* Residue At a Singular Point:

Residue of a complex function $f(z)$ ~~at~~ of a singular point is the measurement of singularity at that perticular point.

Residue = Amount of singularity or Amount of Non-~~Analytiti~~ Analyticity

Case 1:- Residue At a simple Pole (Pole of order 1):-

Method I:-

$$\boxed{\operatorname{Res} f(z=a) = \lim_{z\to a} (z-a) f(z)}$$

Method II:-

If $f(z) = \dfrac{\phi_1(z)}{\phi_2(z)}$

$$\boxed{\operatorname{Res.} f(z=a) = \left.\frac{\phi_1(z)}{\phi_2'(z)}\right|_{z=a}}$$

A-5

Q.17 Determine the poles and corresponding residues of following complex functions.

(i) $f(z) = \dfrac{1-2z}{z(z-1)(z-2)}$

Sol^n

$z=0$, $z=1$, $z=2$ isolated pole of order 1

$$\operatorname{Res.} f(z=0) = \lim_{z\to 0} (\cancel{z}-0) \frac{1-2z}{\cancel{z}(z-1)(z-2)}$$

$$= \frac{1}{2} \quad \text{Ans}$$

$$\operatorname{Res} f(z=1) = \lim_{z\to 1} (\cancel{z-1}) \frac{1-2z}{z(\cancel{z-1})(z-2)}$$

$$= 1. \quad \text{Ans}$$

$$\text{Res} f(z=2) = \lim_{z\to 2} \cancel{(z-2)} \frac{1-2z}{z(z-1)\cancel{(z-2)}}$$

$$= -\frac{3}{2} \quad \underline{\text{Ans}}$$

Case II :- Residue at a pole of order 'n' :-

General formula :-
$$\boxed{\text{Res } f(z=a) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z-a)^n f(z)\right\}\right]}$$

A-5

Q. 17(iv) Determine the poles and corresponding ~~redid~~ residues of following complex functions.

(iv) $f(z) = \dfrac{z^2}{(z+1)^2(z-2)}$

Sol^n

$z = -1 \Rightarrow$ Pole (2)

$z = 2 \Rightarrow$ Pole (1)

$$\cancel{\text{Res} f(z=2) = (z-2)\frac{z^2}{(z+1)^2(z-2)} = \frac{(2)^2}{}}$$

$$\text{Res} f(z=-1) = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{\cancel{(z+1)^2}\cdot\frac{z^2}{\cancel{(z+1)^2}(z-2)}\right\}\right]_{z=-1}$$

$$= \frac{d}{dz}\left(\frac{z^2}{z-2}\right)\Bigg|_{z=-1}$$

$$= \frac{(z-2)\cdot 2z - z^2\cdot 1}{(z-2)^2}\Bigg|_{z=-1} = \frac{(-1-2)(-2)-1}{(-1-2)^2}$$

$$= \frac{-3\times -2-1}{(-3)^2} = \frac{6-1}{9} = \frac{5}{9}$$

$$\text{Res } f(z=2) = \lim_{z\to 2} \cancel{(z-2)} \frac{z^2}{(z+1)^2\cancel{(z-2)}}$$

$$= \frac{(2)^2}{(2+1)^2}$$

$$= \frac{4}{9}$$ Ans

Case III :- Residue at an essential Singular Point :-

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$\boxed{\begin{aligned} \text{Res. } f(z=a) &= b_1 \\ &= \text{Coefficient of } \frac{1}{(z-a)} \text{ in L.S.E. \{Laurent Series expension.\}} \end{aligned}}$$

Q.17(viii) $f(z) = z^2 e^{1/z}$

Soln :-

$z = 0$

$$= z^2\left[1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z^3 \cdot 3!} + \cdots\right.$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{z}\frac{1}{3!} + \cdots$$

L.S.E.

So $\boxed{\text{Res } f(z=0) = \frac{1}{3!}}$ Ans

Residue at removable singular point is zero.

Q.27 The function $f(z) = (z-3)^n \sin\left(\frac{1}{z-3}\right)$ has a residue of $\frac{1}{120}$ at the point $z=3$. The value of n is equal to?

Sol^n

$$f(z) = (z-3)^n \sin\left(\frac{1}{z-3}\right)$$

$$= (z-3)^n \left[\frac{1}{(z-3)} - \frac{1}{(z-3)^3 \cdot 3!} + \frac{1}{(z-3)^5 \cdot 5!} - \cdots\right]$$

$$= (z-3)^{n-1} - \frac{(z-3)^{n-3}}{3!} + \frac{(z-3)^{n-5}}{5!} - \cdots = b_1$$

$\because$ Residue $= \frac{1}{120}$

But here in above expansion $\frac{1}{5!} = b_1$ becoz $5! = 120$

So $(z-3)^{n-5}$ is the first negative term in the above expansion

$$n-5 = -1$$

$$\Rightarrow \boxed{n = 4}$$

$\frac{1}{z-3} = (z-3)^{-1} = (z-3)^{n-5}$

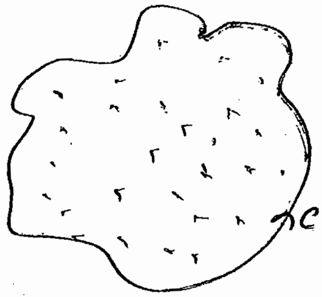
$n-5 = -1$

$n = 4$

* CAUCHY INTEGRAL THEOREM:-

If a complex function $f(z)$ is analytic at all points inside and on a simple closed curve C then -

$$\boxed{\oint_C f(z)\,dz = 0}$$



Simple Curve = Non Intersecting Curve.

* CAUCHY RESIDUE THEOREM:-

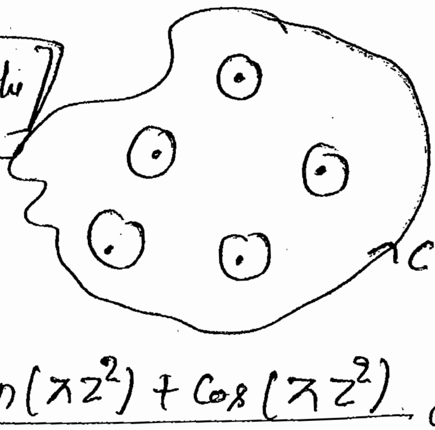
If a complex function $f(z)$ have some singular points inside a simple closed curve C. Then -

for Anticlockwise OR Counterclockwise OR Positive Direction Curve.

$$\boxed{\oint_C f(z)\,dz = 2\pi i\,[\text{Sum of the Residues at the S.P. within } C]}$$

$$\boxed{\oint f(z)\,dz = -2\pi i\,[\text{Sum of the residues at the S.P. within } C]}$$

↳ for Clockwise or negative curve.

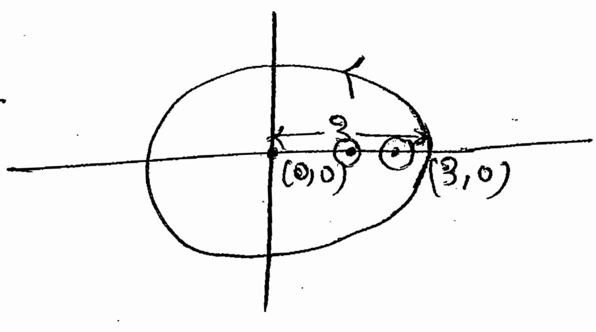


A-4

Q.31 The value of the integral $\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)\,(z-2)}\,dz$ Where 'C' is the circle $|z|=3$ traversed in in the anticlockwise direction, will be:- ?

Sol^n

$$f(z) = \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz$$



C.O.S ! => $(z-1)(z-2) = 0$

S.P : $z = 1$, $z = 2$

$\downarrow$ (1,0) $\quad$ $\downarrow$ (2,0)

At $z=1$

$$\text{Res } f(z=1) = \lim_{z\to 1} (z-1) \times \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}$$

$$= \frac{\sin \pi + \cos \pi}{-1} = \frac{-1}{-1} = 1$$

Residue at $z = 2$

$$\text{Res } f(z=2) = \lim_{z\to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1} = \frac{1}{1}$$

$$= 1$$

$$\oint_c f(z)\,dz = 2\pi i\,[1+1]$$

$$= 4\pi i \quad \text{Ans}$$

A-4

Q.40 If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral $\oint_c \frac{dz}{\sin^2 z}$ is ?

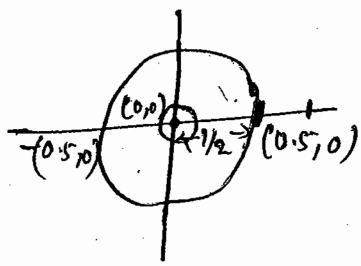
(a) ∞ (b) $2\pi i$ (c) 0 (d) πi

Sol^n

C.O.S. $\Rightarrow \sin^2 z = 0$

$\Rightarrow \sin z = 0$

$\Rightarrow z = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$



$\Rightarrow z = 0, \pm\pi, \pm 2\pi, \ldots$

$\Rightarrow z = 0 \Rightarrow (0,0)$ ✓

$\Rightarrow z = \pi \Rightarrow (\pi, 0)$

So only $z=0$ contributes, other points are outside the circle.

$n \to \infty$

$z = \infty$ so it is a non isolated singular points.
and others are Isolated singular points.
" $z=0$ is isolated pole of order 2."

$$\operatorname{Res} f(z=0) = \frac{1}{(2-1)!} \frac{d}{dz} \left\{ z^2 \cdot \frac{1}{\sin^2 z} \right\} \Big|_{z=0}$$

$$= \frac{d}{dz} \left\{ \frac{z^2}{\sin^2 z} \right\} \Big|_{z=0}$$

$$= \frac{\sin^2 z \cdot 2z - z^2 (2 \sin z \cos z)}{\sin^4 z} \Big|_{z=0}$$

By L'Hospital Rule :-

$=$

Alternative Method :-

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]^2}$$

$$= \frac{1}{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]^2}$$

$$= \frac{1}{z^2}\left[1 - \left\{\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right\}\right]^{-2}$$

$\because$ there is no negative power of z so Residue $= 0$

$\therefore$ Residue $= \text{Co.eff}\left(\frac{1}{z}\right) = 0$

So $\boxed{\oint \frac{dz}{\sin z} = 0}$ Ans

A-4
Q.18 If the real part of a complex analytic function $f(z)$ is given as $u(x,y) = e^{-2xy}\sin(x^2-y^2)$, then $f(z)$ can be written as -

Sol^n $u(x,y) = e^{-2xy}\sin(x^2-y^2)$

$$\frac{\partial u}{\partial x} = e^{-2xy}(-2y)\sin(x^2-y^2) + e^{-2xy}\cos(x^2-y^2)(2x) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^{-2xy}(-2x)\sin(x^2-y^2) + e^{-2xy}\cos(x^2-y^2)(-2y) = \phi_2(x,y)$$

$\phi_1(z,0) = \cos z^2 . 2z$

$\phi_2(z,0) = -2z\sin z^2$

$f(z) = \int [2z(\cos z^2 - 2iz\sin z^2)]dx$

$= \int 2z e^{iz x} dx$

$f(z) = \int [\phi_1 - i\phi_2] dx$

Q.28 The contribution of the point $z=\frac{\pi}{2}$ in evaluation of $\oint \frac{\tan z}{z}\,dz$ (where C is a circle $|z|=2$) is equal to ?

(a) 0 (b) $-4e^{i\pi/2}$ (c) $4e^{i\pi/2}$ (d) $-2/\pi$

Sol^n

$$\oint_c \frac{\tan z}{z}\,dz$$

$$f(z) = \frac{\sin z}{z\cos z}$$

$$\cos z = 0 , \quad z=0$$

$$\Rightarrow z = (2n+1)\frac{\pi}{2}$$

$$n=0 ,\ z=\frac{\pi}{2} , \quad \underline{n=0,\ z=\frac{\pi}{2}}$$

$$\text{Res } f(z=\pi/2) = \frac{\sin z}{\cos z - z\sin z}$$

$$= \frac{1}{0-\frac{\pi}{2}\cdot 1}$$

$$= -\frac{2}{\pi} \text{ Ans}$$

(23) In the Laurent series expansion of $f(z)=\frac{(z^2+3z+2)}{z(z+2)^3}$ about $z=-2$, the highest negative power (<0) terms of $(z+2)$ present in the expansion.

Sol^n

$$f(z) = \frac{(z^2+3z+2)}{z(z+2)^3}$$

$$= \frac{(z^2+z+2z+2)}{z(z+2)^3}$$

$$= \frac{(\cancel{z+2})(z+1)}{z(z+2)^{\cancel{3}}}$$

$$f(z) = \frac{(z+1)}{z(z+2)^2}$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1} \frac{b_n}{(z-a)^n}$$

say $(z+2)=t$, $(z+1)=(t-1)$
$z=t-2$, $f(z)=\frac{(t-1)}{t^2(t-2)}$

$\because -2+2=t \Rightarrow t=0$
So 2 is larger

$$f(z) = \frac{(t-1)}{(-2)t^2(1-\frac{t}{2})}$$

$$= -\frac{(t-1)}{2t^2}\left(1-\frac{t}{2}\right)^{-1}$$

$\because t<2 \therefore \frac{t}{2}<1$

$$\text{So } f(z) = \frac{-(t-1)}{2t^2}\left\{1+\frac{t}{2}+\frac{t^2}{4}+\frac{t^3}{8}\right.$$

$$= -\frac{(t-1)}{2t^2} - \frac{(t-1)}{2t\cdot 2} - \frac{(t-1)}{4\cdot 2} - \cdots$$

$$f(z) = -\frac{(z+1)}{2(z+2)^2} - \frac{(z+1)}{2(z+2)\cdot 2} - \frac{(z+1)}{4\cdot 2}$$

So highest negative power of $(z+2)$ is 2.

Q.26

$$\frac{1}{(1+e^z)^1}$$

$$1-e^z=0$$

$$\Rightarrow e^z=1=e^{2\pi n i}$$

$$z=2n\pi i$$

So it is isolated pole of order 1.

