda_1 \neq \lambda_2$

$$\begin{array}{cc} \lambda_1 & \lambda_2 \\ \downarrow & \downarrow \\ X_1 & X_2 \end{array}$$

$$\underbrace{X_1^{\dagger} X_2 = 0}_{\uparrow \text{ Innerproduct}}$$

(x)

[x] Eigen values of a nilpotent matrix are always zero.

[xi] Eigen values of a idempotent matrix are either 0 or 1.

(xii) Square matrix A of order n×n having all element equal to 1, has eigen values n, 0, 0, 0, ----

Ex —

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \lambda = 3, 0, 0$$

xiii) Square matrix A of order n×n having rows or column that are scalar multiple of particular row or column has eigen values Tr(A), 0, 0, 0, ----

ex- $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \rightarrow \lambda = 14, 0, 0$

CSIR Dec-2011

Ques A 3×3 matrix M has $Tr[M]=6$, $Tr[M^2] = 26$, $Tr[M^3] = 90$. Which of the following can be possible set of eigenvalues of M?

(a) $\{1, 1, 4\}$ (b) $\{-1, 0, 7\}$ (c) $\{-1, 3, 4\}$ ✓ (IV) $\{2, 2, 2\}$

Sol^n-

$$Tr\{M\} = 6 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 6 \quad \text{— (I)}$$
$$Tr\{M^2\} = 26 \Rightarrow \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 26 \quad \text{— (II)}$$
$$Tr\{M^3\} = 90 \Rightarrow \lambda_1^3 + \lambda_2^3 + \lambda_3^2 = 90 \quad \text{— (III)}$$

Since option (c) satifies all three conditions so option (c) is correct.

Ques A 2×2 matrix 'A' has eigenvalues $e^{i\pi/5}$ and $e^{i\pi/6}$. The smallest value of 'n' such that $A^n = I$

(a) 20 (b) 80 (c) 60 ✓ (d) 120.

Sol^n-

$$A \Rightarrow \therefore \lambda = e^{i\pi/5}, e^{i\pi/6}$$

$$A^n \Rightarrow e^{in\pi/5}, e^{in\pi/6}$$

Product of λ_1 and λ_2 $= |A^n|$

$$e^{in\pi/5} \cdot e^{in\pi/6} = 1$$
$$e^{i11n\pi/30} = e^{i2m\pi}$$
$$\Rightarrow \frac{11n\cancel{\pi}}{30} = 2m\cancel{\pi}$$
$$\frac{n}{m} = \frac{60}{11}$$

$$\frac{n}{m} = \frac{60}{11}, \frac{120}{22}, \ldots\ldots$$

So smallest value of $n = 60$ so option (C) is correct.

22/July/2014

Ques Consider a matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$. A 3-dimensional basis formed by eigenvectors of M is

(a) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ ✓ (c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Sol^n

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$|M - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\{\lambda^2 - 1\} = 0$$

So either $1-\lambda = 0$ or $\lambda^2 - 1 = 0$

if $1-\lambda = 0$ then $\lambda = 1$ or $\lambda^2 - 1 = 0$ then $\lambda = \pm 1$

So $\lambda = 1, 1, -1$

for $\lambda = 1$

$$[M - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_2 + x_3 = 0$

$\boxed{x_2 = -x_3}$ and x_1 can be arbitrary.

for $\lambda = -1$

$$(M - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$2x_1 = 0$ So $x_1 = 0$

and $x_2 - x_3 = 0$ and $x_2 = x_3$

Option (b) both conditions satisfies so option b is correct.

Ques 29:- The eigenvalues of $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ are

(a) $(1, i, 1 i)$ (b) $(1, -1, 0, 0)$ ✓ (c) $(1, 1, 1, 1)$ (d) $(1, 1, i, -i)$

Sol^n:- Trace of matrix $= 0$

$\therefore$ Sum of eigen values $= 0$, So option (b) is correct.

Ques 22 For which value of λ, the following matrix $A = \begin{bmatrix} 0 & -4 & 1 \\ 2 & \lambda & -3 \\ 1 & 2 & -1 \end{bmatrix}$ will be invertible in nature?

(a) $\lambda = 4$ (b) $\lambda \neq 8$ ✓ (c) $\lambda \neq 4$ (d) None.

Sol^n:-

$$A = \begin{bmatrix} 0 & -4 & 1 \\ 2 & \lambda & -3 \\ 1 & 2 & -1 \end{bmatrix}$$

The matrix is not invertible when

$$|A| = 0$$

So
$$\begin{vmatrix} 0 & -4 & 1 \\ 2 & \lambda & -3 \\ 1 & 2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 0 + 4\{-2+3\} + 1\{4-\lambda\} = 0$$
$$\Rightarrow \quad 4 + 4 - \lambda = 0$$
$$\Rightarrow \quad \lambda = 8$$

So for invertible $\lambda \neq 8$.

Ques 30 The trace of the matrix of a 2x2 matrix is 1 and determinant 1. Which of the following has to be true?

(a) One of the eigenvalues is 0 (b) One of the eigenvalues is 1
(c) both of the eigenvalues are 1 ✓(d) neither of the eigenvalues are 1.

Sol^n:-
$$\lambda_1 + \lambda_2 = 1$$
$$\lambda_1 \lambda_2 = 1$$

if $\lambda_1 = 1$, $\lambda_2 = 0$ So $\lambda_1 \lambda_2 = 0$

So neither of the eigenvalues are 1. Ans

Degenerate eigen values = equal eigen values.

Qus 87 Consider a 3×3 matrix of the following form:

$$\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

The number of zero eigenvalues for this matrix is -?

(a) 0 (b) 1 (c) 2 ✓ (d) 3

Sol^n:-

$\because$ $$A = \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

If I^{st} Row $\times \frac{a_2}{a_1}$ = II^{nd} Row

I^{st} Row $\times \frac{a_3}{a_1}$ = III^{rd} Row

And I^{st} column $\times \frac{a_2}{a_1}$ = II^{nd} Column

I^{st} Column $\times \frac{a_3}{a_1}$ = III^{rd} Column

So $\boxed{\lambda = (a_1^2+a_2^2+a_3^2),\ 0,\ 0.}$

So no. of zero eigen values = 2. Ans

Ques 38 The eigenvalues of the matrix $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ are ?

(a) $\{1,1,1,1\}$ (b) $\{1,1,-1,-1\}$ ✓ (c) $(0,0,\sqrt2,-\sqrt2)$

(d) $\{\sqrt2,-\sqrt2,\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\}$

Sol^n:- $Tr = 0$

$\because$ $A^2 = I$ so given matrix A is involutary matrix so A is self inverse.

$\Rightarrow \lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$ So option (b) is correct.

* Cayley - Hamilton Theorem :- "Every square matrix satisfies its own characteristic equation or eigen value equation."

e.g. :-

$$A = \begin{bmatrix} 1 & -4 \\ 7 & 2 \end{bmatrix}$$

Eigen value equation -

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -4 \\ 7 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda + 30 = 0$$

$$\boxed{A^2 - 3A + 30I = 0}$$

* Applications :-

I. A matrix satisfies eq^n $A^2 - 3A + 30I = 0$ find eigen values ?

Sol^n $\because$ $A^2 - 3A + 30I = 0$

so $\lambda^2 - 3\lambda + 30 = 0$

after solving the above eigen value eq^n we can easily find eigen values of matrix

$$\lambda = \frac{3 \pm \sqrt{9 - 4 \times 1 \times 30}}{2 \times 1} = \frac{3 \pm \sqrt{9 - 120}}{2}$$

$$\lambda = \frac{3 \pm \sqrt{i^2 111}}{2} = \frac{3 \pm i\sqrt{111}}{2}$$

So $\lambda = \frac{3 + i\sqrt{111}}{2}, \frac{3 - i\sqrt{111}}{2}$

II Calculation of inverse of matrix :-

$$\Rightarrow A^2 - 3A + 30I = 0$$
$$\Rightarrow A^2 - 3A = -30I$$
$$\Rightarrow A^{-1}A^2 - 3A^{-1}A = -30A^{-1}I$$
$$\Rightarrow A - 3I = -30A^{-1}$$
$$\Rightarrow \boxed{A^{-1} = -\frac{1}{30}[A - 3I]}$$

A-1
Ques 38 The eigenvalues of matrix $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ are ?

(a) $\{1,1,1,1\}$ (b) $\{1,1,-1,-1\}$ ✓ (c) $\{0,0,\sqrt{2},-\sqrt{2}\}$ (d) $\{\sqrt{2},-\sqrt{2},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\}$

Sol^n

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$A^2 = I$ {When we A^2 then we find some form of I }

According to Cayley - Hamilton Theorem -

$$\Rightarrow \lambda^2 = 1$$
So $\boxed{\lambda = \pm 1}$ Ans

Ques 40 If a matrix A is such that $3A^3 + 2A^2 + 5A + I = 0$, then A^{-1} is equal to ?

(a) $-(3A^2 + 2A + 5)$ (b) $3A^2 + 2A + 5$ (c) $3A^2 - 5A - 5$
(d) None of these ✓

Sol^n
$$3A^3 + 2A^2 + 5A + I = 0$$
$$3A^3A^{-1} + 2A^2A^{-1} + 5AA^{-1} = -IA^{-1}$$
$$\boxed{A^{-1} = -(3A^2 + 2A + 5I)}$$ So None of these.

Ques41 Let, 'M' be a $n \times n$ diagonalizable matrix which satisfiee the equations $M^2 = M$, then $\mathrm{Det}(M)$ is equal to ?

(a) 0 ✓ (b) 1 (c) n (d) $n-1$

Sol^n $M^2 = M$

According to C-H Theorem

$$\Rightarrow \lambda^2 = \lambda$$
$$\lambda^2 - \lambda = 0$$
$$\Rightarrow \lambda(\lambda - 1) = 0$$
$$\Rightarrow \lambda = 0, 1$$

$\mathrm{Det}(M)$ = Product of eigen values

So $\boxed{\mathrm{Det}(M) = 0}$

OR

$$M^2 = M$$
$$\Rightarrow M^2 - M = 0$$
$$\Rightarrow M(M - I) = 0$$

Either $|M| = 0$ or $|M - I| = 0$

$$\downarrow$$
$$M = I$$

So total product = 0.

Ques42 Let 'P' be a 3×3 Hermitian matrix which satisfies the matrix equation $M^2 - 5M + 6I = 0$. Which of the following are the possible eigenvalues of M?

(a) $(0, 1, 2)$ (b) $(0, 2, 3)$ ✓ (c) $(1, 3, 4)$ (d) $(2, 3, 4)$ ✓

Sol^n

$$M^2 - 5M + 6I = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$
$$\lambda = 2, 3$$

So possible eigen values = $(0, 2, 3)$ & $(2, 3, 4)$

~~So option (b) is correct.~~

Ques[15] The trace of a real 4x4 matrix $U = \exp(A)$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & \pi/4 \\ 0 & 0 & -\pi/4 & 0 \\ 0 & \pi/4 & 0 & 0 \\ -\pi/4 & 0 & 0 & 0 \end{pmatrix}$$ is equal to ?

(a) $2\sqrt{2}$ ✓ (b) $\pi/4$ (c) 4 (d) 2

Soln

$$\mathrm{Tr}\,(\exp(A)) = ?$$

Let

$$\because e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\therefore e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

$$A^2 = A.A = \begin{pmatrix} 0 & 0 & 0 & \pi/4 \\ 0 & 0 & -\pi/4 & 0 \\ 0 & \pi/4 & 0 & 0 \\ -\pi/4 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} 0 & 0 & 0 & \pi/4 \\ 0 & 0 & -\pi/4 & 0 \\ 0 & \pi/4 & 0 & 0 \\ -\pi/4 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0+0-\pi^2/16 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0 & -\pi^2/16 & 0 & 0 \\ 0 & 0 & -\pi^2/16 & 0 \\ 0 & 0 & 0 & -\pi^2/16 \end{pmatrix}$$

$$A^2 = -\frac{\pi^2}{16} I$$

$$A^3 = A^2 . A = \left(-\frac{\pi}{16} I\right) . A$$

$$A^3 = -\frac{\pi^2}{16} A$$

$$A^4 = (A^2)^2 = \left(-\frac{\pi^2}{16} I\right)^2 = \frac{\pi^4}{256} I$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$e^A = I + A + \frac{\left(-\frac{\pi^2}{16}\right)I}{2!} + \frac{\left(-\frac{\pi^2}{16}\right)A}{3!} + \cdots$$

$$e^A = I + A - \frac{1}{2!}\frac{\pi^2}{16} I - \frac{1}{3!}\frac{\pi^2}{16} A + \frac{1}{4!}\frac{\pi^4}{256} I + \cdots$$

$$= I\left[1 - \frac{1}{2!}\frac{\pi^2}{16} + \frac{1}{4!}\frac{\pi^4}{256} + \cdots\right] + A\left[1 + \frac{\pi^2}{16} + \frac{1}{5!}\frac{\pi^4}{256} + \cdots\right.$$

$$= I\left(\cos\frac{\pi}{4}\right) + \frac{A}{\pi/4}\left(\frac{\pi}{4} - \frac{1}{3!}\frac{\pi^3}{64} + \frac{1}{5!}\frac{\pi^5}{1024} + \cdots\right)$$

$$= I\left(\cos\frac{\pi}{4}\right) + \frac{4A}{\pi}\left(\sin \pi/4\right)$$

$$\boxed{\exp(A) = I\cos\left(\frac{\pi}{4}\right) + \frac{4A}{\pi}\sin\left(\pi/4\right)}$$

$$\mathrm{Tr}(\exp(A)) = \mathrm{Tr}\left[I\cos(\pi/4)\right] + \mathrm{Tr}\left[\frac{4A}{\pi}\sin(\pi/4)\right]$$

$\because \mathrm{Tr}(A) = m$

and $\mathrm{Tr}(2A) = 2m$

Similary

$\mathrm{Tr}(\lambda A) = \lambda m$

$$\mathrm{Tr}(\exp(A)) = 4\cos\frac{\pi}{4} + 0 \qquad \{\because \mathrm{Tr}(A) = 0\}$$

$$= 4 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2} \quad \underline{\text{Ans}}$$

- If $\pi/4$ replaced by β then $A = \begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \\ 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{pmatrix}$

$$\exp(A) = I\cos\beta + \frac{A}{\beta}\sin\beta$$

$$\boxed{\mathrm{Tr}(\exp A) = 4\cos\beta}$$

Consider the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Ques 50 The eigenvalues of M are?