Q.15 (ii) $f(z)=\dfrac{1}{z^2-3z+2}$ in the region (i) $|z|<1$ (ii) $|z|>3$

↳ 2 Common

(i) $$f(z)=\frac{1}{z^2-3z+2}$$

$$=\frac{1}{(z-2)(z-1)}=\frac{A}{(z-2)}+\frac{B}{(z-1)}$$

$$=\frac{A}{(-2)(1-\frac{z}{2}]}+\frac{B}{(-1)(1-z)}$$

$$=-\frac{A}{2}\left(1-\frac{z}{2}\right)^{-1}+B(1-z)^{-1}$$

(16) (iii) $f(z) = \tan\left(\frac{1}{z}\right)$ at $z=0$

$$f(z) = \tan\left(\frac{1}{z}\right) = \frac{\sin\left(\frac{1}{z}\right)}{\cos\left(\frac{1}{z}\right)}$$

$$\cos\left(\frac{1}{z}\right) = 0$$

$$\Rightarrow \frac{1}{z} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow z = \frac{2}{(2n+1)\pi} \qquad n = (0, \pm 1)$$

$$z = 0 \Rightarrow n = \infty$$

$$\Downarrow$$

So $z=0$ corresponds to non isolated singular point.

17 (vi) $f(z) = \dfrac{e^z}{z^2(z^2+9)}$ at $(\overset{x}{0}, \overset{y}{-3})$

$$f(z) = \frac{e^z}{z^2(z^2+9)}$$

$$= z^2(z^2+9) = 0$$

$\Rightarrow z=0 \Rightarrow$ Pole of order 2.

$z = +3i$

$\boxed{z = -3i.}$

$(\overset{x}{0}, \overset{y}{-3})$

So $z = -3i$

Q.17(V) $f(z) = \frac{z}{\sin z}$

$\sin z = 0$

$\Rightarrow z = n\pi \; (n = 0, \pm 1, \pm 2, \cdots)$

$\text{Res}\, f(z = n\pi) = \left. \frac{z}{\cos z} \right|_{z = n\pi}$

$= \frac{n\pi}{\cos n\pi}$

$\boxed{(\text{Res}) = (-1)^n n\pi}$

Here $n = \infty$ is non isolated singular point and all the point except $n = \infty$ are simple isolated point of pole d.

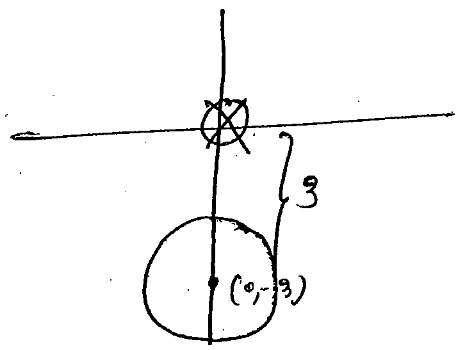
(10) (ii) $\oint_C \frac{dz}{z(z+\pi i)}$; C : Circle of $|z + 3i| = 1$

$f(z) = \frac{1}{z(z+\pi i)}$

S.P. : $z = 0$, $z = -\pi i$

$(0, 0)$ $\times$ $\quad (0, -\pi)$

$(0, -3.14)$



$|z + 3i| = 1$

$(0, -3)$

$I = 2\pi i \left[\text{Res}\, f(z) = -\pi i \right]$

So it is pole of order one.

(18) (V) $\oint \frac{1}{\sinh z} dz$; C : Circle of $|z| = 4$

$f(z) = \frac{1}{\sinh z}$

$\sinh z = 0$

$\Rightarrow -\dfrac{\sin iz}{i} = 0$

Note :- $\left\{ \begin{array}{l} \cosh z = \cos iz \\ \sinh z = i \sin iz \end{array} \right\}$

$\Rightarrow \sin iz = 0$

$\Rightarrow iz = n\pi$

$\Rightarrow z = \dfrac{n\pi}{i} = -n\pi i$

$z = 0$ ✓

$= \pi i$ ✓

$= -\pi i$ ✓ $(0, -\pi)$

10(vii) $f(z) = \oint z^4 e^{1/z} dz$; C; Circle of $|z| = 1$

S.P. $\Rightarrow z = 0$

$f(z) = z^4 \left(1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z^3 \cdot 3!} + \frac{1}{z^4 \cdot 4!} + \frac{1}{z^5 \cdot 5!} \cdots \right)$

So after multiplication first negative term is the residue and here it is $\frac{1}{5!}$

So Res $= \frac{1}{5!}$

So $2\pi i \left(\frac{1}{120} \right) = \frac{\pi i}{60}$ Ans

* Evaluation of $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \Rightarrow$ Here we want to convert above integral in z in place of θ. for this transformation we consider a contour C.

$$C : |z| = 1$$

Here C is a circle center at (0,0) and radius = 1
Here in place of 1 we chose any digit but for mathematical simplicity we consider unit radius.

Now -

$$z = r e^{i\theta}$$

$$\Rightarrow z = e^{i\theta}$$

$$\Rightarrow dz = i e^{i\theta} d\theta$$

$$\Rightarrow dz = i z\, d\theta$$

$$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)}$$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)}$$

Q. 19 (iv) $\int_0^{2\pi} \frac{\sin^2\theta - 2\cos\theta}{2+\cos\theta}\, d\theta$

$f(z) = \int_0^{2\pi} \frac{\sin^2\theta - 2\cos\theta}{2+\cos\theta}\, d\theta$ → we have to convert it in linear form means every part have power 1

$\because 2\sin^2\theta = 1 - \cos 2\theta$

So

$$= \int_0^{2\pi} \frac{\frac{1-\cos 2\theta}{2} - 2\cos\theta}{2+\cos\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\theta - 4\cos\theta}{2+\cos\theta}\, d\theta$$

$(e^{i\theta})^2$

$$= \frac{1}{2}\text{ R.P. of } \int_0^{2\pi} \frac{1 - e^{i2\theta} - 4e^{i\theta}}{2+\cos\theta}\, d\theta$$

$$= \frac{1}{2}\text{ R.P. of } \oint_C \frac{1-z^2-4z}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$ $\because \oint_c f(z)dz = 2\pi i$ (Sum of residu

$$= \frac{1}{\cancel{2}}\text{ R.P. of } \oint_C \frac{(1-z^2-4z)\cancel{2}}{(z^2+4z+1)} \frac{1}{i} dz$$

$$= \text{R.P. of}\left[\frac{1}{i}\oint_C \frac{1-z^2-4z}{z^2+4z+1} dz\right]$$

C.S. $= z^2+4z+1 = 0$

$$z = \frac{-4 \pm \sqrt{16-4\times1\times1}}{2} = \frac{-4\pm\sqrt{12}}{2}$$

$$= \frac{\cancel{2}(-2\pm\sqrt{3})}{\cancel{2}} = -2\pm\sqrt{3}$$

C.S. $= -2+\sqrt{3},\ -2-\sqrt{3}$ $\qquad \sqrt{3} = 1.732$

$z = -2 + 1.732 = -0.268 \quad (-0.268, 0))$

$= -2 - 1.732 = -3.732 \quad (-3.732, 0)$

So $\oint f(z)\,dz = 2\pi i \left[\cancel{\Sigma} \, \mathrm{Res}\, f(z = -2+\sqrt{3}) \right]$

Residue :- at $z \to -2+\sqrt{3}$

$$(z+2-\sqrt{3}) \frac{(1-z^2-4z)}{(z+2+\sqrt{3})(z+2-\sqrt{3})}$$

$$= \frac{1-z^2-4z}{z+2+\sqrt{3}}$$

$$= \frac{1-(\sqrt{3}-2)^2-4(-2+\sqrt{3})}{\cancel{-2}+\sqrt{3}+\cancel{2}+\sqrt{3}}$$

$$= \frac{1-(3-2\cdot 2\sqrt{3}+4)+8-4\sqrt{3}}{2\sqrt{3}}$$

$$= \frac{1-3+\cancel{4\sqrt{3}}-4+8-\cancel{4\sqrt{3}}}{2\sqrt{3}}$$

$$= \frac{9-7}{2\sqrt{3}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\mathrm{Res} = \frac{1}{\sqrt{3}}$

So $\oint_C f(z)\,dz = 2\pi \cancel{i} \times \frac{1}{\sqrt{3}} \times \frac{1}{\cancel{i}}$

$$= \frac{2\pi}{\sqrt{3}} \quad \underline{\text{Ans}}$$

Q.33 The value of integral $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\,d\theta$ will be

Sol^n $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\,d\theta \quad \cos(\sin\theta-n\theta) \to e^{i(\sin\theta - n\theta)}$

$\Rightarrow$ R.P. of $\int_0^{2\pi} e^{\cos\theta} \cdot e^{i(\sin\theta - n\theta)}\,d\theta$

$\Rightarrow$ R.P. of $\int_0^{2\pi} e^{\cos\theta + i\sin\theta} \cdot e^{-in\theta}\,d\theta \quad \because |z|=1, \; z = e^{i\theta}, \; d\theta = \frac{dz}{iz}$

$\Rightarrow$ R.P. of $\oint_c e^z \cdot e^{-n} \frac{dz}{iz}$

$\Rightarrow$ R.P. of $\frac{1}{i}\left[\oint_c \frac{e^z}{z^{n+1}}\,dz\right]$

$\Rightarrow$ ~~R.P. of $\frac{1}{i}$~~

Condition of singularity

$$z^{n+1} = 0$$
$$z = 0$$

$\therefore$ $z=0$ is singular point of order $(n+1)$

So Res at $z=0$

$$= \frac{1}{(n+\cancel{1}-\cancel{1})!}\left[\frac{d^n}{dz^n}\left\{\cancel{z^{n+1}}\frac{e^z}{\cancel{z^{n+1}}}\right\}\right]_{z=0}$$

$$= \frac{1}{n!}\{e^0 = 1\}$$

$$\text{So} = \frac{1}{n!}$$

I = R.P. of $\left[\frac{1}{i} 2\pi i \frac{1}{n!}\right] = \frac{2\pi}{n!}$ Ans

* Evaluation of Real Improper integral using Complex integration :-

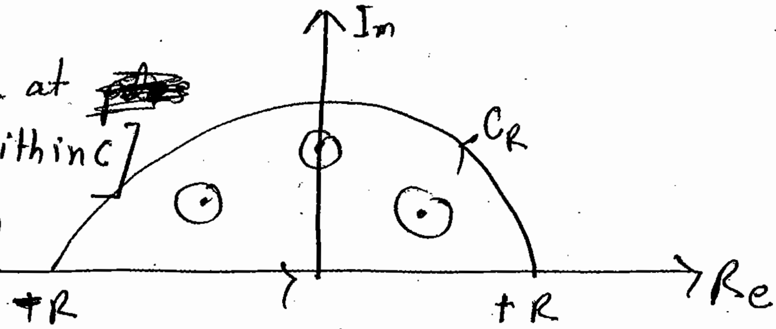
$$\boxed{\int_{-\infty}^{+\infty} f(x)\,dx}$$

Type I :- $f(x)$ will Contain only algebraic function of x.

Then we can choose the corresponding $f(z)$ as the same form of $f(x)$. $\left\{ \text{e.g. } f(x) = \frac{1}{x^6+1},\ f(z) = \frac{1}{z^6+1} \right\}$

(a) If the singular points of the function $f(z)$ does not lies on real axis then

$$\int_{-\infty}^{+\infty} f(z)\,dz = 2\pi i \left[\Sigma \text{ Res. at Poles within } C \right] - i\pi \left[\lim_{z\to\infty} (z f(z)) \right]$$



So
$$\boxed{\int_{-\infty}^{+\infty} f(z)\,dz = 2\pi i \left(\text{Sum of Res at poles within } c \right) - i\pi \left(\lim_{z\to\infty} (z f(z)) \right)}$$

Applicable when points are not at real axis.

Q.20 (i) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^3}\,dx$

$$f(z) = \frac{1}{(1+z^2)^3}$$

S.P. $= (1+z^2) = 0$

So $z = \pm i \rightarrow$ Poles of order 3.

$$\int_{-\infty}^{+\infty} \frac{dz}{(1+z^2)^3} = 2\pi i \left[\operatorname{Res} f(z=i)\right] - i\pi \lim_{z\to\infty} (z f(z))$$

Res at $z \to i$ ~~$2\pi i$~~

$$= \frac{1}{(3-1)!} \frac{d^2}{dz^2} (z-i)^3 \frac{1}{(1+z^2)^3}$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} \left[\cancel{(z-i)^3} \frac{1}{(z+i)^2 \cancel{(z-i)^3}} \right]$$

$$= \frac{1}{2} \frac{d}{dz} \left[-3 (z+i)^{-3-1} (1+0) \right]$$

$$= \frac{1}{2} \left[-3(-4) (z+i)^{-5} \right]$$

$$= \frac{-12\cancel{6}}{\cancel{2}} (2i)^{-5}$$

$$= \frac{3}{16i}$$

$$\lim_{z\to\infty} z \frac{1}{(1+z^2)^3}$$

$$= \lim_{z\to\infty} \frac{z}{z^6 + 3z^4 + 3z^2 + 1}$$

$$= \lim_{z\to\infty} \frac{1}{z^5 + 3z^3 + 3z + \frac{1}{z}}$$

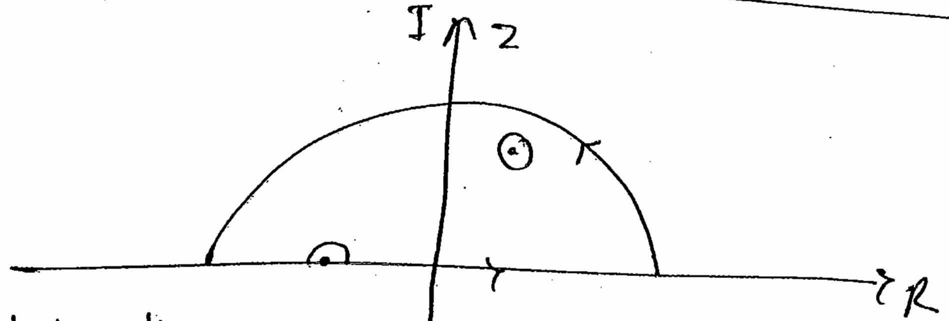
$$= 0$$

So $$\int_{-\infty}^{+\infty} \frac{dz}{(1+z^2)^3} = \cancel{2}\pi \cancel{i} \left(\frac{3}{\cancel{16}_8 \cancel{i}} \right)$$

$$= \frac{3\pi}{8} \text{ Ans}$$

(b) If the singular point of the function $f(z)$ lie on real axis then this integral will be equal to

$$\boxed{\int_{-\infty}^{+\infty} f(z)\,dz = \pi i\left[\Sigma \text{Res. at poles within } c\right] - i\pi\left[\lim_{z\to\infty}(z f(z))\right]}$$



Since for exclude the point not in real axis we have to draw a circle and angle of circle is 2π.