(a) 0, 1, 2 (b) 0, 0, 3 ✓ (c) 1, 1, 1 (d) −1, 1, 3

Sol^n

$\because \quad M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$Tr(M) = 3$

So eigen values $= (3, 0, 0)$ or $(0, 0, 3)$

So option (b) is correct.

Ques 51 The exponential of M simplifies (I is the 3×3 identity matrix)

(a) $e^M = I + \left(\frac{e^3 - 1}{3}\right) M$ (b) $e^M = I + M + \frac{M^2}{2!}$

(c) $e^M = I + 3^3 M$ (d) $e^M = (e-1) M$.

Sol^n

$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots\cdots$

$M^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$

$M^2 = 3M$

$M^3 = M^2 \cdot M = 3 . M . M = 3 . 3M = 3^2 M$

So $M^4 = 3^3 M$.

So $e^{M} = I + M + \frac{3M}{2!} + \frac{3^2 M}{3!} + \frac{3^3 M}{4!} + \cdots$

$$= I + M\left(1 + \frac{3}{2!} + \frac{3^2}{3!} + \frac{3^3}{4!} + \cdots\right)$$

$$= I + \frac{M}{3}\left(3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots\right)$$

$$\boxed{e^{M} = I + \frac{M}{3}(e^3 - 1)}$$

Ans

CSIR Dec-2012

Ques 56 Given a 2×2 unitary matrix satisfying $U^{\dagger}U = UU^{\dagger} = I$ with $\det(U) = e^{i\phi}$, one can construct a unitary matrix V ($V^{\dagger}V = VV^{\dagger} = I$) with $\det V = 1$ from it by-

(a) Multiplying U by $e^{-i\phi/2}$ ✓ (b) Multiplying a single element U by $e^{-i\phi}$

(c) Multiplying any row or column by $e^{-i\phi/2}$ (d) Multiplying U by $e^{-i\phi}$

Soln

$$U^{\dagger}U = UU^{\dagger} = I$$

It is 2×2 unitary matrix

$$\det U = e^{i\phi}$$

$$U \longrightarrow V \longrightarrow \text{unitary}$$
$$\longrightarrow |V| = 1$$

Let

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|U| = ad - bc = e^{i\phi}$$

$$V = Ue^{-i\phi/2}$$

$$V = \begin{pmatrix} ae^{-i\phi/2} & be^{-i\phi/2} \\ ce^{-i\phi/2} & de^{-i\phi/2} \end{pmatrix}$$

$$|V| = ade^{-i\phi} - bce^{-i\phi} = e^{-i\phi}(ad - bc) = e^{-i\phi}\cdot e^{i\phi}$$

$$= e^{0} = 1$$

Ans

CSIR Dec-2013

Ques 57 Consider a $n \times n$ $(n>1)$ matrix A, in which A_{ij} is the product of the indices i and j (namely $A_{ij} = ij$). The matrix A.

(a) has one degenerate eigenvalue with degeneracy $(n-1)$
(b) has two degenerate eigenvalues with degeneracies 2 and $(n-2)$
(c) has one degenerate eigenvalues with degeneracy n
(d) does not have any degenerate eigenvalues.

Sol^n

$$A_{ij} = ij$$

If $n = 3$
So 3×3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

$\lambda = 14, 0, 0$

1 degenerate eigenvalue $\rightarrow$ degeneracy 2.

If $n = 4$
So 4×4

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

$\lambda = 30, 0, 0, 0$

1 deg. eigen value $\longrightarrow$ $g = 3$ ↑ degeneracy.

for $n \rightarrow$

1 deg $\longrightarrow$ $g = (n-1)$ Ans

* Similarity Transformation :-

$$\boxed{B = P^{-1} A P}$$

A, B are similar matrices.
If A, and B matrices are similar then they have -

- → Same Trace
- → Same Det
- → Same eigenvalue
- → Same rank.

Since this transformation transforms the matrix A to its similar matrix B this is why is known as similarity transformation.

* Diagonalization of Matrix using similarity matrices :-

By the proper selection of P we can make the matrix B diagonal and B will be k/a diagonalized matrix corresponding to A.

$A = 3 \times 3$ matrix

So it has 3 eigen values

$$\lambda_1 \;,\; \lambda_2 \;,\; \lambda_3$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} \quad \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{31} \\ x_{32} \\ a_{33} \end{pmatrix}$$

$$P = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ \uparrow \lambda_1 \quad \uparrow \lambda_2 \quad \uparrow \lambda_3 \end{array}$$

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$B = P^{-1}AP$$

$$\boxed{B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}$$

* But every matrix can't be diagonalised.

* Suppose a matrix A of order $m \times n$ has n linearly independent eigenvectors then the matrix can be diagonalised.

* When eigen vectors are linearly dependent then why diagonalisation are not possible?

Proof: Suppose $A = 3 \times 3$ matrix

$$\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \\ \downarrow & \downarrow & \downarrow \\ \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} & \begin{pmatrix} x_{21} \\ x_{22} \\ x_{23} \end{pmatrix} & \begin{pmatrix} x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} \end{array}$$

Linearly Dependent

then $\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = p \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$

So

$$P = \begin{pmatrix} x_{11} & p x_{11} & x_{13} \\ x_{21} & p x_{21} & x_{23} \\ x_{31} & p x_{31} & x_{33} \end{pmatrix}$$

So $\quad P = \begin{pmatrix} x_{11} & x_{11} & x_{13} \\ x_{21} & x_{21} & x_{23} \\ x_{31} & x_{31} & x_{33} \end{pmatrix}$

$\because$ 2 Columns are equal so $|P| = 0$

So $\quad P^{-1} = \frac{\text{adj } P}{|P|}$ is not possible

So diagonalisation is not possible.

23/ July / 2014

Ques 23 If 'U' is unit matrix and the matrix 'A' is defined as $A = I - 2UU^T$, then A^{-1} is equal to ?

(a) $I - 2UU^T$ [✓] (b) $I + 2UU^T$ (c) $2UU^T - I$ (d) $4UU^T$

Sol^n

$$A = I - 2UU^T$$
$$= I - 2II$$
$$= \cancel{I} - 2I$$
$$A = -I \quad = I - 2UU^T$$
$$\because \quad AA^{-1} = I$$
$$\therefore \quad (-I)A^{-1} = I$$
$$A^{-1} = -I$$
$$\therefore \quad \boxed{A^{-1} = I - 2UU^T}$$

Ques 25 The degenerate eigenvalues of the matrix $M = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$ is -

(a) 2 (b) 3 (c) 4 (d) 5 ✓

Sol^n $\quad Tr(M) = 12 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 12$

$Det(M) = 50 \Rightarrow \lambda_1\lambda_2\lambda_3 = 50$

So $(\lambda_1, \lambda_2, \lambda_3) = (5, 5, 2)$ So degenerate eigen value $= 5$ Ans

CSIR June - 2012

Que 54 The eigenvalues of the anti-symmetric matrix $A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$ where n_1, n_2, n_3 are components of an unit vector are

(a) $0, -i, i$ (b) $0, 1, -1$ (c) $0, 1+i, -1-i$, (d) $0, 0, 0$

Soln

$$A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -n_3 & n_2 \\ n_3 & -\lambda & -n_1 \\ -n_2 & n_1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda\{\lambda^2 + n_1^2\} + n_3(-n_3\lambda + n_1 n_2) + n_2(n_1 n_3 - \lambda n_2) = 0$$

$$\Rightarrow -\lambda^3 - n_1^2\lambda - n_3^2\lambda + \cancel{n_1 n_2 n_3} + n_1 \cancel{n_2} n_3 - \lambda n_2^2 = 0$$

$$\Rightarrow -\lambda^3 - \lambda(n_1^2 + n_2^2 + n_3^2) \cancel{+ 2n_1 n_2 n_3} = 0$$

$$\Rightarrow \lambda^3 + \lambda(n_1^2 + n_2^2 + n_3^2) \cancel{- 2n_1 n_2 n_3} = 0$$

$$\because \quad \hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$$

$$\sqrt{n_1^2 + n_2^2 + n_3^2} = 1$$

$$\Rightarrow \lambda^3 + \lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 + 1) = 0$$

$\Rightarrow$ Either $\lambda = 0$ or $\lambda^2 + 1 = 0$

$\lambda = 0$ or $\lambda = \pm i$

So $\lambda = 0, i, -i$ Ans

Que 69 Consider the following transformation in three dimensions

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z$$

(i) Write down the transformation matrix [A].

(ii) Show that, $[A(\theta_1)] \times [A(\theta_2)] = [A(\theta_1+\theta_2)]$

(iii) Is $A(\theta)$ is unitary?

(iv) What are eigenvalues of $A(\theta)$?

Sol^n

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A(\theta)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(i)

$$[A] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ← Transformation matrix.

(ii)

$$A(\theta_1) \times A(\theta_2) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + 0 & \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 + 0 & 0 \\ -\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 + 0 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 + 0 & 0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $\boxed{A(\theta_1) \times A(\theta_2) = A(\theta_1+\theta_2)}$

(iv)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

expanding along III^{rd} row -

$$(1-\lambda)\{\{\cos\theta - \lambda\}^2 + \sin^2\theta\} = 0$$

Either $(1-\lambda) = 0$ or $\{(\cos\theta-\lambda)^2 + \sin^2\theta\} = 0$

When $1-\lambda = 0$ then $\lambda = 1$

and when

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0 \Rightarrow \cos^2\theta + \lambda^2 - 2\lambda\cos\theta + \sin^2\theta = 0$$

$$\Rightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\times1\times1}}{2} = \frac{\cancel{2}\cos\theta \pm \cancel{2}\sqrt{\cos^2\theta - 1}}{\cancel{2}}$$

$$= \cos\theta \pm \sin\theta = e^{\pm i\theta}$$

So $\boxed{\lambda = e^{\pm i\theta}, 1.}$

(iii) If $A(\theta)$ is unitary, then $|A| = 1$

So

$$|A| = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\{\cos^2\theta + \sin^2\theta\}$$

$$= 1 \times 1 = 1$$

$\therefore$ $|A|$ is 1 so $A(\theta)$ is a unitary matrix.

Ques 70: Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$ and hence find —

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^2 - 2A + I$$

Solⁿ

$$\because \quad A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(1-\lambda)(2-\lambda) - 0\} - 1\{0\} + 1\{0 - (1-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)(2 - 2\lambda - \lambda + \lambda^2) + \lambda - 1 = 0$$

$$\Rightarrow 4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 + \lambda - 1 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

According to C-H Theorem —

$$A^3 - 5A^2 + 7A - 8I = 0$$

Given

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$\Rightarrow A^5(A^3 - 5A^2 + 7A - 3I) + A[A^3 - 5A^2 + 7A - 3I] + A^2 + A + I$$

$$\Rightarrow A^5\{0\} + A\{0\} + A^2 + A + I$$

$$\Rightarrow A^2 + A + I$$

Ans

* Power of a Matrix :-

[1] If matrix is a Diagonal Matrix then power of matrix -

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

then

$$D^m = \begin{bmatrix} \lambda_1^m & 0 & 0 \\ 0 & \lambda_2^m & 0 \\ 0 & 0 & \lambda_3^m \end{bmatrix}$$ where m = +ve integer

[2] If matrix is Non-Diagonal :-

If matrix is non-diagonal matrix then we diagonalised the matrix using symmetry transformation.

$$\boxed{A^m = P B^m P^{-1}}$$

* Exponential of a Matrix :-

[1] If given matrix is a diagonal matrix :-

$$\exp(D) = \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix}$$

[2] If given matrix is non-diagonal matrix then -

$$\boxed{\exp(A) = P(\exp B) P^{-1}}$$

* Log of a Matrix :-

[1] If given matrix is a diagonal matrix then-

$$\ln(D) = \begin{bmatrix} \ln\lambda_1 & 0 & 0 \\ 0 & \ln\lambda_2 & 0 \\ 0 & 0 & \ln\lambda_3 \end{bmatrix}$$

[2] If A is non-diagonal matrix then :-

$$\boxed{\ln A = P(\ln B)P^{-1}}$$

* Rank of a Matrix :-

- By deleting any row and any column of a matrix we can get sub matrix of original one.