But when point on the real axis then to excude that point we draw a semicircle and angle of semicircle is π.

Q. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2-1)^3}$

$$f(z) = \frac{1}{(z^2-1)^3}$$

C.O.S $= z^2-1=0$

$z=\pm 1$ both in real axis

So $\displaystyle\int_{-\infty}^{+\infty}\frac{dz}{(z^2-1)^3} = \pi i\left[\text{Res } f(z=1) + \text{Res } f(z=-1)\right] - i\pi \lim_{z\to\infty} z f(z)$

Res $z\to 1$ $\displaystyle \frac{1}{2!}\frac{d^2}{dz^2}\cancel{(z-1)^3}\frac{1}{(z+1)^3\cancel{(z-1)^3}}$

$$= \frac{1}{2}\frac{d}{dz}\left[(-3)(z+1)^{-3-1}\right]$$

Q. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2(x-1)}$

$f(z) = \frac{dz}{(z^2+1)^2(z-1)}$ $\Rightarrow$ c.o.s: $(z^2+1)^2(z-1) = 0$

$z = \pm i \quad (2)$

$z = 1 \quad (1)$

$= 2\pi i \left[Res\, f(z = i) \right] + \pi i \left[Res\, f(z = +1) \right] + i\pi \lim_{z\to\infty} z f(z$

Type II :- If $f(x)$ contains trigonometrical functions (sin or cosine) i.e. alongwith algebric function of x Then we can choose $f(z)$ such that the real or imaginary part of $f(z)$ will be $f(x)$

If $f(x) = \dfrac{\cos 2x}{(x^2+1)^2} \longrightarrow f(z) = \dfrac{e^{i2z}}{(z^2+1)^3}$

(a) If the singular points of $f(z)$ does not liying on the real axis. Then -

$$\boxed{\int_{-\infty}^{+\infty} f(z)\,dz = 2\pi i\,[\text{Sum of the residues}]}$$

(b) If the singular points of $f(z)$ lie on real axis then -

$$\boxed{\int_{-\infty}^{+\infty} f(z)\,dz = \pi i\,[\text{Sum of the residues at poles within}}$$

A-4
Q.41 The value of the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{t^2-R^2}\cos\left(\frac{\pi t}{2R}\right)dt$

$\Downarrow$ R.P. of $\displaystyle\int_{-\infty}^{+\infty} \frac{\exp\left(i\frac{\pi t}{2R}\right)}{t^2-R^2}dt$

Sol^n

$$f(z) = \frac{\exp\left[i\frac{\pi z}{2R}\right]}{z^2-R^2}$$

Poles $= z = \pm R$ on the real axis

$$\int_{-\infty}^{+\infty} f(z)\,dz = \pi i\,[\operatorname{Res} f(z=R) + \operatorname{Res} f(z=-R)]$$

Res at $z \to R$

$$= \cancel{(z-R)} \cdot \frac{e^{i\frac{\pi z}{2R}}}{(z+R)\cancel{(z-R)}}$$

$$= \frac{e^{i\frac{\pi R}{2R}}}{2R} = \frac{e^{i\pi/2}}{2R}$$

$$= \frac{\cos\frac{\pi}{2} + i\sin\pi/2}{2R} = \frac{+i}{2R} \quad \left\{ \because \cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1 \right.$$

Res at $z \to -R$

$$\cancel{(z+R)} \frac{e^{-i\frac{\pi R}{2R}}}{\cancel{(z+R)}(z-R)}$$

$$= \frac{e^{-i\pi/2}}{-2R}$$

$$= \frac{\cos\pi/2 - i\sin\pi/2}{-2R}$$

$$= \frac{i}{2R} \quad \left\{ \because \cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1 \right.$$

$$\int_{-\infty}^{+\infty} f(z)\,dz = 2\pi i \left[\frac{i}{2R} + \frac{i}{2R}\right] = \pi i \times i\left(\frac{\cancel{2}}{\cancel{2}R}\right)$$

$$= -\frac{\pi}{R} \quad \text{Ans}$$

Q. when $\int_0^\infty \frac{1}{t^2 - R^2} \cos\frac{\pi t}{2R}\,dt$

then $\frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{t^2 - R^2} \cos\frac{\pi t}{2R}\,dt$

$$= -\frac{\pi}{2R} \quad \text{Ans}$$

JEST

(2) $\int_0^{\infty} \frac{\ln x}{(x^2+1)^2} dx = ?$

$$f(z) = \frac{\ln z}{(z^2+1)^2}, \quad \int_{-\infty}^{+\infty} \frac{\ln z}{(z^2+1)^2} dz = \ldots\ldots$$

$$\Rightarrow \int_{-\infty}^{0} \frac{\ln z}{(z^2+1)^2} + \int_{0}^{+\infty} \frac{\ln z}{(z^2+1)^2} dz = \ldots$$

$$\Rightarrow \int_{-\infty}^{0} \frac{\ln z}{(z^2+1)^2} dz$$

$$z = -z \Rightarrow dz = -dz$$

$$= \int_{\infty}^{0} \frac{\ln(-z)}{(z^2+1)^2} (-dz)$$

$$= \int_{0}^{\infty} \frac{\ln(-z)}{(z^2+1)^2} dz$$

$$= \int_{0}^{\infty} \frac{\ln(e^{i\pi} z)}{(z^2+1)^2} dz$$

$$= \int_{0}^{\infty} \frac{\ln(e^{i\pi}) + \ln z}{(z^2+1)^2} dz$$

$$= i\pi \int_{0}^{\infty} \frac{dz}{(z^2+1)^2} + \int_{0}^{\infty} \frac{\ln z}{(z^2+1)^2} dz$$

$$\Rightarrow i\pi \int_{0}^{\infty} \frac{dz}{(z^2+1)^2} + 2\int_{0}^{\infty} \frac{\ln z}{(z^2+1)^2} dz = \ldots$$

20 (iv) $\displaystyle\int_0^{\infty} \frac{x^3 \sin mx}{x^4+a^4}\,dx$

$$f(z) = \frac{z^3 e^{imz}}{z^4+a^4} \qquad = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^3 \sin mx}{x^4+a^4}\,dx$$

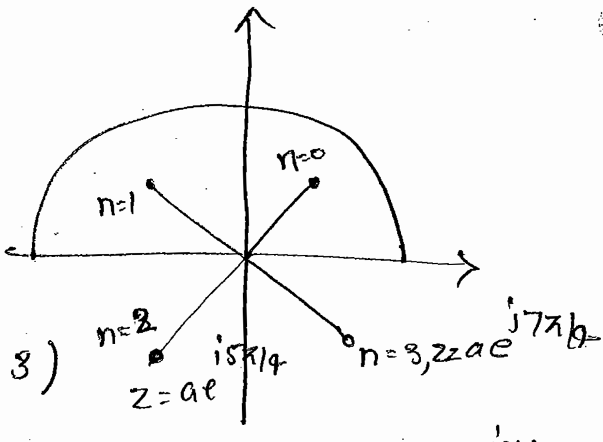
$$= \frac{1}{2}\,\text{I.P. of} \int_{-\infty}^{+\infty} \frac{x^3 e^{imx}}{x^4+a^4}\,dz$$

$$z^4+a^4=0$$

$$z^4=-a^4$$

$$= e^{i(2n+1)\pi}\, a^4$$

$$\Rightarrow z = e^{i(2n+1)\pi/4}\, a \quad (n=0,1,2,3)$$



$$\text{Res}\, f(z=ae^{i\pi/4}) = \frac{\cancel{z^3}\, e^{imz}}{4\cancel{z^3}}\Bigg|_{z=ae^{i\pi/4}} = \frac{1}{4} e^{im(ae^{i\pi/4})}$$

$$\text{Res}\, f(z=ae^{i3\pi/4}) = \frac{1}{4} e^{im(ae^{i3\pi/4})}$$

$$I = \frac{2\pi i}{4}\left[e^{ima\left(\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}\right)} + e^{ima\left(\frac{-1}{\sqrt2}+i\frac{1}{\sqrt2}\right)} \right]$$

$$= \frac{\pi i}{2}\left[e^{-ma/\sqrt2}\left\{ e^{ima/\sqrt2} + e^{-ima/\sqrt2} \right\}\right]$$

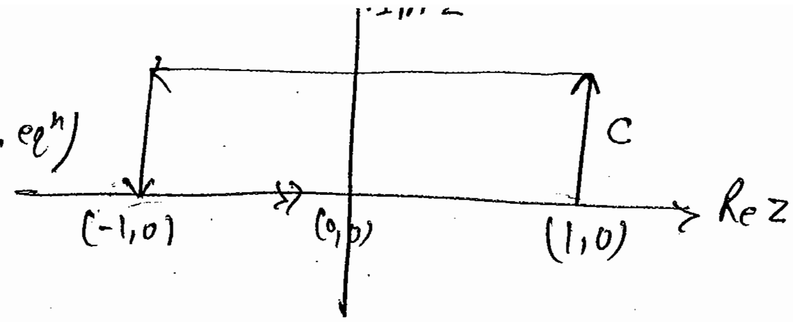
$$= \frac{\pi i}{\cancel{2}}\left[e^{-ma/\sqrt2}\left(\cancel{2}\cos\frac{ma}{\sqrt2}\right)\right]$$

$$\boxed{I = \pi i\, e^{-ma/\sqrt2} \cos\left(\frac{ma}{\sqrt2}\right)}$$

A-4
Q. 37 June 2011

Sol^n

$$\oint_{C+C_R} z^2 e^z dz = 0 \quad (\text{by C-R. eq}^n)$$



$$\Rightarrow \int_C z^2 e^z dz + \int_{C_R} z^2 e^z dz = 0$$

$$\Rightarrow \int_C z^2 e^z dz = -\int_{C_R} z^2 e^z dz$$

$$= -\int_{-1}^{+1} \underset{\text{I}}{x^2} \underset{\text{II}}{e^x} dx = -\left[x^2 . e^x - \int_{-1}^{+1} \underset{\text{I}}{2x} . \underset{\text{II}}{e^x} dx \right]$$

$$= -\left[\{x^2 . e^x\}_{-1}^{+1} - 2\left\{ x . e^x - \int_{-1}^{+1} 1 . e^x dx \right\} \right]$$

$$= -\left[\{x^2 e^x\}_{-1}^{+1} - 2\{ x e^x - e^x \}_{-1}^{+1} \right]$$

$$= -\left[\{1^2 . e^1 - (-1)^2 e^{-1}\} - 2\{\{1 e^1 - e^1\} + \{e^{-1} - e^{-1}\}\} \right.$$

$$= -\left[e - \frac{1}{e} - 2 \right.$$

$$= -\int_{-1}^{+1} x^2 e^x dx = -\left[\int_{-1}^{0} \underset{\text{I}}{x^2} \underset{\text{II}}{e^x} dx + \int_0^1 x^2 e^x dx \right]$$

$$= -\left[\left\{ \{x^2 e^x\}_{-1}^{0} - \int 2x . e^x dx \right\} + \left\{ \{x^2 . e^x\}_0^1 - \int 2x e^x dx \right\} \right]$$

$$= -\left[-e^{-1} - 2\left\{ \{x . e^x\}_{-1}^{0} - \int_{-1}^{0} 1 . e^x dx \right\} + \left\{ e^1 - 2\left\{ \{x . e^x\}_0^1 - \int_0^1 1 . e^x dx \right. \right. \right.$$

$$= -\left[-e^{-1} - 2e^{-1} - 2e^{-1} + \{ e^1 - \cancel{2e^1} + \cancel{2e^1} \} \right]$$

$$= -\left[-5e^{-1} + e^1 \right]$$

$$= \left\{ \frac{5}{e} - e \right\} \quad \underline{\text{Ans}}$$

A-5

46

Sol^n $\int_0^\infty \frac{1}{(y^2+x^2)}\,dx = \frac{\pi}{2y}$ then find $\int_0^\infty \frac{dx}{(y^2+x^2)^2} = ?$

$$\Rightarrow \frac{d}{dy}\left[\int_0^\infty \frac{1}{(y^2+x^2)}\,dx\right] = \frac{d}{dy}\left(\frac{\pi}{2y}\right)$$

$$\Rightarrow \int_0^\infty \left[+ \frac{1}{(y^2+x^2)^2}\,2y \right] dx = \frac{\pi}{2}\left(-\frac{1}{y^2}\right)$$

$$\Rightarrow \int_0^\infty \frac{dx}{(y^2+x^2)^2} = \frac{\pi}{4y^3} \quad \underline{\text{Ans}}$$

* Branch Point Singularity :-

for example -

$$f(z) = \pm \sqrt{z-1}$$

single value of z it have two values of $f(z)$

$\Downarrow$

two branches of $f(z)$

$$f'(z) = \frac{1}{2\sqrt{z-1}}$$ $\because$ It is not defined at $z = 1$.

So $z = 1$ is branch point of $f(z)$

At branch point the value of the two branches of the function becomes equal.

$$\boxed{f(z) = \pm\sqrt{1-1} = \pm 0 = 0, 0}$$

for example :- $f(z) = \sqrt[3]{z+1}\,\sqrt{z-3}$

$\Downarrow \quad \Downarrow$

$3 \times 2 = 6$ different Combinations of z.

$$f'(z) = \frac{1}{3}\frac{1}{(z+1)^{2/3}}\sqrt{z-3} + \sqrt[3]{z+1}\,\frac{1}{2\sqrt{z-3}}$$

$\left.\begin{matrix} z = -1 \\ z = 3 \end{matrix}\right]$ Two branch point.

(38) Which of the following is an analytic function of the complex variable $\Rightarrow z = x+iy$ in the domain $|z| < 2$? , $-2 < z < 2$

(a) $(3+x-iy)^7$

$\Rightarrow (3+\bar{z})^7$

since here we get $\bar{z}$ power of $\bar{z}$ so it is not analytic

(b) $(1+x+iy)^4 (7-x-iy)^3$

$(1+z)^4 (7-z)^3$

Here we get only power of z so it is analytic

(c) $(1-2x-iy)^4 (3-x-iy)^3$

$(1-x\cdot z)^4 (3-z)^3$

$(1-\mathrm{Re}\,z.\,z)^4 (3-z)^3$

Here we get power of Re z also instead of z so it is not analytic.