- Square Sub-Matrix :-
  By deleting equal number of row and column of a matrix we will get square sub-matrix of original one.

- Rank of a matrix is the order of the largest square sub-matrix having non-zero determinant.

$$|3\times3| \neq 0 \Rightarrow \rho(A) = 3$$
$$= 0 \Rightarrow \rho(A) \neq 3$$

Any $|2\times2| \neq 0$ then $\rho(A) = 2$

otherwise $|1\times1|$

- Every matrix is a submatrix of itself.

$$\begin{array}{lcl} 1 \text{ Submatrix} \leftarrow & 3\times 3 \text{ matrix} & \rightarrow \text{Largest sub matrix} \\ & \downarrow & \\ 9 \text{ Submatrix} \leftarrow & 2\times 2 \text{ matrix} & \\ & \downarrow & \\ 9 \text{ Submatrix} \leftarrow & 1\times 1 & \rightarrow \text{Smallest sub matrix.} \end{array}$$

Qus Find the rank of matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

Sol^n

$$\begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{vmatrix}$$

$\Rightarrow 2[-9+8] - 1[0+4] - 1[0-6]$

$\Rightarrow -2-4+6 = 0$

So $|2\times 2|$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0-6 = -6 \neq 0$$

So rank $= 2$. Ans

* Properties of Rank :-

1. The rank of a null matrix is always zero.
2. The rank of a non zero matrix is always greatter than or equal to 1 (≥ 1).
3. The rank of a non-singular matrix of order n is n.
4. The Rank of product of two matrices can not exceeds the rank of either matrix.

$$\overset{3}{\uparrow}\quad \overset{2}{\uparrow}$$
$$A \ \& \ B$$

$$\rho(AB) \leq \rho(A)$$
$$\leq \rho(B)$$

So $\boxed{\rho(AB) \leq 2}$

Ques 96 The rank of the matrix $\begin{pmatrix} \Omega & -1 & 0 \\ 0 & \Omega & -1 \\ -1 & 0 & \Omega \end{pmatrix}$ is 2, for Ω equal to ?

(a) 0 (b) 1 ✓ (c) 2 (d) 4

Sol^n

$$A = \begin{bmatrix} \Omega & -1 & 0 \\ 0 & \Omega & -1 \\ -1 & 0 & \Omega \end{bmatrix}$$

$$\begin{vmatrix} \Omega & -1 & 0 \\ 0 & \Omega & -1 \\ -1 & 0 & \Omega \end{vmatrix} = 0$$

$$\Omega(\Omega^2 - 0) + 1(0-1) + 0 = 0$$
$$\Omega^3 = 1$$
$$\boxed{\Omega = 1}$$

Note :- $\Omega^3 - 1 = 0$

$\Omega = 1, \Omega, \Omega^2$

$(\Omega - 1)(\Omega^2 + \Omega + 1) = 0$

if $(\Omega - 1) = 0$ then $\Omega = 1$

if $(\Omega^2 + \Omega + 1) = 0$ then

$\Omega = \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

Ques 47 The rank of matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$ (a, b, c being all real), is 3 if ?

(a) $a = b = c$

(b) a, b, c are all different but $a+b+c = 0$

(c) Two of the numbers a, b, c are equal but are different from the third

(d) a, b, c are all different and $a+b+c \neq 0$ [✓]

Sol^n

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \quad \cancel{= 1\{bc^3 - b^3c\} - 1\{ac^3 - a^3c\} + 1\{ab^3 - a^3b\}}$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} \qquad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$\Rightarrow (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+ba+a^2 & c^2+ca+a^2 \end{vmatrix}$$

$$\Rightarrow (b-a)(c-a) . 1\{c^2+ca+\cancel{a^2}-b^2-ba-\cancel{a^2}\}$$

$$\Rightarrow (b-a)(c-a)\{(b-c)(a+b+c)\}$$

$$\Rightarrow (b-a)(c-a)(b-c)(a+b+c)$$

Either $b = a$ or $c = a$ or $b = c$ | or $a+b+c = 0$

So option (d) is correct.

* Application Part of Matrix :-

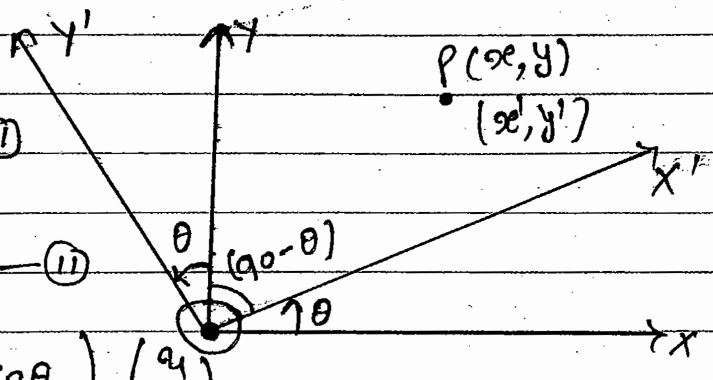
1 Rotation Matrix [2-D] :-

$x' = x\cos\theta + y\cos(90-\theta)$

$x' = x\cos\theta + y\sin\theta$ —— (I)

$y' = x\cos(90+\theta) + y\cos\theta$

$y' = -x\sin\theta + y\cos\theta$ —— (II)



$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R_z(\theta)} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Here $R_z(\theta)$ is called Rotation matrix or Rotation operator corresponding to anticlockwise rotation by angle θ about z-axis.

V.Imp. Property

- Rotational matrix is orthogonal is in nature and its determinant is always +1.

* Rotation of Matrix [3-D] :-

Case I :- Z is fixed :-

$\{x, y, z\} \longrightarrow \{x', y', z'\}$

$x' = x\cos\theta + y\sin\theta$

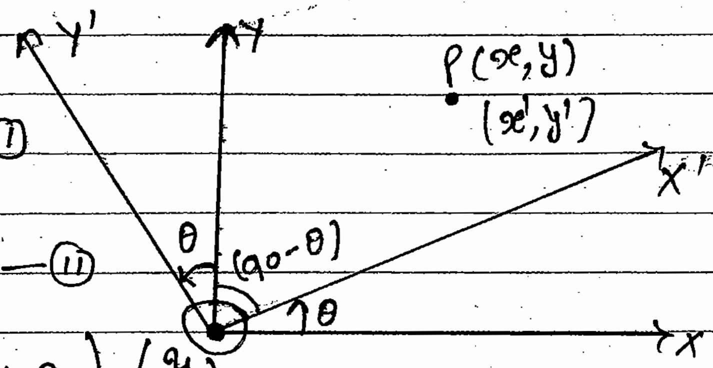
$y' = -x\sin\theta + y\cos\theta$

$z' = z$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

* Application Part of Matrix :-

1. Rotation Matrix [2-D] :-



$$x' = x\cos\theta + y\cos(90-\theta)$$
$$x' = x\cos\theta + y\sin\theta \quad \text{—(I)}$$
$$y' = x\cos(90+\theta) + y\cos\theta$$
$$y' = -x\sin\theta + y\cos\theta \quad \text{—(II)}$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R_z(\theta)} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Here $R_z(\theta)$ is called Rotation matrix or Rotation operator corresponding to anticlockwise rotation by angle θ about z-axis.

V.Imp. Property
- Rotational matrix is orthogonal is in nature and its determinant is always +1.

* Rotation of Matrix [3-D] :-

Case I :- Z is fixed :-

$$\{x, y, z\} \longrightarrow \{x', y', z'\}$$

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

* Application Part of Matrix :-

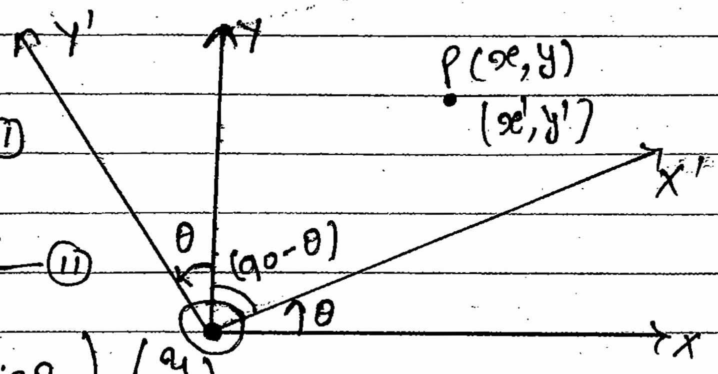
1. Rotation Matrix [2-D] :-

$x' = x\cos\theta + y\cos(90-\theta)$

$x' = x\cos\theta + y\sin\theta$ —— (I)

$y' = x\cos(90+\theta) + y\cos\theta$

$y' = -x\sin\theta + y\cos\theta$ —— (II)



$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R_z(\theta)} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Here $R_z(\theta)$ is called Rotation matrix or Rotation operator corresponding to anticlockwise rotation by angle θ about z-axis.

V.Imp. Property

- Rotational matrix is orthogonal is in nature and its determinant is always +1.

* Rotation of Matrix [3-D] :-

Case I :- Z is fixed :-

$\{x, y, z\} \longrightarrow \{x', y', z'\}$

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

$z' = z$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

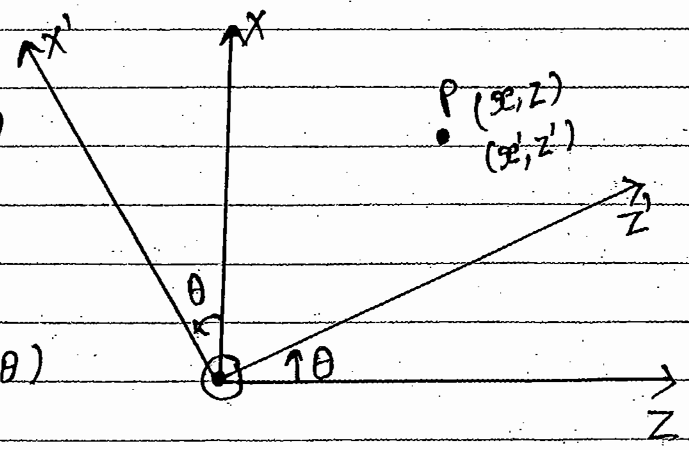
It is also a orthogonal and $\det(R_z(\theta)) = +1$.

Case II $\rightarrow$ X - is fixed :-

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Case III :- Y - axis is fixed :-

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$



$Z' = Z\cos\theta + X\cos(90-\theta)$

$$\boxed{Z' = Z\cos\theta + X\sin\theta}$$

$X' = X\cos\theta + Z\cos(90+\theta)$

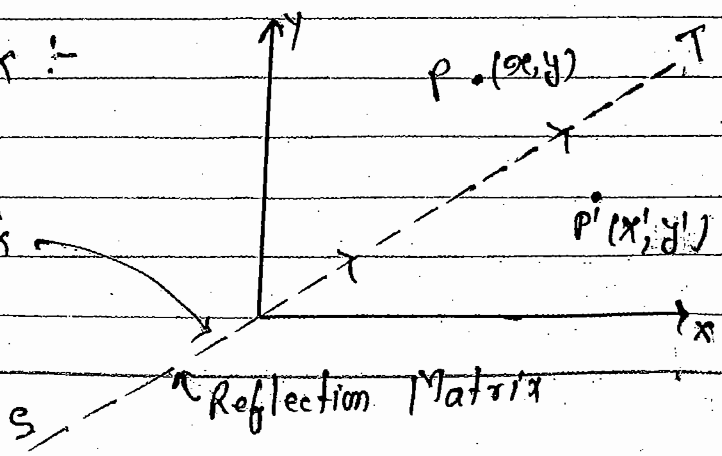
$$\boxed{X' = X\cos\theta - Z\sin\theta}$$

P.

(*) Reflection Matrix :-

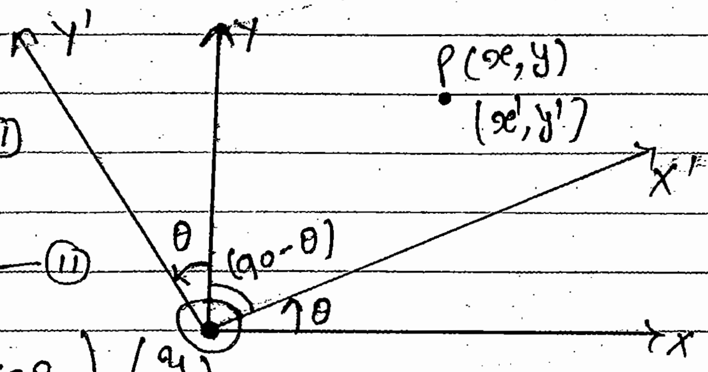
$\vec{ST} = |\vec{ST}|\,\hat{ST}$

$\hat{ST} = \hat{l} = l_x\hat{i} + l_y\hat{j} + l_z\hat{k}$



* Application Part of Matrix :-

1. Rotation Matrix [2-D] :-



$$x' = x\cos\theta + y\cos(90-\theta)$$
$$x' = x\cos\theta + y\sin\theta \quad \text{—(I)}$$
$$y' = x\cos(90+\theta) + y\cos\theta$$
$$y' = -x\sin\theta + y\cos\theta \quad \text{—(II)}$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R_z(\theta)} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Here $R_z(\theta)$ is called Rotation matrix or Rotation operator corresponding to anticlockwise rotation by angle θ about z-axis.

V.Imp. Property

- Rotational matrix is orthogonal is in nature and its determinant is always +1.

* Rotation of Matrix [3-D] :-

Case I :- Z is fixed :-

$$\{x, y, z\} \longrightarrow \{x', y', z'\}$$

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

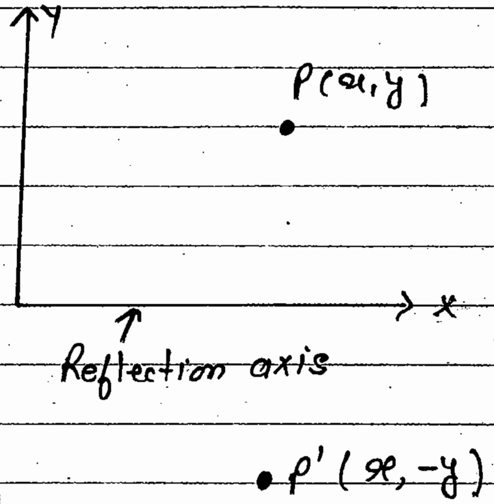
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} l_x^2 - l_y^2 & 2l_x l_y \\ 2l_x l_y & l_y^2 - l_x^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For X-axis :-

$\hat{I} = \hat{i}$

Here $l_x = 1$, $l_y = 0$

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\boxed{\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$$

True

* Property of Reflection Matrix :-

- Reflection Matrix is a orthogonal matrix and its determinant is -1.

Ques Suppose $\vec{P}$ is vector in the X, Y, Z coordinate system and the co-ordinate axis are rotated in the anticlockwise direction about Y-axis by an angle 60° what will be the vector in the new co-ordinate system x', y', z'

$$\vec{P} = 2\hat{i} + 3\hat{j} + 2\hat{k}$$

Sol^n

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos 60 & 0 & -\sin 60^\circ \\ 0 & 1 & 0 \\ \sin 60^\circ & 0 & \cos 60^\circ \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} (1-\sqrt{3}) \\ 3 \\ (1+\sqrt{3}) \end{pmatrix}$$

So, $\boxed{(1-\sqrt{3})\hat{i}' + 3\hat{j}' + (1+\sqrt{3})\hat{k}'}$ Ans

Ques Consider the transformation -

(i) $x' = 0.6x + 0.8y$
$y' = -0.8x + 0.6y$

(ii) $x' = 0.6x + 0.8y$
$y' = 0.8x - 0.6y$

Sol^n:-

Ques 67 If α is the characteristic root (eigen value) of a non-singular matrix, then show that, $|A|/\alpha$ is a characteristic root of (adj. A).

Sol^n $\because$ α is the eigen value of matrix A.

Eigen value of matrix $A = \alpha$
" " " " $Adj\,A = ?$

$$\because \quad A^{-1} = \frac{Adj\,A}{|A|}$$

$$\therefore \quad Adj\,A = |A|\,A^{-1}$$

So if Eigen value of matrix $A = \alpha$
then " " " $A^{-1} = \dfrac{1}{\alpha}$

So " " " $|A|\,A^{-1} = \dfrac{|A|}{\alpha}$

that means Eigen value of matrix $Adj\,A = \dfrac{|A|}{\alpha}$ **Ans**

Ques 68 Represent the following transformations -
$x_1 = 3y_1 + 2y_2$, $x_2 = -y_1 + 4y_2$, $y_1 = z_1 + 2z_2$, $y_2 = 3z_1$

by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2

Sol^n
$$\left.\begin{aligned} x_1 &= 3y_1 + 2y_2 \\ x_2 &= -y_1 + 4y_2 \end{aligned}\right\} \longrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\left.\begin{aligned} y_1 &= z_1 + 2z_2 \\ y_2 &= 3z_1 \end{aligned}\right\} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3+6 & 6+0 \\ -1+12 & -2+0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

So

$$\boxed{\begin{aligned} x_1 &= 9z_1 + 6z_2 \\ x_2 &= 11z_1 - 2z_2 \end{aligned}}$$

Ans

Ques 71 The matrix $A = \begin{pmatrix} p & r \\ r & q \end{pmatrix}$ is transformed in the diagonal form $D = M^{-1}AM$, where $M = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$

Find the value of ϕ which gives the diagonal transformation

Sol^n $\because$ $A = \begin{pmatrix} p & r \\ r & q \end{pmatrix}$ & $M = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$

Date :- 14/Aug/2014

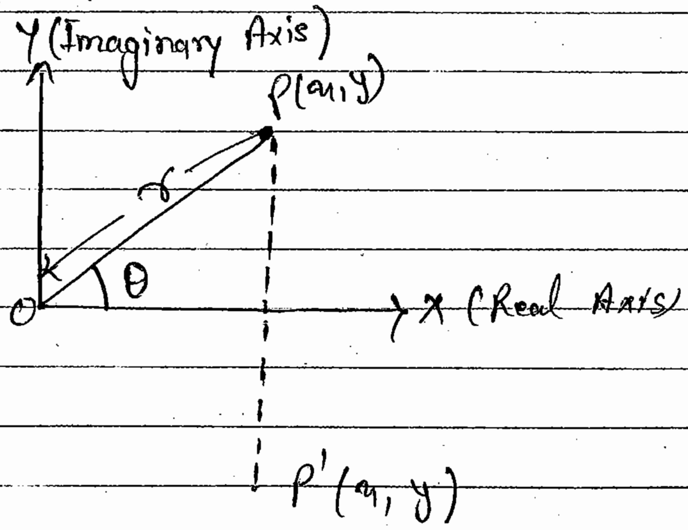
Complex Analysis

* Complex Numbers :-

$$Z = \underbrace{x}_{\text{Real}} + i\underbrace{y}_{\text{Imaginary}}$$

$$|Z| = \text{modulus of } Z = \sqrt{x^2+y^2} = r$$

$$\arg Z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$



θ is the angle which OP makes with real axis.

So the conjugate of a complex number is nothing but the reflection about real axis.

(1) $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$

(2) $|Z_1 - Z_2| \geq |Z_1| - |Z_2|$

(3) $|Z_1 Z_2| = |Z_1| \, |Z_2|$

(4) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(5) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(6) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

$z = x + iy$ (Cartesian Co-ordinate)

$z = r\cos\theta + ir\sin\theta$

$\boxed{z = r(e^{i\theta})}$ [Polar form]

r = modulus
θ = arg.

* Square Root of a Complex Number:-

Square Root of a complex number is again a complex number.

$\sqrt{x+iy} = a+ib$

$x+iy = (a+ib)^2$
$x+iy = a^2 - b^2 + 2iab$

Comparing Real and Imaginary part

$x = a^2 - b^2$
$y = 2ab$

$$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$$

$$(a^2+b^2)^2 = x^2 + y^2$$

$$\Rightarrow \boxed{a^2+b^2 = \sqrt{x^2+y^2}} \quad \text{--- (I)}$$

$$\Rightarrow \boxed{a^2-b^2 = x} \quad \text{--- (II)}$$

(I) + (II)

$$2a^2 = x + \sqrt{x^2+y^2}$$

$$a^2 = \frac{x+\sqrt{x^2+y^2}}{2}$$

$$\boxed{a = \pm\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{1/2}}$$

11^{ly}

$$\boxed{b = \pm\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{1/2}}$$

Here 4 combinations of a and b are found but in square Root only two combinations are found. So which two combination (+ +, − −, + −, − +) is consider this sugest the value $y = 2ab$ which is given in ques.

When $y = +ve \Rightarrow 2ab = +ve$ So + +, − − is taken

When $y = -ve \Rightarrow 2ab = -ve$ So + −, − + is taken.

Que. Solve $\sqrt{3-4i}$

$$\sqrt{3-4i} = a+ib$$

Comparing :- $\sqrt{x+iy} = a+ib$

$y = -ve$

$\because$ y = Negative (-4)
So we take opposite sign of a and b in square root.

$$a = \pm\left(\frac{3+5}{2}\right)^{1/2}$$

$$\boxed{a = \pm 2}$$

$$b = \pm\left(\frac{5-3}{2}\right)^{1/2}$$

$$\boxed{b = \pm 1}$$

$2-i \quad , \quad -2+i$

So Square Root -

$2-i \quad , \quad -2+i$ Ans

* De-Movire's Theorem :-

$$\boxed{(\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta}$$

* Complex Cube Root of Unity :-

$$x^3 = 1$$

$\Rightarrow \quad x^3 - 1^3 = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

$\Rightarrow \quad x = 1 \text{ or } x^2+x+1 = 0$

$x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$= \dfrac{-1 \pm \sqrt{3}\, i}{2}$

$x = \dfrac{-1+\sqrt{3}\,i}{2}, \ \dfrac{-1-\sqrt{3}\,i}{2}$

So $\underbrace{\boxed{x = 1}}_{1}, \underbrace{\boxed{x = \dfrac{-1+\sqrt{3}\,i}{2}}}_{\omega}, \underbrace{\boxed{x = \dfrac{-1-\sqrt{3}\,i}{2}}}_{\omega^2}$

$x = 1, \omega, \omega^2$

* Relations :-

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

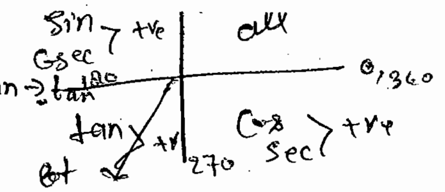
* $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

* $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

* $\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$

* $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

even multiple of $\frac{\pi}{2}$ then
Cos $\to$ -Cos , Sin $\to$ -Sin, tan $\to$ tan
Odd multiple of $\frac{\pi}{2}$ then
Cos $\to$ Sin, Sin $\to$ Cos, tan $\to$ Cot.



Q.1 If $x+iy = (1-i\sqrt{3})^{100}$, then the co-ordinate of point $P(x,y) = ?$

Sol^n

$$P(x,y) \quad x+iy = (1-i\sqrt{3})^{100}$$

$$= 2^{100}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{100}$$

{ in the bracket multiply and divide by 2 and take 2 Common on outside the bracket.

$$= 2^{100}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^{100}$$

$$= 2^{100}\left(\cos\frac{100\pi}{3} - i\sin\frac{100\pi}{3}\right)$$

$$= 2^{100}\left[\cos\left(33\pi+\frac{\pi}{3}\right) - i\sin\left(33\pi+\frac{\pi}{3}\right)\right]$$

$$= 2^{100}\left[\cos\left(66\times\frac{\pi}{2}+\frac{\pi}{3}\right) - i\sin\left(66\times\frac{\pi}{2}+\frac{\pi}{3}\right)\right]$$

$$= 2^{100}\left[-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$$= 2^{100}\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$$= -2^{99}, + i\,2^{99}.\sqrt{3}$$

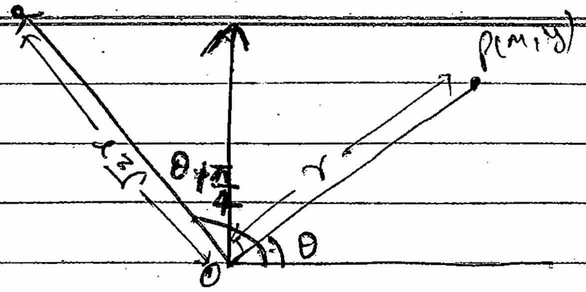
So $\boxed{x = -2^{99}, \quad y = \sqrt{3}.2^{99}}$

Q.5 Multiplying a complex number z by 1+i rotates the radius vector of z by an angle of ?

(a) 90° Clockwise (b) 45° antoclockwise ✓ (c) 45° clockwise
(d) 90° anticlockwise.

$$\arg((1+i)\cdot z) = \arg(1+i) + \arg z = \tan^{-1}\frac{\pi}{4} + \theta$$

$$= \frac{\pi}{4} + \theta.$$



Say $z = re^{i\theta} = x + iy$

When $\underbrace{(1+i)}_{z_1}\ \underbrace{z}_{z_2}$

$$|(1+i)\cdot z| = \sqrt{2}r \quad \text{—(1)}$$

OP = Radius Vector.

$$\arg (1+i).z = \left(\frac{\pi}{4} + \theta\right)$$

$\left\{ \tan^{-1}\left(\frac{y}{x}\right) = \left(\frac{1}{1}\right) \right.$
$\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$

So Radius vector is rotatee 45° anticlockwise.