(7) $(x+iy-1)^{1/2}$

$= (z-1)^{1/2}$ it is a multivalued function

$= \sqrt{z-1}$

It is not analytic at $z=1$ $\therefore$ domain is $|z|<2$ means $-2<z<2$ so it lies in the domain so it is not analytic.

Note :- for Analyticity the concept of power of z method is valid only for single valued function not in multivalued function. For multivalued function check differentiability.

(14)

$f(z) = \tan^{-1} z$ $\qquad |z|<1$

Taylor series about $z=0$

$\Rightarrow \frac{df}{dz} = \frac{1}{1+z^2} = (1+z^2)^{-1}$

$= 1 - z^2 + z^4 - z^6 + \cdots$

$$\Rightarrow \boxed{\tan^{-1} z = f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} \cdots}$$

16 (VI) $f(z) = \frac{z}{1+z^4}$

C.O.S. $= z^4+1 = 0$

$\Rightarrow z^4 = -1 = e^{i(2n+1)\pi/4}$

DIFFERENTIAL EQUATIONS

An equation that contains derivatives is k/a Differential Equation.

e.g. $\boxed{\dfrac{dy}{dx} + 2y = \cos x}$

* **Order of the differential Equation:-** It is the order of the highest derivative term present in the differential equation.

* **Degree of the differential equation:-** Degree is the power of the highest order derivative term present in the differential equation.

$$\boxed{\frac{dy}{dx} + 2y = \cos x}$$

$\Downarrow$

1st order, 1st degree differential equation.

(1) $3y = x\dfrac{dy}{dx} + \sin\left(\dfrac{dy}{dx}\right)$

order = 1, Degree can not be determined. because it is an infinite series.

$$3y = x\frac{dy}{dx} + \left[\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3 \frac{1}{3!} + \cdots\right]$$

(2) $\sqrt{2\left(\dfrac{dy}{dx}\right)^3 + 4} = \left(\dfrac{d^2y}{dx^2}\right)^{3/2}$

order = 2, Degree = $\cancel{\dfrac{d^2y}{dx^2}}$ 3, differential equation.

* First Order Differential Equation :-

Any first order differential equation can be written as-

$$\frac{dy}{dx} = f(x,y)$$

$$\Downarrow$$

$$\Rightarrow y = f(x) \longleftarrow \text{Solution.}$$

* Physical Interpretation :-

If $y = f(x) \longrightarrow$ curve

$\frac{dy}{dx} \Rightarrow$ slope of tangent to the curve.

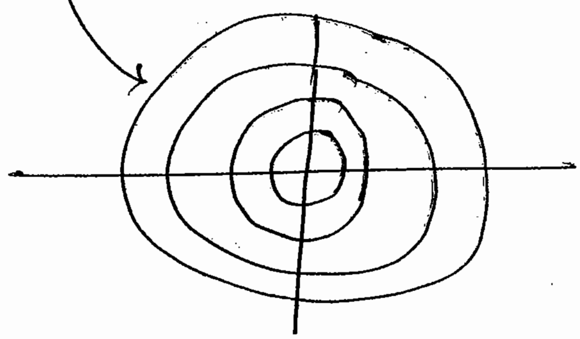
Here $\boxed{y = f(x)} \longrightarrow$ It denotes the family of curve.

$\boxed{\frac{dy}{dx} = f(x,y)} \longrightarrow$ slope of tangent to the curve.

Let an equation $x^2+y^2=c$, this equation represent family of circle of centere at $(0,0)$ and radius is $\sqrt{c}$! It is not a single circle, it is family of circle.

$$\boxed{x^2+y^2 = c}$$

$(0,0)$, $r = \sqrt{c}$



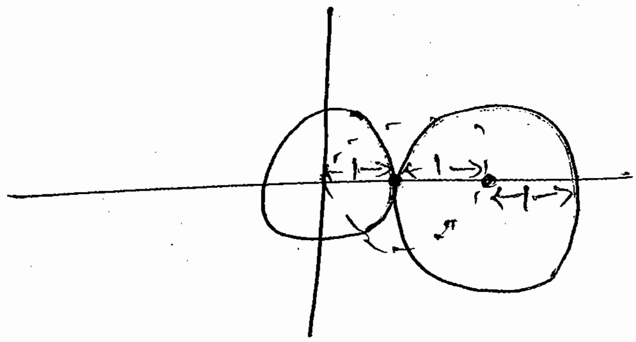
Let an another equation -

$$(x-h)^2 + (y-k)^2 = a^2 \quad \text{here } r = a = \text{Constt.}$$

It represents family of circles of curve of radius 'a' and centre at (h, k) which is variable. So centre may change.

When

$$(x-h)^2 + (y-k)^2 = 1$$



Case - I Separation of Variables :-

$$\frac{dy}{dx} = f(x,y) = \frac{\phi_1(x)}{\phi_2(y)}$$

$$\Rightarrow \boxed{\int \phi_1(x)\,dx = \int \phi_2(y)\,dy}$$

A-7

Q.1 (ii) Solve the Differential equation (separation of variables):-

(ii) $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$ given that $y\left(\frac{\pi}{3}\right) = 0$

$$\Rightarrow \int \frac{\cos x}{\sin x}dx + \int \frac{e^y}{(e^y+1)}dy = 0$$

dp, =r

$$\Rightarrow \ln(\sin x) + \ln(e^y+1) = \log c$$

let $\sin x = p$, $\cos x\,dx = dp$

and $e^y+1 = q$, $e^y dy = dq$

$$\Rightarrow \ln\left[\sin x(e^y+1)\right] = \ln c$$

$$\Rightarrow \boxed{\sin x(e^y+1) = c}$$

$\because \quad x = \frac{\pi}{3}, \quad y = 0$

$\therefore \quad C = \sin\frac{\pi}{3}\,(e^{0}+1)$

$C = \sqrt{3}$

$\Downarrow$

So $\boxed{\sin x\,(e^{y}+1) = \sqrt{3}}$ Ans

(iv) $\frac{dy}{dx} = (4x+y+1)^2$ given that $y(0)=1$

let $4x+y+1 = z$

$4 + \frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 4$

Now

$\frac{dz}{dx} - 4 = z^2$

$\frac{dz}{dx} = 4 + z^2$

$\int \frac{dz}{4+z^2} = \int dx$

$\int \frac{dz}{2^2+z^2} = \int dx$

$\frac{1}{2}\tan^{-1}\frac{z}{2} = x + C$

$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + C$

when $x=0,\ y=1$

$\frac{1}{2}\tan^{-1}\frac{2}{2} = 0 + C$

$\frac{1}{2}\tan^{-1}\tan\frac{\pi}{4} = C$

$\therefore \boxed{C = \frac{\pi}{8}}$

So $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + \frac{\pi}{8}$

$\Rightarrow \boxed{\tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x + \frac{\pi}{4}}$

A-6

Q.63 The solution of the D.E. $\frac{dx}{dt} = x^2$ with the initial condition $x(0)=1$ will blow up as t tend to -

(a) 1 (b) 2 (c) 1/2 (d) ∞.

Sol^n $\frac{dx}{dt} = x^2$ $\quad x(0)=1$

$\int \frac{dx}{x^2} = \int dt$

$\Rightarrow -\frac{1}{x} = t + c$

$x(0) = 1 \Rightarrow -1 = 0 + c \Rightarrow c = -1$

$-\frac{1}{x} = t - 1$

$\Rightarrow \frac{1}{x} = 1 - t$

$\Rightarrow \boxed{x = \frac{1}{1-t}}$

So at $t=1$ x blow up (means tends to ∞).

So $\boxed{t=1}$ ans

Case II :- Homogeneous Type :-

$$\boxed{\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}}$$

If $f_1(x,y)$ and $f_2(x,y)$ have same degree.

Ex-
$$\frac{dy}{dx} = \frac{\overbrace{x^2}^{\text{degree '2'}} - \overbrace{y^2}^{\text{degree '2'}} + \overbrace{xy}^{\text{degree '2'}}}{\underbrace{y^2}_{\text{deg. '2'}} + \underbrace{xy}_{\text{degree '2'}}} = \frac{\text{Total degree } 2}{\text{Total degree } 2}$$

So it is first order Homogeneous differential equation

Note :- "In addition and subtraction degree remain's same but in multiplication degree added up and in division degree subtracted."

* Solution of Homogeneous D.E. :-

(1) $\boxed{y = vx} \Rightarrow v = \frac{y}{x} \leftarrow \sin\frac{y}{x}$

$$\Rightarrow \boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

(2) $x = vy \Rightarrow v = \frac{x}{y} \leftarrow e^{x/y}$

$$\Rightarrow \boxed{\frac{dx}{dy} = v + y\frac{dv}{dy}}$$

A-7

Q.2 Solve the Differential Equation (Homogeneous type) :-

(iii) $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$ given that $y(0) = 1$.

Sol^n $(1+e^{x/y})dx + e^{x/y}\left(1+\frac{x}{y}\right)dy = 0$

$$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{1+e^{x/y}}$$

$$\Rightarrow v + y\frac{dv}{dy} = \frac{e^{v}(v-1)}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{e^{v}(v-1)}{1+e^{v}} - v$$

$$= \frac{-e^{v}-v}{1+e^{v}}$$

$$\Rightarrow \int \frac{e^{v}+1}{e^{v}+v}\,dv = -\int \frac{dy}{y}$$

Let $e^{v}+v = p$

$dv(e^{v}+1) = dp$

$$\Rightarrow \ln(e^{v}+v) = -\ln y + \ln c$$

$$\Rightarrow \ln\left(e^{x/y}+\frac{x}{y}\right) = \ln\left(\frac{c}{y}\right)$$

$$\Rightarrow e^{x/y} + \frac{x}{y} = \frac{c}{y}$$

$\because$ at $x=0$, $y=1$

So $e^{0} + 0 = c$

$$\Rightarrow \boxed{c=1}$$

So $\boxed{e^{x/y} + \frac{x}{y} = \frac{1}{y}}$ Ans

A-6

Q.5 Suppose that, a tumour in a rat is approximately spherical and its rate of growth is proportional to its diameter. If the ~~tumour~~ has diameter 5 m.m.. When detected at 8mm, after three months later, what will the diameter be after another three months?

(a) 10.1 m.m. (b) 12.8 m.m. (c) 11.6 m.m. (d) 13.2 m.m.

Sol^n $\because$ toumor is in 3-D spherical shape so rate of growth is change in volume. with time.

$$\frac{dv}{dt} \propto 2r \qquad (2r = \text{diameter})$$

$t=0 \Rightarrow d = 5$ m.m.

$t = 3 \Rightarrow d = 8$ m.m.

$t = 6 \Rightarrow d = ?$

$$\frac{dv}{dt} = 2kr \quad -① \quad \Big/\ (k = \text{proportionality constant})$$

$\because V = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = \frac{4}{\cancel{3}}\pi . \cancel{3}r^2\frac{dr}{dt}$$

$$\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt} \quad -⑪$$

So from ①

$$\cancel{4}\pi r^{\cancel{2}}\frac{dr}{dt} = \cancel{2}k\cancel{r} \Rightarrow r^2\frac{dr}{dt} = \frac{2k}{4\pi}r$$

~~$2\pi r\, dr = kdt$~~

$$r^2\frac{dr}{dt} = c_1 r \qquad \left(\text{let } c_1 = \frac{2k}{4\pi}\right)$$

$$\int r\,dr = c_1\int dt$$

$$\Rightarrow \frac{r^2}{2} = c_1 t + c_2$$

$$\Rightarrow \frac{d^2}{8} = c_1 t + c_2$$

at $t=0$ $d = 5$ at $t = 3$, $d = 8$

$$\boxed{\frac{25}{8} = c_2} \qquad \frac{64}{8} = 3c_1 + \frac{25}{8}$$

$$\boxed{c_1 = \frac{13}{8}}$$

$$\frac{d^2}{8} = \frac{13}{8}t + \frac{25}{8}$$

$$\Rightarrow d^2 = 13t + 25$$

at $t = 6 \Rightarrow d^2 = 103$

$$\boxed{d \approx 10.1\ m.m.}$$ Ans

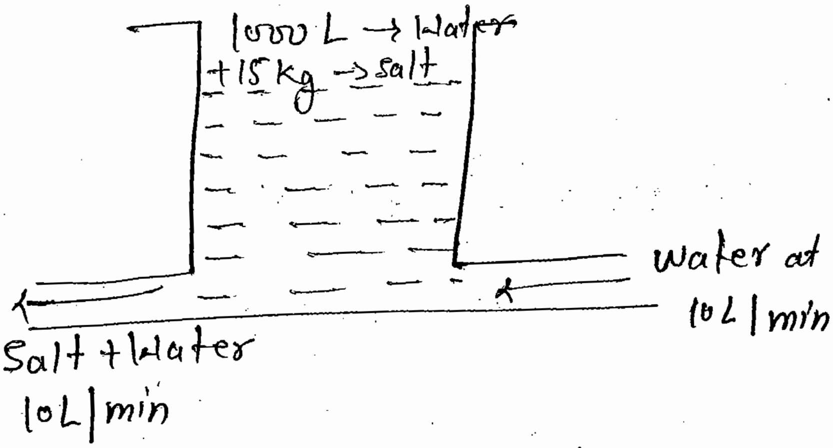
A-6

Q.6 A tank holds 1000 liters of water, in which 15 kg of salt is dissolved. Pure water enter the tank at the rate of 10 liters/minutes. The solution is kept throughout mixed and drained from the tank at the same rate. If 'm' is the mass of the salt in the tank at any time 't' which of the following options decribes the rate of change of the mass of the salt in the tank?