Q.2 The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.

Sol^n

$$\left(\frac{1+i}{1-i}\right)^n = \left(\frac{(1+i)^2}{1+i^2}\right)^n \qquad \left\{ \text{multiplying by complex conjugate} \right.$$

$$= \left(\frac{1+i^2+2i}{2}\right)^n$$

$$= (i)^n$$

$$= \{\sqrt{-1}\}^n$$

Smallest integer is $\boxed{n=2}$ Ans

A-5

Q.4 Find the area of the triangle enclosed by the vectors $z, iz, z-iz$.

Sol^n $z, iz, z-iz$

Radius Vectors form the Δ

$y = 1$

$\arg i = \theta = \tan^{-1}\frac{1}{0} = \tan^{-1}\left(\tan\frac{\pi}{2}\right)$

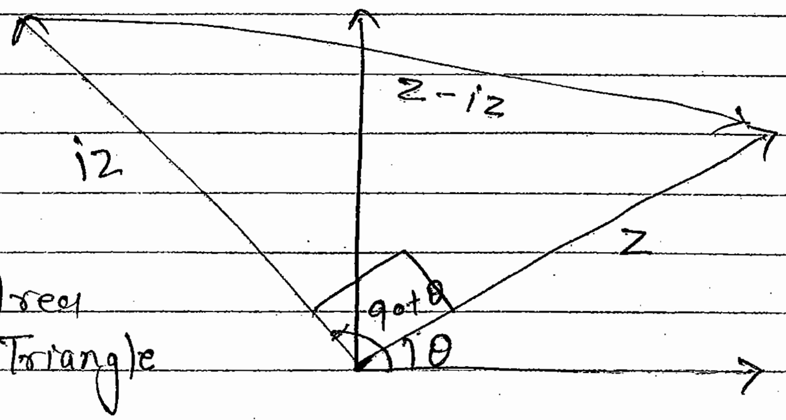
$\boxed{\arg i = \frac{\pi}{2}}$

So

$z = re^{i\theta}$

$iz = re^{i(\theta + \frac{\pi}{2})}$

$\because$ the angle b/w z and iz is $\frac{\pi}{2}$ so it is right angle Triangle So. Area of Right angle Triangle



Area of Right Angle Triangle is -

$= \frac{1}{2} \times \text{Base} \times \text{height}$

$= \frac{1}{2} \times r \times r$

$= \frac{1}{2} \times r^2$

$= \frac{1}{2}|z|^2$ **Ans**

A-4
Q.6 If 1, ω, ω^2 be cube roots of unity, then the sum of the following series -
$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$ ---- to 2n factor will be -

$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$ ---- 2n factor

$1+\omega+\omega^2 = 0$, $\omega^3 = 1$

So $= (-2\omega)(-2\omega^2)(-2\omega)$ ---- 2n factor

$= (-2\omega)^n(-2\omega^2)^n$

$= (4\omega^3)^n$

$= 2^{2n}$ Ans option (b) is correct.

$i^2 + i^4 + i^6 - - - -$ up to $(2n+1)$ term

A-4 The sum of following series is - ?

Q.3 ~~If 1, ω, ω² are the cube roots of unity then (1+ω+ω²)⁵ + (1+ω−ω²)⁵ will be -~~

(a) i (b) $-i$ (c) 1 (d) -1 ✓

$i^2 + i^4 + i^6 - - - -$ $(2n+1)$ term

$\Rightarrow \underbrace{-1+1}-1 - - - - - -$ $(2n-1)$ term

$\Rightarrow -1$ Ans {∵ $2n$ term is cancil out.

A-4

Q.7 If $z = (\lambda+3) + i\sqrt{5-\lambda^2}$ (λ is real parameter and $i = \sqrt{-1}$) then the locus of z will be -

(a) Circle ✓ (b) ellipse (c) parabola (d) hyperbola

Solⁿ

$$z = (\lambda+3) + i\sqrt{5+\lambda^2}$$

$$x = \lambda+3 \Rightarrow \lambda = x-3$$

$$y = \sqrt{5-\lambda^2}$$

$$\Rightarrow y^2 = 5-\lambda^2$$

$$\Rightarrow y^2 = 5-(x-3)^2$$

$$\Rightarrow (x-3)^2 + y^2 = 5$$

which is the eqⁿ of circle.

A-4

Q.9 The expression $\sinh^{-1} x$ can be simplified to -

(a) $\ln(x+\sqrt{x^2+1})$ (b) $\ln(x+\sqrt{x^2-1})$ (c) $\ln(x-\sqrt{x^2-1}$ (d) $\ln(x-\sqrt{x^2+1})$

Solⁿ The expression $\sinh^{-1} x$

$$y = \sinh^{-1} x \Rightarrow x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - e^{-y}$$

$$2x = e^y - \frac{1}{e^y} \Rightarrow 2x = \frac{e^y \cdot e^y - 1}{e^y}$$

$$e^{2y} - e^y \cdot 2x - 1 = 0$$

let $p = e^y$

So

$$p^2 - 2xp - 1 = 0$$

$$p = \frac{2x \pm \sqrt{4x^2 - 4\times 1\times(-1)}}{2\times 1}$$

$$p = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = p = x \pm \sqrt{x^2+1}$$

$$y = \log\left(x \pm \sqrt{x^2+1}\right)$$

$$\boxed{y = \log\left(x + \sqrt{x^2+1}\right)}$$

{$\because$ + log = exist but, − log = doesn't exist.
So we take +ve sign.}

A-5
Q.2 If $m = e^{i\theta}$, $n = e^{i\phi}$, then show that ~~$1+m+m^2 = e^{i\theta}(1+2\cos\theta)$~~. $\frac{m-n}{m+n} = i\tan\left(\frac{\theta-\phi}{2}\right)$.

$$\because m = e^{i\theta}, \quad n = e^{i\phi}$$

$$\therefore \frac{m}{n} = \frac{e^{i(\theta-\phi)}}{1}$$

$$\frac{m+n}{m-n} = \frac{e^{i(\theta-\phi)}+1}{e^{i(\theta-\phi)}-1}$$

$\left\{ \because \frac{a}{b} = \frac{c}{d} \quad \therefore \frac{a+b}{a-b} = \frac{c+d}{c-d} \right.$ Componendo dividendo.

$$= \frac{\cancel{e^{i(\theta-\phi)/2}}}{\cancel{e^{i(\theta-\phi)/2}}}\left[\frac{e^{i(\theta-\phi)/2} + e^{-i(\theta-\phi)/2}}{e^{i(\theta-\phi)/2} - e^{-i(\theta-\phi)/2}}\right]$$

$$\frac{m+n}{m-n} = \frac{\cancel{2}\cos(\theta-\phi)/2}{\cancel{2}i\sin(\theta-\phi)/2}$$

$$\boxed{\frac{m-n}{m+n} = i\tan(\theta-\phi)/2}$$

A-5

Q.3 Find the complex numbers z such that $\arg(z+1) = \frac{\pi}{6}$ & $\arg(z-1) = \frac{2\pi}{3}$.

$z-1 = x+iy-1$

$z+1 = x+iy+1$

$$\arg(z+1) = \frac{\pi}{6} \Rightarrow \tan^{-1}\frac{y}{x+1} = \pi/6$$

$$\arg(z-1) = \frac{2\pi}{3} \Rightarrow \tan^{-1}\frac{y}{x-1} = \frac{2\pi}{3}$$

So $\frac{y}{x+1} = \tan\frac{\pi}{6}$ —— (I) $= \tan 30 = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{y}{x-1} = \tan\frac{2\pi}{3}$ —— (II) $= \tan 120 = -\sqrt{3}$

After solving get x and y and then we get $z = x+iy$. $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$

Q.4 If $1, \omega, \omega^2$ are the cube root of unity then $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5$ will be -

Sol^n $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5$

$(-2\omega)^5 + (-2\omega^2)^5 = 2^5(-\omega^2-\omega)$

$= 32(1+\cancel{\omega}-\cancel{\omega})$

$= 32$

A-5

Q.5 If $2\cos\theta = m + \frac{1}{m}$ and $2\cos\phi = n + \frac{1}{n}$, then prove that $m^p n^q + \frac{1}{m^p n^q} = 2\cos(p\theta + q\phi)$

Solⁿ

$$2\cos\theta = m + \frac{1}{m}, \quad 2\cos\phi = n + \frac{1}{n}$$

$$m^p n^q + \frac{1}{m^p n^q} = ?$$

$$\Rightarrow 2\cos\theta = m + \frac{1}{m}$$

$$\Rightarrow m^2 - 2m\cos\theta + 1 = 0$$

$$\Rightarrow m = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2.1}$$

$$= \frac{\cancel{2}\cos\theta \pm \cancel{2}\sqrt{-\sin^2\theta}}{\cancel{2}}$$

$$= \cos\theta \pm i\sin\theta$$

$$m = e^{\pm i\theta}$$

Similarly

$$n = e^{\pm i\phi}$$

$$\Rightarrow m^p n^q + \frac{1}{m^p n^q} = e^{ip\theta} \cdot e^{iq\phi} + e^{-ip\theta} \cdot e^{-iq\phi}$$

$$= e^{i(p\theta + q\phi)} + e^{-i(p\theta + q\phi)}$$

$$= 2\cos(p\theta + q\phi)$$

Ans

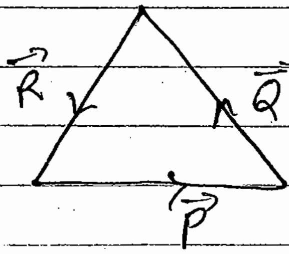
Test Question

Test-1

Q.1

Sol^n

$$\vec{P} + \vec{Q} + \vec{R} = 0$$



$\Rightarrow \vec{P} + \vec{Q} + \vec{R} = 0$

So
$$\left.\begin{aligned} |\vec{P}| + |\vec{Q}| &> |\vec{R}| \\ |\vec{Q}| + |\vec{R}| &> |\vec{P}| \\ |\vec{R}| + |\vec{P}| &> |\vec{Q}| \end{aligned}\right\}$$

So option (a) ~~$\left(1, \frac{3}{2}, \frac{1}{2}\right)$~~ $\left(2, \frac{1}{2}, 2\right)$ is correct.

Q.2

Sol^n

$$\nabla^2 f(r) = P f''(r) + \frac{m}{r^n} f'(r)$$

$$\nabla^2 f(r) = \vec{\nabla} \cdot \left(\vec{\nabla} f(r) \right)$$

$$= \vec{\nabla} \left[f'(r) \, \hat{r} \right] = \vec{\nabla} \cdot \left[f'(r) \frac{\vec{r}}{r} \right]$$

scalar

$$= \vec{\nabla}\left(\frac{f'(r)}{r} \right) \cdot \vec{r} + \frac{f'(r)}{r} \left(\vec{\nabla} \cdot \vec{r} \right)$$

$\because \vec{\nabla} \cdot \vec{r} = 3$

$$= \frac{r f''(r) - f(r) \cdot 1 (\hat{r} \cdot \vec{r})}{r^2} + \frac{3 f'(r)}{r}$$

$$= f''(r) - \frac{f'(r)}{r} + 3 \frac{f'(r)}{r}$$

$$= f''(r) + 2 \frac{f'(r)}{r}$$

$$f''(r) - \frac{2f'(r)}{r}$$

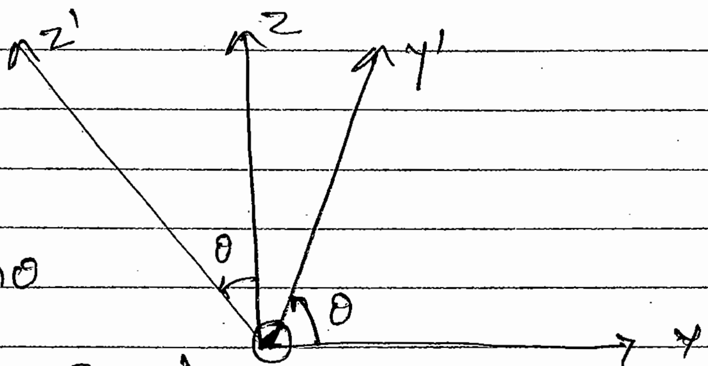
Comparing $p=1,\ m=2,\ n=1$

So $\boxed{m=n+p}$ Ans

Test

Q.4.

Sol^n



$y' = y\cos\theta + z\sin\theta$

$\Rightarrow z' = z\cos\theta - y\sin\theta$

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\text{So } R\left(\frac{\pi}{2}\right) = \begin{pmatrix} \cos\pi/2 & \sin\pi/2 \\ -\sin\pi/2 & \cos\pi/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$ for Clock wise.

$$\lambda = \pm i$$

So it is Traceless and its Eigenvalues of matrix will be purely Imaginary, Eigenvectors of the matrix are orthogonal.

but it is not singular.

because $i \times -i = +1$ not zero

So (d) is correct.

Q.6.

Sol^n

$$\vec{\nabla}[\vec{a}\cdot\vec{r}] = \vec{a}$$

$$\vec{\nabla} \times [\vec{a} \times \vec{r}] = 2\vec{a}$$

$$\Rightarrow \vec{\nabla}[\{(\vec{a}\times\vec{b})\times\vec{c}\}\cdot\vec{r}]$$

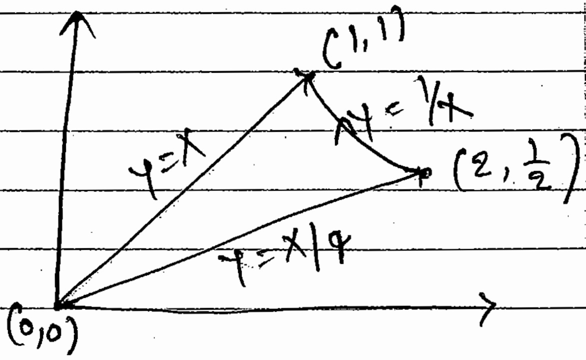
$\Rightarrow$ $\perp^{or}$ to c

and lie in plane of $\vec{a}$ & $\vec{b}$.

So option (b) and (d) is correct.

Q.12

Sol^n



$$A = \frac{1}{2}\oint (x\,dy - y\,dx)$$

non zero Case I $y = \frac{x}{4}$

$dy = \frac{dx}{4}$

Case II :- $y = \frac{1}{x} \rightarrow dy = -\frac{1}{x^2}\,dx$

Case III :-

$y = x \Rightarrow dy = dx$

$y = \frac{1}{x}$
$y = \frac{x}{4}$
$\frac{1}{x} = \frac{x}{4}$
$x = \pm 2$
$x = 2$ (in first quard.)
So $y = \frac{1}{2}$
$\because y = x,\ y = \frac{1}{x}$
$x = \frac{1}{x}$
$x^2 = 1$
$x = \pm 1$,
So $x = 1$
$y = 1$

Test -
Q. 11

$$\int_S \vec{F} \cdot d\vec{s} = 0$$

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{F}) \, dV = 0$$

$$\Rightarrow \int_V (2p + q - 3r) \, dV = 0$$

$$\Rightarrow (2p + q - 3r) \int_V dV = 0$$

$$\because \int_V dV \neq 0$$

$$\therefore \boxed{2p + q - 3r = 0} \quad \underline{Ans}$$

Q. 13

$$A = \begin{pmatrix} 0 & -p & q \\ p & 0 & -r \\ -q & r & 0 \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 + \lambda(p^2 + q^2 + r^2) = 0$$

$$\therefore a^3 = 64$$

$\Rightarrow \quad x^3 - 64 = 0$

$\Rightarrow \quad (x-4)(x^2 + 4x + 16) = 0$

So $\quad x = \underset{\substack{\downarrow \\ p}}{4},\ \underset{\substack{\downarrow \\ q}}{4\omega},\ \underset{\substack{\downarrow \\ r}}{4\omega^2}$

$$p^2 + q^2 + r^2 = 16 + 16\omega^2 + 16\omega^4$$
$$= 16(1 + \omega^2 + \omega)$$
$$= 0$$

So $\lambda = 0, 0, 0$ <u>Ans</u>

Q.17

A

$$\vec{A} = \frac{1}{r^2}(-y\hat{i} + x\hat{j}) = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$$

undefined at (0,0)

$$\vec{\nabla} \times \vec{A} = 0$$

$$\oint_C \vec{A} \cdot d\vec{l}$$

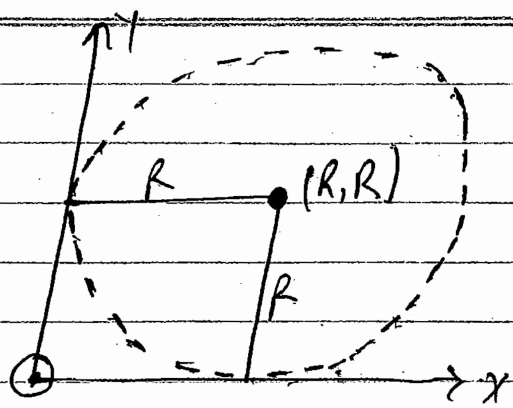
$$C: (x - x_0)^2 + (y - y_0)^2 = R^2$$

$$|x_0| > R$$
$$|y_0| > R$$

$\because \quad |x_0| = R \Rightarrow x_0 = -R, R$

$\quad\ \ |y_0| = R \Rightarrow y_0 = -R, R$

$\because \ |x_0| > R$
$|y_0| > R$



So circle shift outward but function is undefined at (0,0) which is outside of the circle so

$$\boxed{\oint \vec{A}\cdot d\vec{l} = 0}$$

So ans is (c).

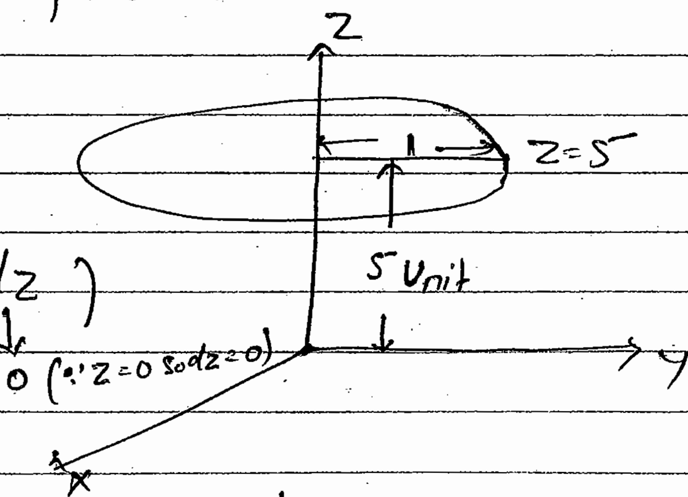
(16)

Sol^n

$$F = ay\hat{i} + z\hat{j} + x\hat{k}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \pi$$

$$C : x^2+y^2 = 1 \ , \ z = 5$$



$$\oint \vec{F}\cdot d\vec{r} = \pi$$

$$\Rightarrow \int (a y\,dx + z\,dy + x\,dz)$$

0 ($\because z = 0$ so $dz = 0$)

$$\int_{2\pi}^{0} a\sin\theta(-\sin\theta)d\theta + \underbrace{5\cos\theta\, d\theta}_{0}$$

$\because \sin 0 = 0$
$\sin 2\pi = 0$

$$a\int_0^{2\pi} (\sin^2\theta)d\theta = \pi$$

$$a\pi = \pi \qquad \text{so } \boxed{a = 1} \text{ Ans.}$$

Q.7

$n \times n$

$a_{ij} = m\,(\det(I_n)) = m$

$$\begin{pmatrix} m & m & m \\ m & m & m \\ m & m & m \end{pmatrix} \begin{pmatrix} m & m & m & \cdots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

$\lambda = 3m, 0, 0$

$\boxed{\lambda = mn, 0, 0}$ Ans

* MILNE-THOMSON METHOD :-

Case I :- $u(x,y)$ is given :-

Steps :-

1) $$\frac{\partial u}{\partial x} = \phi_1(x,y), \quad \frac{\partial u}{\partial y} = \phi_2(x,y)$$

2) Calculate $\phi_1(z,0)$, $\phi_2(z,0)$ by putting $x = z$, $y = 0$

3) $$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$$

Case II :- If Imaginary Part is given -

$v(x,y)$ is given

Step 1 :- $$\frac{\partial v}{\partial x} = \psi_1(x,y), \quad \frac{\partial v}{\partial y} = \psi_2(x,y)$$

Step 2 :- Calculate $\psi_1(z,0)$, $\psi_2(z,0)$

Step 3 :- $$f(z) = \int [\psi_2(z,0) + i\psi_1(z,0)]\,dz$$

A-5

Q.9(i) $v = e^x(x\cos y - y\sin y)$

Sol^n

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y) = \psi_1(x,y)$$

$$\frac{\partial v}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) = \psi_2(x,y)$$

$$\psi_1(z,0) = e^z . z + e^z$$

$$\psi_2(z,0) = 0$$

So $$f(z) = \int 0 + i\overbrace{(e^z z + e^z)}^{\frac{d}{dz}(e^z . z)}\,dz$$

$$= i[\underbrace{e^z . z - \cancel{e^z}}\, + \cancel{e^z}] + c$$

$$\boxed{f(z) = i e^z . z + c}$$

(ii) $u - v = (x-y)(x^2+4xy+y^2)$

Sol^n: $\because \quad f(z) = u + iv$

$$i f(z) = iu - v$$

$$(1+i) f(z) = (u-v) + i(u+v)$$

$$\Rightarrow \quad F(z) = U + iV$$

$$U = (u-v) = (x-y)(x^2+4xy+y^2)$$

$$\frac{\partial U}{\partial x} = (x-y)(2x+4y) + 1(x^2+4xy+y^2) = \phi_1(x,y)$$

$$\frac{\partial U}{\partial y} = (-1)(x^2+4xy+y^2)(x-y)(4x+2y) = \phi_2(x,y)$$

$$\phi_1(z,0) = z^2 + 2z^2 = 3z^2$$

$$\phi_2(z,0) = -z^2 + 4z^2 = 3z^2$$

So $\quad F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz$

$$= \int (3z^2 - i\,3z^2)\, dz$$

$$= \cancel{3}\left[\left(\frac{z^3}{\cancel{3}} - \frac{iz^3}{\cancel{3}}\right)\right]$$

$$= z^3 - iz^3$$

$$F(z) = z^3(1-i)$$

$$\because \quad f(z) = \frac{F(z)}{(1+i)}$$

$$\boxed{f(z) = \frac{z^3(1-i)}{(1+i)}}$$

* Power Series Expansion :-

If a complex function $f(z)$ is analytic at $z=a$ then we can do power series expansion ~~abou~~ of $f(z)$ about $z=a$.

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots$$

" The power series expansion of $f(z)$ about $z=a$ may converge within a disc of complex plane or in the intire complex plane or at a perticular point of the complex plane".

Ex- (1) Within a Disk:-

$$1) \quad \sum_{n=0}^{\infty} z^n = \underset{0th}{1} + \underset{1^{st}}{z} + \underset{2^{nd}}{z^2} + \cdots\cdots$$

for convergence and divergence of series :-

$\left|\frac{t_{n+1}}{t_n}\right| < 1$, Converges $\qquad t_{n+1} = (n+1)^{th}$ term of series

$\left|\frac{t_{n+1}}{t_n}\right| > 1$, then series diverges. $\qquad t_n = n^{th}$ term of series.

So
$$\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{z^{n+1}}{z^n}\right| = |z|$$

So series will converge if $|z|<1$

here $|z|<1$ What denote ?

Say $\quad C : |z-a| = r$

denotes circle whose center $(0,a)$ and radius r.

$$|x+iy-a| = r$$

$$\Rightarrow \cancel{\sqrt{(x+iy}}\;\cancel{/}$$

$$\Rightarrow \sqrt{(x-a)^2+y^2} = r$$

$$(x-a)^2 + y^2 = r^2$$

$$|z - ia| = r$$

$\downarrow$

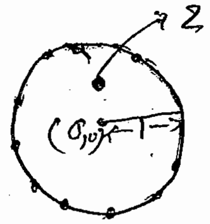
center $(0, a)$, radius r.

another example :-

$|z| = 1$

$\hookrightarrow$ circle $(0,0)$, $r=1$

$\hookrightarrow$ $x^2+y^2=1$



So $|z| < 1$ represents :-

$\hookrightarrow$ $x^2+y^2 < 1$

So Series will converge for the points within the circle $|z| = 1$.

Means when we put any value of z which lie inside the circle of radius 1 then this series gives a finite value. (means converges).

(2) Entire Complex Plane :-

$$Ex - \sum_{n=0}^{\infty} \frac{z^n}{n!} = \overset{0^{th}}{1} + \overset{1^{st}}{z} + \overset{2^{nd}}{\frac{z^2}{2!}} + \overset{3^{rd}}{\frac{z^3}{3!}} + \cdots$$

Sol^n

$$\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{z^{n+1}}{\cancel{(n+1)!}\,(n+1)}\ \frac{\cancel{n!}}{z^n}\right| = \frac{|z|}{(n+1)}$$

$n \to \infty$ (when no. of terms are infinite) $\Rightarrow \left|\frac{t_{n+1}}{t_n}\right| = 0 < 1$ (infinite series gives the sum of series is finite.)

So series will converge for all values (finite value) of z therefore the series will converge in the intire complex plane.