Soln

$t = 0 \Rightarrow 15$ kg

$t = t \Rightarrow m$ kg



$\because \frac{dm}{dt} \propto$ Amount of salt getting out of the tank per minute

In 1000 L $\longrightarrow$ m kg

10 L $\longrightarrow$ $\frac{m}{1000} \times 10$ kg $= \frac{m}{100}$ kg

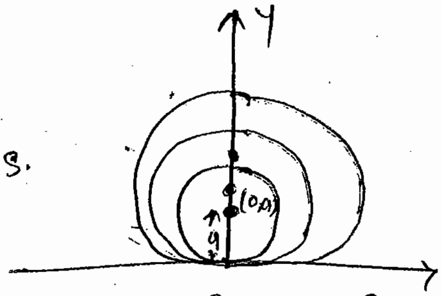
$$\frac{dm}{dt} \propto \frac{m}{100}$$

$$\boxed{\frac{dm}{dt} = -\frac{m}{100}}$$ Ans

Q.8 The differential equation of all circles passing through the origin and having their centers on the y-axis

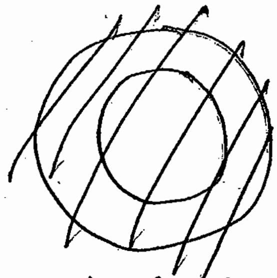
Sol^n Differential equation of circles means solution of this ~~cir~~ differential equation gives the family of circles.

D.E.
$\downarrow$
family of circles.



$x^2+(y-a)^2=a^2$
$\downarrow$
1st order D.E

Note:- The number of variable parameter will be the order of corresponding differential equation.



$\because$ Here we change only one parameter (radius) then center is automatically change and we get family of curve having centers on the y-axis and ~~center~~ passing through the origin. So Here order of differential equation is one,

$$x^2+(y-a)^2=a^2$$

$$\Rightarrow \cancel{2}x+\cancel{2}(y-a)\frac{dy}{dx}=0$$

$$\Rightarrow (y-a)\frac{dy}{dx}=-x$$

$$\Rightarrow \boxed{(y-a)=-\frac{x}{dy/dx}}$$

$$\Rightarrow \boxed{a=y+\frac{x}{dy/dx}}$$

$$x^2+\frac{x^2}{\left(\frac{dy}{dx}\right)^2}=\left[y+\frac{x}{\left(\frac{dy}{dx}\right)}\right]^2$$

$$\Rightarrow x^2 - y^2 = \cancel{\frac{x^2}{\left(\frac{dy}{dx}\right)^2}} - \cancel{\frac{x^2}{\left(\frac{dy}{dx}\right)^2}} + \frac{2xy}{\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow x^2 - y^2 = \frac{2xy}{(dy/dx)}$$

$$\Rightarrow \boxed{(x^2 - y^2)\left(\frac{dy}{dx}\right) = 2xy}$$

* Case 3:- Linear Differential Equation :-

A differential equation is said to be linear if the power of the dependent variable and it's derivatives are one.

First Order Linear Differential Equation:-

Standard form of Linear Differential Equation of first order is -

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

Here $P(x)$ and $Q(x)$ are either constants or function of x.

$$\boxed{I.F. = e^{\int P(x)dx}}$$

$$\underbrace{(I.F.)\frac{dy}{dx} + (I.F.)P(x)y}_{} = (I.F.)Q(x)$$

$$\Rightarrow \frac{d}{dx}[y(I.F.)] = (I.F.)Q(x)$$

$$\Rightarrow \int d[y(I.F.)] = \int (I.F.)Q(x)dx$$

Short Trick.

$$\boxed{y(I.F.) = \int (I.F.)Q(x)dx}$$

If $\boxed{\dfrac{dx}{dy} + P(y)\,x = Q(y)}$

$$\boxed{I.F. = e^{\int P(y)\,dy}}$$

$$\Rightarrow \frac{d}{dy}\left[x(I.F.)\right] = (I.F.)\,Q(y)$$

A-7

Ques 3 Solve the differential equation [Linear type]

(iii) $(x^3-x)\dfrac{dy}{dx} - (3x^2-1)y = x^5 - 2x^3 + x$ given that $y(e) = e$.

Solⁿ

$$\frac{dy}{dx} - \frac{(3x^2-1)}{(x^3-x)}y = \frac{x^5-2x^3+x}{(x^3-x)}$$

$$= \frac{\cancel{x}(x^4-2x^2+1)}{\cancel{x}(x^2-1)} = \frac{(x^2-1)^{\cancel{2}}}{\cancel{(x^2-1)}}$$

$$= (x^2-1)$$

$$I.F. = e^{-\int \frac{3x^2-1}{x^3-x}dx}$$

$$= e^{-\ln(x^3-x)}$$

$$= e^{\ln(x^3-x)^{-1}}$$

$$= \frac{1}{(x^3-x)}$$

So $$y\frac{1}{x^3-x} = \int \frac{1}{(x^3-x)}(x^2-1)\,dx$$

$$\Rightarrow \boxed{y \cdot \frac{1}{x^3-x} = \ln x + c}$$

when $x = e$, $y = e$ $\Rightarrow \dfrac{e}{e^3-e} = \ln e + c \Rightarrow \dfrac{\cancel{e}}{\cancel{e}(e^2-1)} = c$

$$c = \frac{1}{e^2-1}$$

So $$\boxed{\frac{y}{x^3-x} = \ln x + \frac{1}{e^2-1}}$$ Ans

Q.23 If $y(x)$ satisfies the differential equation $\frac{dy}{dx}+2y=2+e^{-x^2}$ with $y(0)=0$, then $\lim_{x\to\infty} y(x)$ is equal to -

(a) 0 (b) 1 (C) 2 (d) -1

Sol^n

$$\frac{dy}{dx}+2y = 2+e^{-x^2} \quad \text{with } y=0 \text{ when } x=0$$

This is linear differential equation of order 1.

$$I.F. = e^{2x}$$

$$\Rightarrow y\cdot e^{2x} = \int (2e^{2x}+e^{2x-x^2})dx$$

$$y\cdot e^{2x} = e^{2x} + \int e^{2x-x^2}dx$$

$$y = 1+e^{-2x}\boxed{\int e^{2x-x^2}dx}$$

$\therefore$ Here e^{-2x} is decaying function of x and after integraction of $\int e^{2x-x^2}dx$ gives some function of x but we know when x is very large then exponential function is dominating. Here $x=\infty$ so $e^{-\infty}=0$ after multiplication second term is zero.

$\lim_{x\to\infty} y = 1$

So $\boxed{\lim_{x\to\infty} y = 1}$ Ans

standared Integration,

$$= e^{2x-x^2}$$

$$= e^{-(x^2-2x+1)+1}$$

$$= e^{-(x-1)^2}\cdot e$$

$$\int e^{-x^2}dx \;---$$

Case 4: Non - Linear $\Rightarrow$ Linear

$$\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)}$$

→ no function of x and y with $\frac{dy}{dx}$

→ no function of y on R.H.S.

A-7
Q3 (v) $\quad 3\frac{dy}{dx} + \frac{3y}{x} = 2x^4 y^4$ ← Nonlinear equation (Bernoly eq^n)

Sol^n Here y^4 is unwanted form so we devide by y^4 on total equation.

$$\Rightarrow \frac{3}{y^4}\frac{dy}{dx} + \frac{3}{x}\frac{1}{y^3} = 2x^4$$

Say $\frac{1}{y^3} = z \Rightarrow -\frac{3}{y^4}\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow -\frac{dz}{dx} + \frac{3}{x} z = 2x^4 \Rightarrow \boxed{\frac{dz}{dx} - \frac{3}{x} z = -2x^4}$$ ← linear equation

$$\text{I.F.} = e^{\int -\frac{3}{x} dx} \Rightarrow \quad \text{I.F.} = e^{-3\log_e x} = e^{\log_e x^{-3}}$$

$$\Rightarrow \text{I.F.} = \frac{1}{x^3}$$

So $\boxed{(\text{I.F})\, y = \int (\text{I.F.})\, Q(x)\, dx}$

$$\frac{1}{x^3} z = \int \frac{1}{x^3}(-2x^4) = -2\int \frac{x^4}{x^3} dx = -2\int x\, dx$$

$$\frac{1}{x^3} z = -\cancel{2}\frac{x^2}{\cancel{2}} + c = -x^2 + c$$

$$\frac{1}{x^3 y^3} + x^2 = c$$

when $x = 1,\ y = 1$

$$1 + 1 = c \Rightarrow \boxed{c = 2}$$

$$\frac{1}{x^3}\cdot\frac{1}{y^3} = -x^2 + 2$$

$$\boxed{\frac{1}{y^3} = -x^5 + 2x^3}$$ Ans

(vii) $r \sin\theta - \frac{dr}{d\theta}\cos\theta = r^2$ given that $r\left(\frac{\pi}{4}\right) = 1$

Sol^n

$$-\frac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$$ ← Non-Linear Differential Eq^n.

So divide by $r^2\cos\theta$

$$\Rightarrow -\frac{1}{r^2}\frac{dr}{d\theta} + \left(\frac{1}{r}\right)\tan\theta = \sec\theta.$$

Say $\boxed{\frac{1}{r} = z}$ $\quad -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{dz}{d\theta}$

$$\Rightarrow \boxed{\frac{dz}{d\theta} + z\tan\theta = \sec\theta}$$ ← Linear Differential Equation

$$I.F. = e^{\int P(x)\,dx} = e^{\int \tan\theta\,d\theta} = e^{\log_e \sec\theta} = \sec\theta$$

$$\boxed{I.F. = \sec\theta}$$

So $(I.F.)y = \int (I.F.)\,Q(x)\,dx$

$$\sec\theta . z = \int \sec\theta . \sec\theta . d\theta.$$

$$\sec\theta . z = \int \sec^2\theta\,d\theta = \tan\theta + c$$

$$\sec\theta . \frac{1}{r} = \tan\theta + c$$

when $\theta = \frac{\pi}{4}$ then $r = 1$

$$\sec\frac{\pi}{4} \cdot \frac{1}{1} = \tan\frac{\pi}{4} + c$$

$$\sqrt{2} = 1 + c$$

$$\boxed{c = \sqrt{2} - 1}$$

So solution is

$$\boxed{\frac{1}{r}\sec\theta = \tan\theta + (\sqrt{2} - 1)}$$ Ans

Q.7 The rate at which a hot body cools is proportional to the difference in temparature between it and surroundi (Newton's law of Cooling). A body is heated to 110°C and placed in air at 10°. It's temperature becomes 60° after 1 hour. After what additional time it cool down to 30°?

Sol^n

$$\frac{d\theta_B}{dt} \propto (\theta_B - \theta_S)$$

θ_B = temp. of body

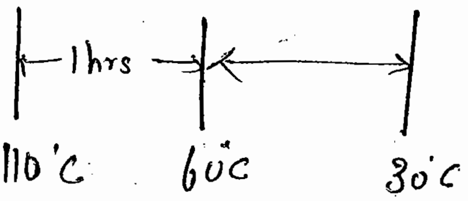
θ_S = temp. of surrounding

at $t=0$, $\theta_B = 110°C$ $\theta_S = 10°C$ θ_S remains constant.

at $t = 1h$, $\theta_B = 60°C$ $\theta_S = 10°C$

at $t = ??$, $\theta_B = 30°C$ $\theta_S = 10°C$

$$\frac{d\theta_B}{dt} = -K(\theta_B - \theta_S)$$



$$\int \frac{d\theta_B}{(\theta_B - \theta_S)} = -K\int dt$$

$$\Rightarrow \ln(\theta_B - \theta_S) = -Kt + C$$

at $t=0$, $\theta_B = 110°C$, $\theta_S = 10°C$

$$\Rightarrow \boxed{C = \ln(100)}$$

at $t = 1h$, $\theta_B = 60°C$, $\theta_S = 10°C$

$$\Rightarrow \ln(50) = -K + \ln 100$$

$$\Rightarrow \boxed{K = \ln 2}$$

So $\ln(\theta_B - \theta_S) = -\ln 2 . t + \ln(100)$

At $\theta_B = 30°C$, $\theta_S = 10°C$

$$\boxed{t = \frac{\ln 5}{\ln 2} \text{ hrs}}$$

So $t = \left(\frac{\ln 5}{\ln 2} - 1\right)$ hrs **Ans** {becoz tim after 1 hr

* Case-5 Exact Differential Equation :-

This type of Differential equation can be obtain just by differentiating its solution.

$$\boxed{Mdx + Ndy = 0} \quad \text{—— (I)}$$

This equation will be exact if -

$$\boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

$$\Rightarrow \boxed{\phi(x, y) = \text{Constant}}$$

$$\Rightarrow d\phi = 0$$

$$\Rightarrow \boxed{\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0} \quad \text{—— (II)}$$

Compairing (I) and (II)

$$M = \frac{\partial \phi}{\partial x}, \quad N = \frac{\partial \phi}{\partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \underline{\frac{\partial^2 \phi}{\partial y \partial x}} \qquad \Rightarrow \frac{\partial N}{\partial x} = \underline{\frac{\partial^2 \phi}{\partial x \partial y}}$$

$\because$ L.H.S. is same. so R.H.S must be same.

So $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$$\Rightarrow \frac{\partial \phi}{\partial x} = M \Rightarrow \phi = \int M dx + f_1(y)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = N \Rightarrow \phi = \int N dy + f_2(x)$$

$\boxed{\phi(x, y) = \text{Combination of all different terms} + C}$

C → Numerical Constt.

Direct Method:-

Sol^n:- $\boxed{\int M\,dx + \int N_y\,dy = \text{Constt}}$ where N_y = Part of N that does not contain

or

Sol^n:- $\boxed{\int M_x\,dx + \int N\,dy = \text{Constt}}$ M_x = Part of M that does not contain y.

A-7

Q.4 Solve the Differential Equation (exact equation):-

(ii) $[1+\ln(xy)]dx + [1+\frac{x}{y}]dy = 0$ given that $y(1)=1$

Sol^n

$[1+\ln(xy)]dx + [1+\frac{x}{y}]dy = 0$ at $x=1, y=1$

Comparing with $M\,dx + N\,dy = 0$

$M = 1+\ln(xy)$ $\quad N = 1+\frac{x}{y}$

$\int[1+\ln(xy)]dx + \int 1\,dy = 0$ $\quad N_y = 1$

$\Rightarrow x + \int \underset{I}{\ln(xy)} \cdot \underset{II}{1}\,dx + y = c$

$\Rightarrow x + \ln(xy) \cdot x - \int \frac{1}{xy} \cdot \cancel{y} \cdot x\,dx + y = c$

$\Rightarrow \cancel{x} + x\ln(xy) - \cancel{x} + y = c$

$\Rightarrow \boxed{x\ln(xy) + y = c}$

$\because x=1, y=1$

$1 + 1\log 1 - \cancel{1} + \cancel{1} = c$

$\boxed{c=1}$

So sol^n is -

$\boxed{x\ln(xy) + y = 1}$ Ans.

IInd Method:-

$\int(1+\ln x)dx + \int\left(1+\frac{x}{y}\right)dy = 0$

$\int \ln(xy)+1 = 1+\ln x + \ln y$

$\boxed{M_x = 1+\ln x}$

$\Rightarrow x + \int \ln x \cdot 1\,dx + y + x\ln y = c$

$\Rightarrow x + \ln x \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\,dx + y + x\ln y = c$

$$\Rightarrow \cancel{x} + \underline{x \ln x} - \cancel{x} + y + \underline{x \ln y} = c$$

$$\Rightarrow \boxed{x \ln(xy) + y = c}$$

* **Second Order Linear D.E. (Homogeneous) :-**

Standard form -

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0} \quad \text{---} (*)$$

1 - Dimensional Schrodinger Equation for x-independent potential.

$$\left(-\frac{\hbar^2}{2m}\right)\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\psi = 0$$

$$\Rightarrow \boxed{\frac{d^2\psi}{dx^2} + \underbrace{\frac{2m}{\hbar^2}(E-V)}_{Q}\psi = 0}$$

Assume a trial solution -

$$y = ce^{mx}$$

$$\Rightarrow \frac{dy}{dx} = cm \cdot e^{mx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = c \cdot m^2 e^{mx}$$

Putting the value of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in eqn (*)

$$\Rightarrow c \cdot e^{mx}\left[m^2 + Pm + Q\right] = 0$$

$c \cdot e^{mx} = 0 \Rightarrow$ Zero solution or Trivial Solution.

Non Trivial Solution :-

$$m^2 + Pm + Q = 0$$

$\Rightarrow m = m_1, m_2$ {roots :- m_1 and m_2 may be real or Complex}

Case - 1 :- m_1 and m_2 are real and unequal.

Sol^n = $$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case - 2 :- m_1 and m_2 are real and equal to m.

then sol^n :-
$$\boxed{y = (c_1 + c_2 x) e^{m x}}$$

Case-3 :- m_1 and m_2 are Complex numbers.

then sol^n -
$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$
Here m_1 and m_2 is complex.

Case - 4 :- m_1 and m_2 are complex conjugate of each other.

$$m_1 = \alpha + i\beta$$
$$m_2 = \alpha - i\beta$$

So
$$\boxed{y = e^{\alpha x}[A \cos\beta x + B \sin\beta x]}$$
where $A = C_1 + C_2$
$B = i(C_1 - C_2)$

{This result found by putting the value of m_1 & m_2 in previous equation.

Ex- Particle in 1-D box -

$$V(x) = 0 \quad 0 < x < a$$
$$= \infty \quad \text{otherwise.}$$

$0 < x < a$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + K^2\psi = 0$$

$$\psi = c e^{mx} \Rightarrow m^2 + K^2 = 0$$
$$m = \pm iK.$$

Sol^n = $\psi = e^{\alpha x}[A \cos\beta x + B \sin\beta x]$

$$\psi = A \cos Kx + B \sin Kx$$

Here $\alpha = 0$ and $m_1 = iK$
$\beta = K$ $\quad m_2 = -iK$

roots are complex conjugat

Q.9 Solve the Differential equations: (Homogeneous type):-

(ii) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$ given that $y(0) = 2$, $y'(0) = 1$

Sol^n $\downarrow$ $\downarrow$ $\downarrow$ $y=1$

$$m^2 + 6m + 13 = 0$$

$$m = \frac{-6 \pm \sqrt{36 - 4\times1\times13}}{2\times1} = \frac{-6 \pm \sqrt{36-52}}{2}$$

$$\boxed{m = -3 \pm 2i} \qquad \alpha = -3, \ \beta = 2$$

So $y = e^{-3x}[A\cos 2x + B\sin 2x]$

$\because$ $y(0) = 2 \Rightarrow x = 0, \ y = 2$

$$2 = e^0[A\cos 0 + B\sin 0]$$

$$\boxed{2 = A}$$

$$\frac{dy}{dx} = -3e^{-3x}[-2A\sin 2x + 2B\cos 2x]$$

$\because$ $\left.\frac{dy}{dx}\right|_{x=0} = 1$

So $1 = -3e^0[-2A\sin 0 + 2B\cos 0]$

$$1 = -3\times 2B$$

$$\boxed{B = -\frac{1}{6}}$$

So $\boxed{y = e^{-3x}\left[2\cos 2x - \frac{1}{6}\sin 2x\right]}$ Ans

$\cancel{\frac{dy}{dx}}$ $y = \underset{\mathrm{I}}{e^{-3x}}\,\underset{\mathrm{II}}{2\cos 2x} + \underset{\mathrm{I}}{e^{-3x}}\,\underset{\mathrm{II}}{B\sin 2x}$

$$\frac{dy}{dx} = -2e^{-3x}\sin 2x\times 2 + 2\cos 2x.\,e^{-3x}.(-3) + e^{-3x}.\,B\cos 2x\,(2) + B\sin 2x\, e^{-3x}(-3)$$

$$\left.\frac{dy}{dx}\right]_{x=0} = 1$$

$$1 = 0 + 2\times(-3) + 2B$$

$$\boxed{B = \frac{7}{2}}$$

So $\boxed{y = e^{-3x}\left[2\cos 2x + \frac{7}{2}\sin 2x\right]}$ Ans

Q. 61

Solⁿ :-

$$\frac{d^2 f}{dx^2} - (3-4i)f = 0$$

$$\Rightarrow m^2 - (3-4i) = 0$$

$$\Rightarrow m = \sqrt{3-4i}$$

$$= \sqrt{4-1-4i}$$

$$= \sqrt{(2)^2 + (i)^2 - 2.2.i}$$

$$= \sqrt{(2-i)^2} = \pm(2-i)$$

So $m \Rightarrow 2-i, -2+i$

$$\boxed{m = \pm(2-i)}$$

Here m_1 and m_2 are Complex but not Complex Conjugate due to which in book P and Q are real but here it is Complex so this problem arises.

$$y = C_1 e^{(2-i)x} + C_2 e^{-(2-i)x}$$

At $x=0$, $f=1$

$$\boxed{1 = C_1 + C_2} \quad \text{—— (1)}$$

As $x \to \infty$ $f(x) \to 0$

$$\Rightarrow 0 = C_1 \underset{\substack{\downarrow \\ 0}}{(\infty)} + C_2(0)$$

$$\Rightarrow f(x) = e^{-(2-i)x}$$

$$\Rightarrow f(\pi) = e^{-(2-i)\pi}$$

$$f(\pi) = e^{-2\pi} . e^{i\pi}$$

$$\boxed{f(\pi) = -e^{-2\pi}}$$

$\{ \because e^{i\pi} = -1 \}$

Note:-
Here does not mean $0 \times \infty = 0$
Here we let $C_1 = 0$ becoz in $C_1 = 0$ then $C_1 e^{(2-i)x}$ is 0. We over-come the prob. of infinity, becoz when we put $x = 0$ $C_1 e^{(2-i)x}$ does not exist.

A-6 CSIR - June 2014

Q.64 Consider the differential equation -

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

with initial condition $x(0) = 0$ and $\dot{x}(0) = 1$. The solution $x(t)$ attains its maximum value when 't' is

(a) 1/2 (b) 1 (c) 2 (d) ∞

Sol^n

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

$x(0) = 0$

$\dot{x}(0) = 1 \Rightarrow \left.\frac{dx}{dt}\right|_{x=0} = 1$

$$m^2 + 2m + 1 = 0$$

$\boxed{m = -1, -1}$ real and equal.

$$x(t) = \underset{\mathrm{I}}{(C_1 + C_2 x)} \underset{\mathrm{II}}{e^{-t}}$$

$x = 0$, $t = 0$

$\boxed{C_1 = 0}$

$$x(t) = (0 + C_2 x) e^{-t}$$

$$x(t) = C_2 x e^{-t}$$

$$\dot{x}(t) = C_2[e^{-t} - t e^{-t}]$$

when $\dot{x}(0) = 1$

So $\boxed{C_2 = 1}$

$$\boxed{x(t) = te^{-t}}$$

For maxi^{m} $\frac{dx}{dt} = 0$

$\Rightarrow \boxed{t=1}$

A-6
Q.20

$$\frac{dy}{dx} - 2x = f(x) \quad [x \in R]$$
$$y(0) = 0$$

Where $f(x) = 0 \quad x \le 0$
$\quad\quad = 1 \quad x > 0$

$x \le 0 \quad y_1 = x^2 + c_1 \Rightarrow y_1(0) = 0 \Rightarrow \boxed{c_1 = 0}$

$\boxed{y_1 = x^2} \quad x \le 0$

$x > 0$: $y_2 = x^2 + x + c_2$

Continuity Conditions:-

$y_1(x=0) = y_2(x=0)$

$\Rightarrow \boxed{0 = c_2}$

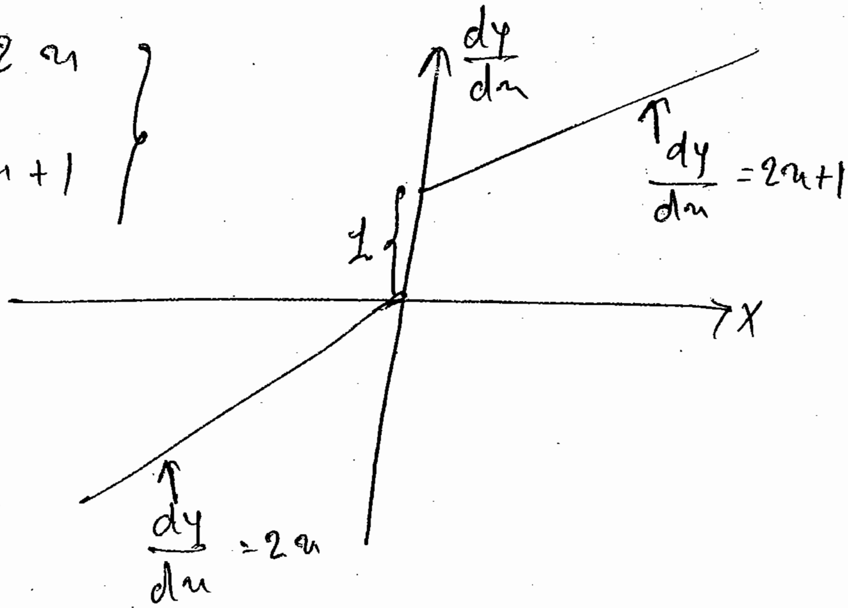
$x \le 0$: $\frac{dy_1}{dx} = 2x$

$x > 0$: $\frac{dy_2}{dx} = 2x+1$

$\left.\frac{dy_1}{dx}\right|_{x=0} = 0$

$\left.\frac{dy_2}{dx}\right|_{x=0} = 1$

Not Continuous at $x=0$
option (a).



A-6
Q-29

Sol^n $\frac{dy}{dx} = \frac{1}{1+x^2}$

Method I :- $\because \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ $x \le 1$

here $x = 1$

$y = \tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

$y = \tan^{-1}\tan\frac{\pi}{4}$

$\boxed{y = \frac{\pi}{4}}$ Since sum of this series is a finite quantity so series Converges to $\frac{\pi}{4}$.

Method II :- $\frac{dy}{dx} = \frac{1}{1+x^2}$

$\Rightarrow \int dy = \int \frac{dx}{1+x^2} \Rightarrow y = \tan^{-1}x + c$

$\Rightarrow dy = (1+x^2)^{-1}dx$

$= (1 - x^2 + x^4 - x^6 + \cdots)dx$

$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + c$

$\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots}$

$\Rightarrow y = \tan^{-1}x + c$

H-6

Q.23

Solⁿ $\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$

first we have to find $P(x,y)$

$\Rightarrow \frac{d^2x}{dt^2} = -\frac{dy}{dt} = -x$

$\Rightarrow \frac{d^2x}{dt^2} + x = 0$

$\Rightarrow m^2+1=0$

$\Rightarrow \boxed{m = \pm i}$

$\Rightarrow x(t) = A\cos t + B\sin t$

$\Rightarrow y(t) = A\sin t - B\cos t$

Squaring and adding -

$\boxed{x^2+y^2 = (A^2+B^2)}$ Trajectory of particle is Circle.

A-7

Q.9 Solve the D.E. (Homogeneous type):-

(IV) $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$

$m^3 - 2m^2 + 4m - 8 = 0$

$\Rightarrow m^2(m-2) + 4(m-2) = 0$

$\Rightarrow (m-2)(m^2+4) = 0$

$\Rightarrow m = \underset{\text{Real}}{2}, \pm 2i \leftarrow$ Complex.

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

$y = C_1 e^{2x} + C_2 e^{2ix} + C_3 e^{-2ix}$

$$y(\infty) = C_1(\infty) + C_2(\infty) + C_3(0)$$

$$C_1 \to 0 \qquad C_2 \to 0$$

So C_1 and C_2 must be zero for existing this condition. when C_1 and C_2 is 0 so $C_1 e^{2x}$ and $C_2 e^{2ix}$ does not exist.

So $y = C_3 e^{-2ix}$

$$y' = -2i \; e^{-2ix} C_3$$

$$2i = (-2i) C_3$$

$$\boxed{C_3 = -1}$$

So $\boxed{y = -e^{-2ix}}$

A-6

Q.26 :- If $y = e^{2t}$ is a solution to $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + ky = 0$ what is the value of K?

Sol^n

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dt} + ky = 0$$

$$\Rightarrow 4e^{2t} - 5.2e^{2t} + ke^{2t} = 0$$

$$\Rightarrow (K-6)e^{2t} = 0$$

$$\Rightarrow \boxed{K = 6}$$

A-6

Q.35 Which one of the following differential equations represents all circles with radius 'a'?

Sol^n

Assume centre $= (h, k)$

given radius $= a =$ Constant.

So ~~we~~ ∵ in a eq^n of circle we have only 2 parameter - radius and center ∵ here radius is constant so to get a family of circle we can only vary center (h, k) ∴ Here two parameters are variable. So order of differential equation is 2.