*③ At A perticular point of Complex Plane:-

Ex- $\sum_{n=0}^{\infty} z^n . n! = 1 + z + \underbrace{z^2 . 2! + z^3 . 3! + \cdots}_{\text{at } z=0 \text{ it is } =0}$ So sum = 1 (finite)

Sol^n $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{z^{n+1} . (n+1)!}{z^n . n!}\right| = (n+1)|z|$

$$n \to \infty \Rightarrow \left|\frac{t_{n+1}}{t_n}\right| \longrightarrow \infty$$

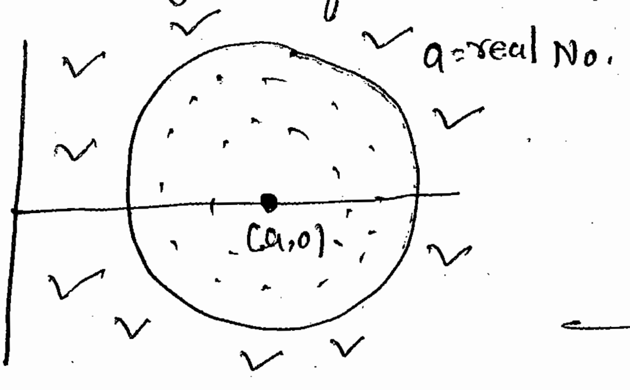
$z = 0$ this series will converge.

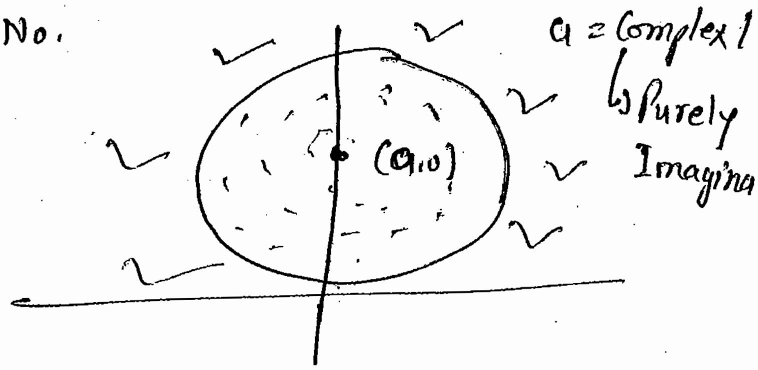
* Radius of Convergence of Power Series:-

Suppose a complex function $f(z)$ is analytic at $z = a$, and the corresponding power series expansion is -

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n}$$

Such that this power series converges for $|z-a| <$ And Diverges for $|z-a| > R$ where a = real no.





Then the R is k/a The Radius of Convergence of the power series.

The methametical formula for Calculating the Radius of Convergence of a power series.

$$\boxed{R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}$$ R = Radius of Convergence

1) $\sum_{n=0}^{\infty} z^n \Rightarrow$ $|z| < 1$ Converge

$|z| > 1$ Diverge

So $\boxed{R = 1}$

by the formula:-

$$R = \lim_{n \to \infty} \left| \frac{1}{1} \right| = 1$$

So $\boxed{R = 1}$

2) $\sum_{n=0}^{\infty} \frac{z^n}{n!} \rightarrow$ entire Complex plane

$\downarrow$

Circle of $R = \infty$

$\hookrightarrow$ Radius of Convergence.

So $\boxed{R = \infty}$

by the formula:-

$$R = \lim_{n \to \infty} \left(\frac{1}{n!} \frac{(n+1)!}{1} \right)$$

$$= \lim_{n \to \infty} \left((n+1) \right)$$

$$\boxed{R = \infty}$$

3) A point is nothing but circle of Radius 0

So $\boxed{R = 0}$

by formula :-

$$R = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \frac{1}{n+1} = \frac{1}{\infty} = 0$$

$$\boxed{R = 0}$$

Region of Convergence :-

$$\boxed{|z-a| < R}$$

<u>Ques</u> $f(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z-3i)^n$

Calculate Radius of Convergence and Region of Convergen of the power series.

<u>Sol^n</u> $f(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z-3i)^n$

$a = 3i$

$$R = \lim_{n\to\infty} \left| \frac{(2n)!}{(n!)^2} \cdot \frac{[(n+1)!]^2}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| \qquad \left\{ \because (2n+2)! = (2n+2)(2n+1).2n! \right.$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}{\cancel{n^2}\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} \right|$$

$$\boxed{R = \frac{1}{4}}$$ This is the radius of Convergence of the given Power series.

Region of Convergence :-

$$|z-a| < R$$

So $\boxed{|z-3i| < \frac{1}{4}}$ $\quad\left\{ \because a = 3i,\ R = \frac{1}{4} \right.$

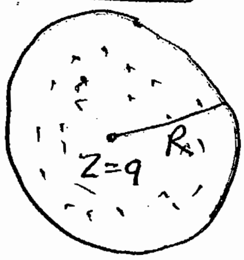
* Taylor Series Expansion :-

If a complex function $f(z)$ is analytic at all points inside and on the circle $C : |z-a| = R$ Then the Taylor Series expansion of $f(z)$ for all point inside the circle about $z=a$ will be :-

$$\boxed{f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots}$$

order of the derivative

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n}$$



A-4 CSIR Dec-2012

Q.42 The taylor series expansion of the function $\ln(\cosh x)$, where x is real, about point $x=0$ starts with the following terms.

(a) $-\frac{1}{2}x^2 + \frac{1}{12}x^4 + \cdots$ $\quad$ (b) $\frac{1}{2}x^2 - \frac{1}{12}x^4 + \cdots$

(c) $-\frac{1}{2}x^2 + \frac{1}{6}x^4 + \cdots$ $\quad$ (d) $\frac{1}{2}x^2 + \frac{1}{6}x^4 + \cdots$

Sol^n $\quad f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$

$$+ \frac{f''''(0)}{4!}(x-0)^4 + \cdots$$

$f'(x) = \frac{1}{\cosh x} \cdot \sinh x = \tanh x$

$f''(x) = \mathrm{sech}^2 x$

$f'''(x) = 2\,\mathrm{sech}\,x \cdot (-\mathrm{sech}\,x \cdot \tanh x) = -2\underbrace{\mathrm{sech}^2 x}_{II}\,\underbrace{\tanh x}_{I}$

$f''''(x) = -2\left[\tanh x \cdot 2\,\mathrm{sech}\,x \cdot (-\mathrm{sech}\,x \cdot \tanh x) + \mathrm{sech}^2 x \cdot \mathrm{sech}^2 x\right]$

$= \left[+\{4\tanh^2 x\, \mathrm{sech}^2 x \cancel{+} 2\,\mathrm{sech}^4 x\right]$

$= 4 \times 0 - 2 \times 1 = -2.$

$= \cancel{1 \times 1 + 2 = +2}$

So $f(x) = 0 + 0 \cdot x + \frac{1}{2!}x^2 + 0 \cdot x^3 + \frac{(-2)}{4!}x^4 + \cdots$

$= \frac{1}{2!}x^2 + \frac{2}{4!}x^4 = \frac{1}{2}x^2 - \frac{\cancel{2}}{4 \times 3 \times \cancel{2} \times 1}x^4$

$$\boxed{f(x) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \cdots}$$

A-5

Q.12 Find the first three terms of the Taylor's series expansion of $f(z) = \frac{z}{(z-3)(z-4)}$ about $z=2$. Find the region of convergence.

Sol^n

$f(z) = \frac{z}{(z-3)(z-4)}$

$\frac{z}{(z-3)(z-4)} = \frac{A}{(z-3)} + \frac{B}{(z-4)}$

$\frac{z}{\cancel{(z-3)(z-4)}} = \frac{A(z-4) + B(z-3)}{\cancel{(z-3)(z-4)}}$

$z = z(A+B) - 4A - 3B$

$A + B = 1$

$-4A - 3B = 0$

$\cancel{4A} + 4B = 4$

$-\cancel{4A} - 3B = 0$

$B = 4$

So $A = -3$

$$f(z) = \frac{-3}{(z-3)} + \frac{4}{(z-4)} \Rightarrow f(z=2) = \frac{-3}{2-3} + \frac{4}{2-4} = \frac{+3}{+1} + \frac{4^2}{-2} = \cancel{8}$$

$$=1 \qquad 3-2=1$$

$$\boxed{f(z) = f(2) + f'(2)(z-2) + \frac{f''(2)}{2!}(z-2)^2 + \cdots}$$

$$f'(z) = \frac{+3}{(z-3)^2} - \frac{4}{(z-4)^2} \Rightarrow f'(z=2) = \frac{3}{1} - \frac{4}{4} = 2$$

$$f''(z) = \frac{-6}{(z-3)^3} + \frac{8}{(z-4)^3} \Rightarrow f''(z=2) = \frac{5}{2}$$

$$f'''(z) = \frac{+18}{(z-3)^4} - \frac{24}{(z-4)^4} \Rightarrow f'''(z=2) = \frac{11}{2}$$

So $f(z) = \underset{a_0}{1} + \underset{a_1}{2(z-2)} + \underset{a_2}{\frac{5}{2}(z-2)^2} + \underset{a_3}{\frac{11}{4}(z-2)^3} + \cdots$

$$\underset{n=0}{\frac{a_0}{a_1} = \frac{1}{2}}, \quad \underset{n=1}{\frac{a_1}{a_2} = \frac{4}{5} = 0.8}, \quad \underset{n=2}{\frac{a_2}{a_3} = \frac{10}{11}} \approx 0.91 \cdots\cdots \frac{a_n}{a_{n+1}} \approx 1$$

So $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$$= \lim_{n\to\infty} |1| = 1$$

So $\boxed{R=1}$ So Radius of Convergence $R \approx 1$

So Region of Convergence:-

$$|z-a| < R$$

So $\boxed{|z-2|<1}$ Ans

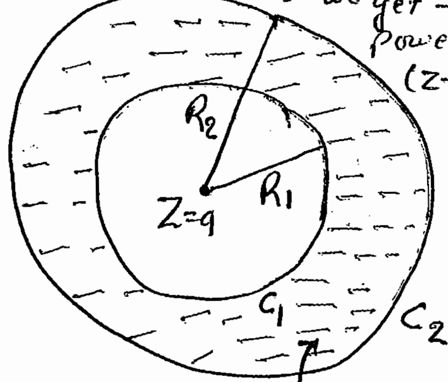
* LAURENT SERIES :-

If a Complex function is analytic at all points on two concentric circle $C_1 : |z-a| = R_1$ and $C_2 : |z-a| = R_2$. And it is also analytic at the points in the ~~angula~~ annular region between the circle. Then the Laurent Series expansion of $f(z)$ at $z=a$ for all point ~~b/w anpu~~ in the annular region b/w C_1 and C_2 is given by -

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}}$$

→ Principal Part.

At $z=a$, $f(z)$ is not analytic so we get -ve powers of $(z-a)$



annular Region.

CSIR June - 12

A-4

Q.40 The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $\underbrace{1 \le |z| \le 2}_{\text{Analytic}}$, and around $\underbrace{z=1}_{\text{not analytic}}$, is ?

Sol^n:- Assume $z-1 = t$

So $f(z) = \frac{1}{t(t-1)} = \frac{1}{t} \frac{1}{(-1)(1-t)} = -\frac{1}{t} (1-t)^{-1}$

$f(z) = -\frac{1}{t}(1-t)^{-1}$

$\because (1-x)^{-1} = 1 + x + x^2 + \cdots$ When $|x| \le 1$

$f(z) = -\frac{1}{t}(1 + t + t^2 + \cdots)$

$= -\frac{1}{t} - 1 - t - t^2 - \cdots$

So $\boxed{f(z) = \underbrace{\frac{-1}{(z-1)}}_{\text{Negative Power of } (z-1)} \underbrace{-1-(z-1)-(z-1)^2}_{\text{Positive Power of } (z-1)}}$

So $\boxed{f(z) = \frac{1}{(1-z)} - z - (1-z)^2 \; ---}$ Ans

A-5

Q.15 Expand the following function in Laurent series.

(i) $f(z) = \frac{1}{z(z+2)^3}$ about the pole $z=-2$

(ii) $f(z) = \frac{1}{z^2-3z+2}$ in the region (i) $|z|<1$ (ii) $|z|>3$

Solⁿ (1) Say $z+2=t$

$\Rightarrow f(z) = \frac{1}{t^3(t-2)}$

$\because z=-2$

$-2+2=t \Rightarrow t=0$ So 2 is larger

$\Rightarrow f(z) = \frac{1}{(-2)t^3(1-\frac{t}{2})} = \frac{-1}{2t^3}\left(1-\frac{t}{2}\right)^{-1}$

$\Rightarrow \because t < 2$

$\therefore \frac{t}{2} < 1$

So $f(z) = \frac{-1}{2t^3}\left[1+\frac{t}{2}+\frac{t^2}{4}+\frac{t^3}{8}+\cdots\right]$

$= \frac{-1}{2t^3} - \frac{1}{4t^2} - \frac{1}{8t} - \frac{1}{16} - \cdots$

$$\boxed{f(z) = \frac{-1}{2(z+2)^3} - \frac{1}{4(z+2)^2} - \frac{1}{8(z+2)} - \frac{1}{16} - \cdots}$$

A-4

Q.24 The Laurent series expansion of the complex function

$$f(z) = \frac{z}{(z-1)(z-2)}$$

in the region $|z-1| > 1$ can be expressed as ?