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2$$

$$\Rightarrow \quad \cancel{2}(x-h) + \cancel{2}(y-k)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x-h) + (y-k)\frac{dy}{dx} = 0$$

Again −

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-k)\frac{d^2y}{dx^2} = 0$$

$$(y-k) = -\left\{\frac{1+\left(dy/dx\right)^2}{\left(d^2y/dx^2\right)}\right\} \quad \text{— (I)}$$

$$(x-h) = -(y-k)\frac{dy}{dx} \quad \text{— (II)}$$

$$(x-h) = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\left(\frac{d^2y}{dx^2}\right)}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 \Big/ \left(\frac{d^2y}{dx^2}\right)^2 - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}\cancel{\frac{dy}{dx}} = a^2$$

$$\cancel{\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right] \Big/ \frac{d^2y}{dx^2}\left[1-\frac{dy}{dx}\right] = 0}$$

$$\cancel{\left(1-\frac{dy}{dx}\right) = \frac{1+\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)} \quad \frac{\left(\frac{d^2y}{dx^2}\right)}{\left(1+\left(\frac{dy}{dx}\right)^2\right)}}$$

$$\cancel{\left(1-\frac{dy}{dx}\right)\left(1+\left(\frac{dy}{dx}\right)^2\right) = \frac{d^2y}{dx^2}}$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\boxed{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2}$$

* Second order Linear D.E. (Non - Homogeneous):-

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x)}$$

function of x or Constant.

Total solution = Complementary function (C.F.) + Particular Integral (P.I.)

Same as the sol^n of Corresponding homogeneous D.E.

$$\underbrace{(D^2 + P.D + Q)}_{f(D)\ =\ \text{function of differential operator}}\, y = R(x)$$

<u>Case I</u> :- $R(x) = e^{ax}$

$$\boxed{P.I. = \frac{1}{f(D)} e^{ax}}$$

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

if $f(a) = 0$

then $$\boxed{P.I. = x \frac{1}{f'(a)} \cdot e^{ax}}$$

If $f'(a) = 0$

$$\boxed{P.I. = x^2 \frac{1}{f''(a)} e^{ax}}$$

A-7

Q.10 Solve the D.E. (Non-Homogeneous type)

(i) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^x \cosh 2x$

Sol^n

$$(D^2 - D - 6)y = e^x \cosh 2x$$

$\cancel{C.F.}$ $\Rightarrow$ $D^2 - 3D + 2D - 6 = 0$

$$D(D-3) + 2(D-3) = 0$$

$$(D-3)(D+2) = 0$$

$$D = 3, -2$$

$$\boxed{C.F. = c_1 e^{3x} + c_2 e^{-2x}}$$

for P.I :- $\cancel{P.I.}$ $\Rightarrow$ $e^x \cosh 2x = e^x\left(\frac{e^{2x} + e^{-2x}}{2}\right) = \frac{1}{2}\left(e^{3x} + e^{-x}\right)$

$\cancel{P.I.}$ $\Rightarrow$ $\frac{1}{2} e^{3x} + \frac{1}{2} e^{-x}$

So $P.I. = \left(\frac{1}{D^2 - D - 6}\right)\left(\frac{1}{2} e^{3x} + \frac{e^{-x}}{2}\right)$

$$= \frac{1}{2}\left[\frac{1}{D^2 - D - 6} e^{3x} + \frac{1}{D^2 - D - 6} e^{-x}\right]$$

$$= \frac{1}{2}\left[x \cdot \frac{1}{2D - 1} e^{3x} + \frac{1}{4} e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{5} e^{3x} - \frac{1}{4} e^{-x}\right] = \frac{x}{10} e^{3x} - \frac{1}{8} e^{-x}$$

So $\boxed{T.S. = c_1 e^{3x} + c_2 e^{-x} + \frac{x}{10} e^{3x} - \frac{1}{8} e^{-x}}$ Ans

Case II : If $R(x)$ = Polynomial function of x

$$\boxed{P.I. = \frac{1}{f(D)}[\text{Polynomial of } x]}$$

$\frac{1}{f(D)}$ ⟶ expand this into series.

$$\boxed{P.I. = \underbrace{(1 + D + 2D^2 + 3D^3 + \cdots)}_{0}\ \underbrace{[R(x)]}_{x^2}}$$

A-7

Q. 10 (iii) $\quad 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = x^2 - 2x$

Sol^n $\quad$ A.E. $= (2D^2 + 3D + 4)y = x^2 - 2x$

$$2D^2 + 3D + 4 = 0$$

$$D = \frac{-3 \pm \sqrt{9 - 4\times 2\times 4}}{2\times 2} = \frac{-3 \pm \sqrt{9-32}}{4} = \frac{-3 \pm i\sqrt{23}}{4}$$

$$D = \frac{-3 + i\sqrt{23}}{4}, \quad \frac{-3 - i\sqrt{23}}{4} \qquad \text{Here } \alpha = \frac{-3}{4}, \beta = \frac{\sqrt{23}}{4}$$

So $\boxed{C.F. = e^{-\frac{3}{4}x}\left(C_1 \cos\frac{\sqrt{23}}{4}x + C_2 \sin\frac{\sqrt{23}}{4}x\right)}$

$$P.I = \frac{1}{2D^2 + 3D + 4}(x^2 - 2x)$$

$$= \frac{1}{4\left[1 + \frac{2D^2 + 3D}{4}\right]}(x^2 - 2x)$$

$$= \frac{1}{4}\left[1 + \frac{2D^2 + 3D}{4}\right]^{-1}(x^2 - 2x)$$

$$\cancel{= \frac{1}{4}\left[1 - \left(\frac{2D^2 + 3D}{4}\right)\right](x^2 - 2x)}$$

$$\cancel{= \frac{1}{4}\left[(x^2 - 2x) - \left(\frac{2D^2 + 3D}{4}\right)(x^2 - 2x)\right]}$$

$D(x^2 - 2x) = 2x - 2$

$D^2(x^2 - 2x) = 2$

$D^3(x^2 - 2x) = 0$

and higher order term $= 0$

$$= \frac{1}{4}\left[1 - \frac{2D^2+3D}{4} + \frac{(2D^2+3D)^2}{16} - \cdots\right](x^2-2x)$$

$$= \frac{1}{4}\left[(x^2-2x) - \frac{1}{4}(4+6x+6) + \frac{1}{16}18\right]$$

$$= \frac{1}{4}\left[x^2-2x-1+\frac{3\cancel{6}x}{\cancel{4}_2}+\frac{\cancel{6}}{\cancel{4}}+\frac{18}{16}\right]$$

$$= \frac{x^2-2x}{4} - \frac{3x-1}{8} + \frac{18}{16}$$

$$= \frac{16x^2-32x-24x+8+18}{64} = \frac{16x^2-56x+26}{64}$$

$$= \frac{\cancel{2}(8x^2-28x+13)}{\cancel{64}\ 32} = \frac{8x^2-28x+13}{32}$$

$$\boxed{P.I. = \frac{8x^2-28x+13}{32}}$$

T.S. = C.F. + P.I.

So $$\boxed{y = e^{-\frac{3}{4}x}\left(C_1 \cos \frac{\sqrt{23}}{4}x + C_2 \sin\frac{\sqrt{23}}{4}x\right) + \frac{1}{32}(8x^2-28x+13)}$$

Ans

*Case III If $R(x) = \cos ax$ / $\sin ax$ then

$$\boxed{P.I. = \left(\frac{1}{f(D^2)}\right)\cos ax} \text{ or } \boxed{P.I. = \left(\frac{1}{f(D^2)}\right)\sin ax}$$

$$\boxed{P.I. = \frac{1}{f(-a^2)}\cos ax}$$

If $f(-a^2) = 0$ then

$$\boxed{P.I. = x\frac{1}{f'(-a^2)}\cos ax}$$

H-T

Q.10(V) $\frac{d^2y}{dx^2}+4y = 2\cos x\cos 3x$

$(D^2+4)y = 2\cos x\cos 3x$

$D^2+4=0 \qquad \alpha=0,\ \beta=2$

$D = \pm 2i$

So $y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

$A = C_1+C_2,\ B = i(C_1-C_2)$

$\boxed{2\cos A\cos B = \cos(A+B)+\cos(A-B)}$

$\boxed{C.F. = C_1\cos 2x + C_2\sin 2x}$

$P.I. = \frac{1}{(D^2+4)}2\cos x\cos 3x = \frac{1}{(D^2+4)}\cos 4x + \cos 2x$

$= \frac{1}{(D^2+4)}\cos 4x + \frac{1}{(D^2+4)}\cos 2x$

$= \frac{1}{(-16+4)}\cos 4x + x\frac{1}{2D}\cos 2x$

$= \frac{-1}{12}\cos 4x + \frac{x}{2}\frac{\sin 2x}{2}$

$P.I. = -\frac{1}{12}\cos 4x + \frac{x}{4}\sin 2x$

$$\boxed{T.S. = \underbrace{C_1\cos 2x + C_2\sin 2x}_{C.F.} \underbrace{-\frac{1}{12}\cos 4x + \frac{x}{4}\sin 2x}_{P.I}}$$

10(Vi) $\frac{d^3y}{dx^3}+y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$

$\because\ 2\cos^2\frac{\theta}{2} = 1+\cos\theta$

$\therefore\ \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$

So $\frac{d^3y}{dx^3}+y = \frac{1}{2}+\frac{1}{2}\cos^2\left(\frac{x}{2}\right)+e^{-x}$

$$\frac{1}{(D^3+1)}\left(\frac{1}{2}x^{0}+\frac{\cos x}{2}+e^{-x}\right)$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{(D^3+1)}e^{0x}\right]+\frac{1}{2}\left[\left(\frac{1}{(-1)D+1}\right)\cos x\right]+\frac{1}{(D^3+1)}e^{-x} \quad \text{—(1)}$$

$$\Rightarrow \frac{1}{(-1)D+1}\cos x = \frac{1}{1-D}\cos x = \frac{1+D}{1-D^2}\cos x \quad \begin{cases}\text{multi-}\\ \text{ply by}\\ 1+D\end{cases}$$

$$= \frac{1}{2}\left(\frac{1+D}{2}\right)\cos x$$

$$= \frac{1}{4}(\cos x - \sin x)$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{D^3+1}\right)e^{0x} = \frac{1}{2}\left(\frac{1}{0+1}\right) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{(D^3+1)}e^{-x} = \frac{1}{3D^2}e^{-x} = x\frac{1}{3}e^{-x}$$

for C.F. ⇒

Auxileary Equation $D^3+1=0$

when $(D+1)=0$ $\cancel{D^3/z}$ $\quad (D+1)(D^2-D+1)=0$

the $D=-1$ or when $D^2-D+1=0$

then $D = \dfrac{+1\pm\sqrt{1^2-4\times1\times1}}{2\times1}$

$\boxed{D = \dfrac{1\pm i\sqrt{3}}{2}}$ roots are complex Conjugate.

So $\boxed{C.F. = c_1 e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)}$

T.S. = C.F. + P.I.

So $\boxed{y = c_1 e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{2} + \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}xe^{-x}}$

Case IV :- If $R(x) = e^{ax}\phi(x)$

Polynomial of x or $\cos ax / \sin ax$

then
$$\boxed{P.I. = \frac{1}{f(D)}\left[e^{ax}\phi(x)\right]}$$

$$\boxed{P.I. = e^{ax}\frac{1}{f(D+a)}\left[\phi(x)\right]}$$

A-7

Q.10 (vii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^3 e^{2x}$

Sol^n A.E. $= (D^2 - 4D + 4)y = x^3 e^{2x}$

$D^2 - 4D + 4 = 0$

$D^2 - 2D - 2D + 4 = 0$

$D(D-2) - 2(D-2) = 0$ $\Rightarrow (D-2)(D-2) = 0$

$D = 2, 2$ Roots are same.

$$\boxed{C.F. = (C_1 + C_2 x)e^{2x}}$$

$$P.I. = \frac{1}{D^2 - 4D + 4}\, x^3 e^{2x}$$

$$= \frac{1}{(D-2)^2}(x^3 . e^{2x})$$

$$= e^{2x}\frac{1}{D^2}(x^3)$$

$$\boxed{P.I. = e^{2x} . \frac{x^5}{20}}$$

$$\boxed{T.S. = (C_1 + C_2 x)e^{2x} + e^{2x} . \frac{x^5}{20}}$$ Ans

M-6

Q.34 The particular integral of $\frac{d^4y}{dx^4} - y = \cos x \cosh x$ is?