Soln:

$$f(z) = \frac{z}{(z-1)(z-2)}$$

$$\cancel{|z-1|} \quad |z-1| > 1$$

$$z-1 = t$$

$$|t| > 1$$

$$\cancel{\frac{z}{(z-1)(z-2)}} = \frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{A(z-2)+B(z-1)}{\cancel{(z-1)(z-2)}}$$

$$\Rightarrow A(z-2) + B(z-1) = z$$

$$A = -1 \quad , \quad B = 2$$

$$f(z) = -\frac{1}{z-1} + \frac{2}{(z-2)} = \frac{-1}{t} + \frac{2}{t(1-\frac{1}{t})}$$

$$= -\frac{1}{t} + \frac{2}{t}\left(1-\frac{1}{t}\right)^{-1}$$

$$= -\frac{1}{t} + \frac{2}{t}\left[1 + \frac{1}{t} + \frac{1}{t^2} + \cdots\right]$$

$$= -\frac{1}{t} + \frac{2}{t} + \frac{2}{t^2} + \frac{2}{t^3} + \cdots$$

$$= \frac{1}{t} + \frac{2}{t^2} + \frac{2}{t^3} + \cdots$$

$$\boxed{f(z) = \frac{1}{(z-1)} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \cdots}$$

this is expansion about $z=1$ at $z=1$ then funcn is not analytic

Q.22 Suppose a complex function $f(z)$ such that $f(1) = 1$, $f'(1) = 1$ $f''(1) = 1$ and all other higher derivatives of $f(z)$ is 0. at $z = 1$. The value of $f(z)$ at $z = \frac{1}{2}$ will be - ?

Sol^n Here function is analytic about $z = 1$

So by Taylor Series expansion -

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots$$

$$= 1 + 1(z-1) + \frac{1}{2!}(z-1)^2 + 0 + 0 + 0 \cdots$$

$$\Rightarrow \boxed{f(z) = z + \frac{1}{2}(z-1)^2}$$

at $z = \frac{1}{2}$

So $f\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - 1\right)^2 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{4}\right)$

$= \frac{5}{8}$ Ans

* Singular Points of a Complex function :-

⇒ **Singular Point :-** "These are the points at which function is not analytic."

Singular Points

- Isolated
- Non-Isolated

(1) Isolated Singular Point :-

If a complex function $f(z)$ is not analytic at $z=a$ and there is no other singular point in the infinitesimaly small neighbourhood about $z=a$. Then $z=a$ is a isolate singular points of the complex function $f(z)$.

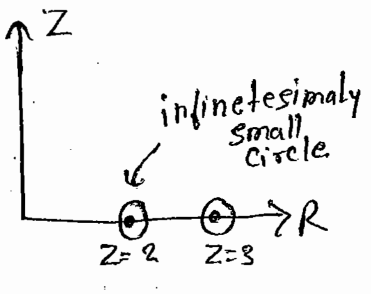
for Ex :-

$$f(z) = \frac{1}{(z-3)(z-2)}$$

C.O.S. (Condition of Singularty) = $(z-3)(z-2) = 0$

$\Rightarrow z = 3, \quad z = 2$

$(3,0) \quad (2,0)$



$z = 3$ and $z = 2$ are two isolated singular points

In the case of isolated singular points the condition of singularity is satisfied by a finite number of points.

2. Non-Isolated Singular Points:-

If the complex function $f(z)$ is not analytic at $z=a$ and there are other ~~singular~~ singular points in the infinetisimally small neaghbourhood about $z=a$ then $z=a$ is a non-isolated singular point of the complex function $f(z)$.

Ex– $$f(z) = \frac{1}{\sin\frac{\pi}{z}}$$

Condition of singularity:-

$$\sin\frac{\pi}{z} = 0 \Rightarrow \frac{\cancel{\pi}}{z} = n\cancel{\pi}$$

$$\Rightarrow z = \frac{1}{n} \quad (n=0, \pm1, \pm2, \dots)$$

When:-

$n=0 \Rightarrow z=\infty$

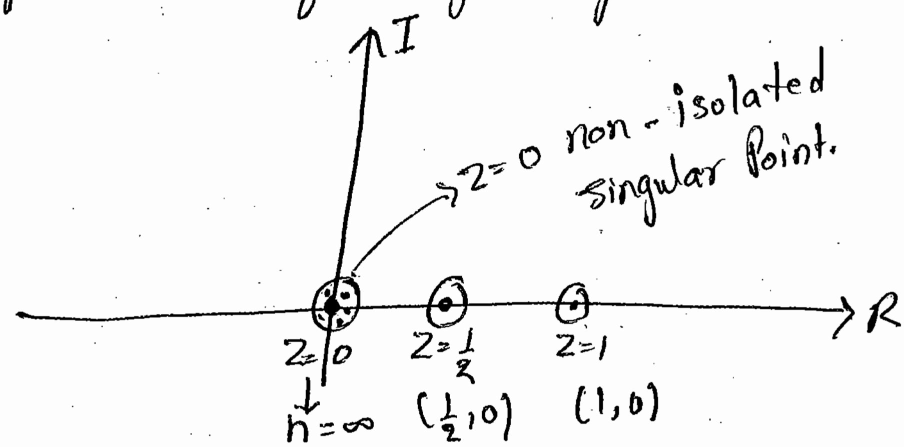
$n=\pm1 \Rightarrow z=\pm1$

$n=\pm2 \Rightarrow z=\pm\frac{1}{2}$

$\vdots$

$n=\infty \Rightarrow z=0$

So there is infinite no. of singular points.



But $$\left.\begin{aligned} n=\infty, &\quad z=0 \\ n=\infty-1, &\quad z=0 \\ n=\infty-2, &\quad z\approx0 \end{aligned}\right\}$$

So $z=0$ will be a non isolated points.

In case of non-isolated singular points the condition of singularity is satisfies by infinite many points.

The point corresponding to $n = \infty$ will be a non isolated singular points. And all the other points are isolated singular points.

Ex- $f(z) = \dfrac{1}{\sin \pi z}$

C.O.S. : $\sin \pi z = 0$

$\Rightarrow \cancel{\pi} z = n \cancel{\pi}$

$\Rightarrow z = n \quad (n = 0, \pm 1, \pm 2, \cdots)$

$n = \infty \Rightarrow z = \infty \Rightarrow$ N.I.S.P.

Isolated Singular Points

- Essential
- Non-Essential or Pole
- Removable

Suppose a complex function $f(z)$ has a isolated singular point at $z = a$ then the Laurent Series expansion of $f(z)$ about $z = a$ is -

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{\text{Principal Part of the expansion}},$$

1. Isolated Essential Singular Points:-

If the principal part of Laurent Series Expansion contains infinite no. of negative power term of $(z-a)$ then $z=a$ is a Isolated Essential Singular Point of $f(z)$.

e.g:- $f(z) = e^{\frac{1}{z-4}}$

Sol^n C.O.S.: $z-4=0 \Rightarrow z=4$

So $z=4$ is isolated singular point

So $f(z) = 1 + \frac{1}{z-4} + \frac{1}{(z-4)^2 \cdot 2!} + \cdots$

$\because$ Here $(z=4)$ have infinite no. of terms of ~~$(z-4)$~~ negative power of $(z-4)$.

So it is a Isolated Essential Singular point.

2. Isolated Non-Essential Singular Points or Isolated Poles:-

If the principal part of the Laurent Series expansion contain finite no. of negative power terms of $(z-a)$ then $z=a$ is a non-essential singular point or Pole of $f(z)$.

e.g. :- $f(z) = \frac{1}{z(z+2)^3}$

Sol^n C.O.S.: $z(z+2)^3 = 0$

$\Rightarrow z = 0, -2$

So $z=0$ and -2 are two isolated singular points.

$z=-2$

$$f(z) = \underbrace{\frac{1}{2(z+2)^3} - \frac{1}{4(z+2)^2} - \frac{1}{8(z+2)}}_{\text{negative power}} \underbrace{- \frac{1}{16} - \cdots}_{\text{Positive Powers}}$$

So $z=-2$ is a Non Essential Singular Point.

Order of Pole :-

$z=-2$ is a pole of order 3.

{because here heighest negative power of z is 3.}

"The heighest negative power present in the Laurent Series Expansion will be order of the pole."

A-4

Q.40 $\quad f(z) = \dfrac{1}{(z-1)(z-2)}$

option (D) is correct because here heighest negative power of z is 1.

(3) Isolated Removable Singular Points :-

If the principal part of Laurent series expansion of $f(z)$ contains no negative power term of $(z-a)$ then $z=a$ is a removable singular point of $f(z)$.

e.g :- $f(z) = \dfrac{\sin z}{z}$

C.O.S. ! $z=0$

$$f(z) = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

So here there is no negative power of z
So $z=0$ is a removable singular point of $f(z)$.

$$\lim_{z\to 0} \frac{\sin z}{z} = 1 \quad \text{(finite)}$$

A.5

Q.16 Discuss the nature of singularity regarding the following complex functions.

(iii) $f(z) = \dfrac{\cot \pi z}{(z-a)^2}$ at $z=a, z=\infty$

Sol^n

$$f(z) = \frac{\cos \pi z}{\sin \pi z (z-a)^2}$$

C.O.S. ! $\sin \pi z (z-a)^2 = 0$

either $\sin \pi z = 0$ or $(z-a)^2 = 0$

$\cancel{\pi} z = n\cancel{\pi}$ $\qquad$ $\boxed{z=a}$ Isolated singular point.

$\boxed{z=n}$ $\qquad$ $\Downarrow$

$z = \infty \Rightarrow n = \infty$ $\qquad$ Pole of order 2 $\{\because (z-a)^2$ is Polynomial$\}$

So it is a Non-Essential Isolated singular point. $\qquad$ or Non Essential Singular Point

So $z=a \longrightarrow$ I. Pole of order 2
$z=\infty \longrightarrow$ N.I.S.P.

(iv) $f(z) = \sin\left(\dfrac{1}{1-z}\right)$

Sol^n

C.O.S. ! $(1-z) = 0$
$z = 1$ Isolated singular point

$\because \sin\left(\frac{1}{1-z}\right) = \left[\frac{1}{1-z} - \frac{1}{(1-z)^3 \cdot 3!} + \cdots\right]$

So they have infinite no. of negative power term.

So $z=1$ is a isolated Essential singular point.

(vii) $f(z) = \dfrac{\sin z}{(z-\pi)^2}$

Sol^n

C.O.S.: $(z-\pi)^2 = 0$

$z-\pi = 0$

$\boxed{z_0 = \pi}$ Isolated singular point

By Laurent Series Expansion:

put $z-\pi = t$

$z = t+\pi$

So $f(z) = \dfrac{-\sin t}{t^2}$

$= -\frac{1}{t^2}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right]$

$= -\frac{1}{t} + \frac{t}{3!} - \frac{t^3}{5!} + \cdots$

$f(z) = -\dfrac{1}{(z-\pi)} + \dfrac{(z-\pi)}{3!} - \dfrac{(z-\pi)^3}{5!} + \cdots$

Here heighest negative power $= 1$

So

$z = \pi$ is Isolated pole of order 1.

{ this is an exceptional case }
because after seing $f(z)$ we
Can say it is of order 2 but it
is not true.

(vi) $f(z) = \dfrac{z}{1+z^4}$

Note: If $z^4 = +1 = e^{i2n\pi}$
$z^4 = -1 = e^{i(2n+1)\pi}$

Sol^n C.O.S. : $z^4+1=0$

$\Rightarrow z^4 = -1 = e^{i(2n+1)\pi}$

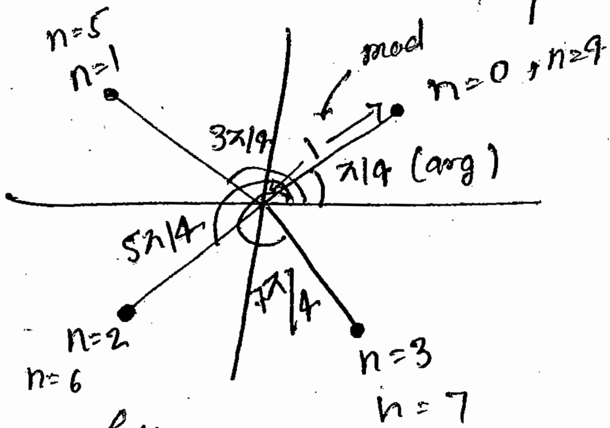
$\Rightarrow \boxed{z = e^{i(2n+1)\pi/4}}$ $n=0,1,2,3$ {becoz order is 4}

$z = e^{i\pi/4}$

$z = e^{i3\pi/4}$

$z = e^{i5\pi/4}$

$z = e^{i7\pi/4}$



$z = re^{i\theta}$

$(z^4+1) = (z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})$

follow the solution so if we take $n=4,5,6,7$ then ans. is same.
or if we take $n=0,\pm1,2$ give the same ans.

$\Rightarrow$ So it is isolated singular point of order 1.