Sol^n $\quad \frac{d^4y}{dx^4} - y = \cos x \cosh x = \underbrace{\frac{1}{2} e^{x}\cos x}_{} + \underbrace{\frac{1}{2} e^{-x}\cos x}_{}$

$$\boxed{\begin{aligned}(1+x)^4 &= 1+4x+6x^2+4x^3+x^4\\ (1-x)^4 &= 1-4x+6x^2-4x^3+x^4\end{aligned}}$$

$$\Rightarrow \frac{1}{2}\frac{1}{(D^4-1)}\left(e^{x}\cos x\right)$$

$$\Rightarrow \frac{1}{2} e^{x} \frac{1}{(D+1)^4-1} \cos x$$

$$\Rightarrow \frac{1}{2} e^{x} \frac{1}{(\cancel{1}+4D+6D^2+4D^3+D^4-\cancel{1})} \cos x$$

$$\Rightarrow \frac{1}{2} e^{x} \frac{1}{(-1)^2+\cancel{4D(-1)}+6(-1)+\cancel{4D}} \cos x$$

$$\Rightarrow \frac{1}{2} e^{x}\left(\frac{-1}{5}\cos x\right) = \boxed{-\frac{1}{10} e^{x}\cos x}$$

$$\Rightarrow \frac{1}{2} e^{-x}\cos x = \frac{1}{2}\frac{1}{(D^4-1)}\left(e^{-x}\cos x\right)$$

$$\Rightarrow \frac{1}{2} e^{-x} \frac{1}{(D-1)^4-1} \cos x$$

$$\Rightarrow \frac{1}{2} e^{-x} \frac{1}{(\cancel{1}-4D+6D^2-4D^3+D^4)-\cancel{1}} (\cos x)$$

$$\Rightarrow \frac{1}{2} e^{-x} \frac{1}{\cancel{-4D}+6(-1)\cancel{-4D(-1)}+(-1)^2} \cos x$$

~~$\Rightarrow \frac{1}{2} e^{-x} \frac{1}{3+8D} \cos x$~~

~~$\Rightarrow \frac{1}{2} e^{-x} \frac{1}{(3+8D)}\frac{(3-8D)}{(3-8D)} \cos x$~~

$$\cancel{= -\frac{1}{2} e^{x} \cdot \frac{(3-8D)\cos x}{(9-64D^2)}}$$

$$\cancel{= -\frac{1}{2} e^{x} \cdot \frac{(3-8D)\cos x}{9-64(-1)} = -\frac{e^{x}}{2}\frac{(3-8D)\cos x}{73}}$$

$$\cancel{= -\frac{1}{2} e^{x}\left[\frac{3}{73}\cos x + \frac{8\sin x}{73}\right]}$$

So $+\frac{1}{2} e^{-x}\cos x = \frac{1}{2} e^{-x}\left(-\frac{1}{5}\cos x\right)$

So T.S. $= -\frac{1}{10} e^{x}\cos x - \frac{1}{10} e^{-x}\cos x$

$$= -\frac{1}{5}\cos x\left(\frac{e^{x}+e^{-x}}{2}\right) = -\frac{1}{5}\cos x \cosh x$$

$$\boxed{y = -\frac{1}{5}\cosh x \cos x}$$

A-6

Q.40

Sol^n

$$\frac{d^2y}{dt^2} + y = \sin t \qquad y(0) = 0$$

Auxileary Equation :-

$$(D^2+1)y = \sin t$$

$$D^2+1 = 0$$

$$D = \pm i$$

$\alpha = 0, \beta = 1$

$$\boxed{C.F = A\cos t + B\sin t}$$

$$P.I. = \frac{1}{(D^2+1)}\sin t$$

$$= \frac{+t}{2(D)} \sin t$$

$$\boxed{P.I. = -\frac{t}{2}\cos t}$$

Total Solution = C.F. + P.I.

$$y(t) = A\cos t + B\sin t - \frac{t}{2}\cos t$$

initial condition :- when $t=0$, $y=0$

$$0 = \cancel{y(0)} = A + 0 - 0$$

$$\boxed{A = 0}$$

So $$\boxed{y(t) = B\sin t - \frac{t}{2}\cos t}$$

A-6

Q.88 The solution of differential eq^n $yy' + y^2 - x = 0$ where c is a constant is ?

Sol^n

$$y\frac{dy}{dx} + y^2 = x$$

$y^2 = z$

$$2y\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{1}{2}\frac{dz}{dx} + z = x$$

$$\Rightarrow \frac{dz}{dx} + 2z = 2x$$

$$\Rightarrow (D+2)z = 2x$$

$$D+2 = 0$$

$$D = -2$$

So $\boxed{C.F. = ce^{-2x}}$

$$P.I. = \frac{1 \times 2x}{(D+2)} = \frac{\cancel{2}x}{\cancel{2}(1+\frac{D}{2})} = (1+\frac{D}{2})^{-1}x$$

$$= (1 - \frac{D}{2})x$$

$$= x - \frac{Dx}{2} = \left[x - \frac{1}{2}\right]$$

$\boxed{P.I. = x - \frac{1}{2}}$

Total Solution = C.F. + P.I.

$$= ce^{-2x} + x - \frac{1}{2}$$

$\boxed{\text{Total Solution} = ce^{-2x} + x - \frac{1}{2}}$ Ans

H-6 Q(3) The half-life of a radioactive substance is the time requir for one-half of that substance to decay. The amount of ^{14}C, an isotope of Carbon present at a future time t (in months) is given by $A(t) = 100\exp[-0.0338t]$. The half-life of the material in monts is-

Sol^n $$A(t) = 100\exp(-0.0338t)$$

$$\frac{dA}{dt} \propto A \Rightarrow A(t) = A_0\exp(-\lambda t) \quad \text{So } \lambda = 0.0338$$

$$T_{1/2} = \frac{\ln 2}{\lambda}$$

$$\Rightarrow \boxed{T_{1/2} = \frac{\ln 2}{0.0338}} \text{ Ans}$$

Q.7 The size of a population 'P' is modeled by the differential equation $\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$. For what values of P, Population is increasing? -

Sol^n $$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

$$\frac{dP}{dt} = +ve$$

$$\boxed{0 < P < 4200} \text{ Ans}$$

$$P\left(1 - \frac{P}{4200}\right) > 0$$

$$P > 0, \left(1 - \frac{P}{4200}\right) > 0 \Rightarrow P < 4200$$

$$P < 0, \quad \left(1 - \frac{P}{4200}\right) < 0$$

$$\boxed{0 < P < 4200} \text{ Ans}$$

option (b) is correct.

(28) The general solution to $\frac{d^2z}{dt^2}+6\frac{dz}{dt}+9z=0$ is z. Which of the following ~~options is~~ options are correct.?

Sol^n

$$\frac{\partial^2 z}{\partial t^2}+6\frac{\partial z}{\partial t}+9z=0$$

$$\Rightarrow m^2+6m+9=0$$

$$\Rightarrow m=-3,-3$$

$$z=(A+Bt)e^{-3t}$$

as $\boxed{t\to\infty \Rightarrow z\to 0}$

(a) As $t\to\infty$, $z\to A$ for any value of B

(b) The behaviour of z as $t\to\infty$ depends on A, B

(c) ✓ As $t\to\infty$, $z\to 0$ for any value of A and B.

(d) As $t\to\infty$, $z\to\infty$ for any value of A and B.

So option c is correct.

A-6

Q.33 The particular integral of differential equation $y''+y'+3y=5\cos(2x+3)$ is ?

Sol^n

$$y''+y'+3y=5\cos(2x+3)$$

$(Ax+B)$

$$P.I.=\frac{1}{D^2+D+3}5\cos(2x+3)=\frac{5\times 1}{-4+D+3}\cos(2x+3)$$

$$=5\frac{1}{(D-1)}\cos(2x+3)$$

$$=5\frac{(D+1)}{(D^2-1)}\cos(2x+3)=\frac{5(D+1)\cos(2x+3)}{(-4-1)}$$

$$=\frac{\cancel{5}(D+1)\cos(2x+3)}{-\cancel{5}}$$

$$=-(D+1)\cos(2x+3)$$

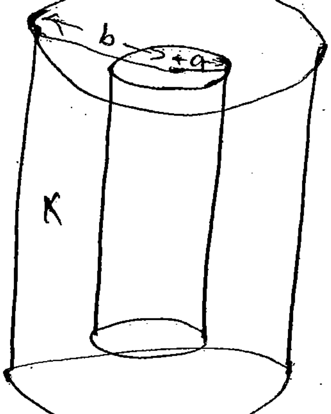
$$\boxed{P.I.=2\sin(2x+3)-\cos(2x+3)}$$

A

A-7 Q.8 In one-dimensional steady state heat conduction for a hollow cylinder with thermal conductivity k in the region $a \le r \le b$, the ~~therm~~ temp. T_r at a distance $r\,(a \le r \le b)$ is given by $\frac{d}{dr}\left[r\frac{dT_r}{dr}\right]=0$, subjected to the condition $T_r = T_1$ at $r=a$, $T_r = T_2$ at $r=b$. Determine the steady state temperature distribution T_r?

Sol^n.

$a \le r \le b$



$$\frac{d}{dr}\left[r\frac{dT_r}{dr}\right]=0$$

$$\Rightarrow r\frac{dT_r}{dr} = C_1$$

$$\Rightarrow \int dT_r = C_1\int\frac{dr}{r}$$

$$\Rightarrow \boxed{T_r = C_1 \ln r + C_2} \quad (a)$$

at $r=a$, $T_r = T_1$

at $r=b$, $T_r = T_2$

$$T_1 = C_1 \ln a + C_2$$
$$-T_2 = -C_1 \ln b \pm C_2$$
$$T_1 - T_2 = C_1 \ln\left(\frac{a}{b}\right)$$

$$\boxed{C_1 = \frac{(T_1 - T_2)}{\ln\left(\frac{a}{b}\right)}}$$

$$C_2 = T_1 - \frac{(T_1 - T_2)}{\ln\left(\frac{a}{b}\right)}\ln a$$

$$C_2 = \frac{\ln\left(\frac{a}{b}\right)T_1 - (T_1 - T_2)\ln a}{\ln\left(\frac{a}{b}\right)}$$

$$= \frac{T_1\cancel{\ln a} - T_1\ln b - T_1\cancel{\ln a} + T_2\ln a}{\ln\left(\frac{a}{b}\right)}$$

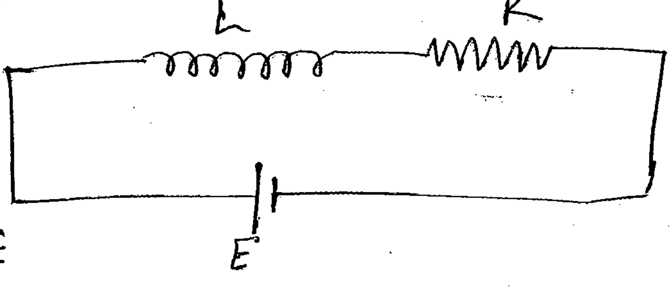
$$C_2 = \frac{T_2\ln a - T_1\ln b}{\ln\left(\frac{a}{b}\right)}$$

So from (a)

$$T_r = \frac{(T_1 - T_2)}{\ln\left(\frac{a}{b}\right)}\ln r + \frac{T_2\ln a - T_1\ln b}{\ln\left(\frac{a}{b}\right)}$$

A-7 (5) A inductance 'L' and resistance 'R' are connected in series with e.m.f. 'E'. If the current is zero at time $t=0$, find the current as a function of time. Also find the current at the end of 0.01 sec. If $E = 100$ Volts, $L = 100$ H, $R = 20$ ohms.

Sol^n

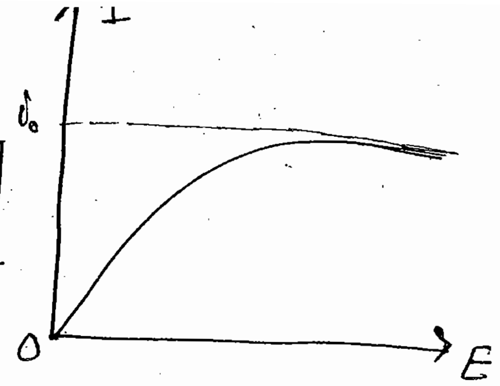


$$L\frac{di}{dt} + iR = E$$

$$\Rightarrow \frac{di}{dt} + i\frac{R}{L} = \frac{E}{L}$$

at $t=0$, $i=0$

$$\Rightarrow \boxed{i(t) = i_0(1-e^{-R/L\,t})}$$



$$i_0 = \frac{E}{R}$$

So $$\boxed{i(t) = \frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)}$$

A-7

Q.10 Solve the D.E. (Non-homogeneous type):-

(ix) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$

$D^2 - 2D + 1 = 0$

$D^2 - D - D + 1 = 0$

$D(D-1) - 1(D-1) = 0$

$(D-1)(D-1) = 0$

$D = 1, 1$ Roots are Same.

$$\boxed{C.F. = (C_1 + C_2 x)e^x}$$

P.I. $= \frac{1}{D^2 - 2D + 1} x\sin x$ $\qquad$ $R(x) = e^{ax}\phi(x)$

$= \cancel{I.P. \text{ of } \frac{1}{D^2 - 2D + 1}(x e^{ix})}$

$= I.P.$ of $\left[e^{ix} \frac{1}{(D+i)^2 - 2(D+i) + 1} (x) \right]$

$$= \text{I.P. of } \left[e^{ix} \frac{1}{D^2 + 2Di - \cancel{1} - 2D - 2i + \cancel{1}} x \right]$$

$$= \text{I.P. of } \left[e^{ix} \frac{1}{-2i\left(1 - \frac{D^2 + 2Di + 2D}{2i}\right)} \right] x$$

$$= \text{I.P. of } \left[e^{ix}\left(\frac{-1}{2i}\right) \left(1 - \frac{D^2 - 2Di + 2D}{2i}\right)^{-1} x \right]$$

$$= \text{I.P. of } \left[e^{ix}\left(\frac{-1}{2i}\right) \left(1 + \frac{D^2 + 2Di - 2D}{2i}\right) x \right]$$

$$= \text{I.P. of } \left[e^{ix}\left(\frac{-1}{2i}\right) \left(x + 0 + 1 - \frac{2}{2i}\right) \right]$$

$$= \text{I.P. of } \left[-\frac{x e^{ix}}{2i} + \frac{e^{ix}}{2i} + \frac{e^{ix}}{2i^2} \right]$$

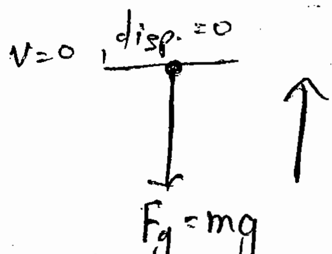
$$= \text{I.P. of } \left[\frac{i^2 x e^{ix}}{2\cancel{i}} + \frac{i^2 e^{ix}}{2\cancel{i}} + \frac{e^{ix}}{2} \right]$$

$$= \text{I.P. of } \left[\frac{i x e^{ix}}{2} + \frac{i e^{ix}}{2} - \frac{e^{ix}}{2} \right]$$

$$\boxed{\text{P.I.} = \frac{1}{2} x \cos x + \frac{1}{2} \cos x - \frac{1}{2} \sin x}$$

Q.6 A particle falls freely under gravity from a great height, there being a resistance proportional to the velocity of fall. Find the velocity and displacement as function of time.

Sol^n



$F_R \propto v$

$\Rightarrow F_R = Kv$

Equation of motion:-

$$\Rightarrow m\frac{dv}{dt} = F_g - F_R = mg - kv$$

$$\Rightarrow \frac{dv}{dt} = g - \frac{K}{m}v$$

$$\Rightarrow \frac{dv}{dt} + \frac{K}{m}v = g$$

$$\Rightarrow v = v(t)$$

* Method of Variation of Parameters:-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x)$$

Like:- if m_1, m_2 are real and unequal.

$$y = c_1 \underbrace{e^{m_1 x}}_{y_1} + c_2 \underbrace{e^{m_2 x}}_{y_2}$$

$$\boxed{C.F. = Ay_1 + By_2}$$

y_1 & y_2 are two linearly Independent Solution of D.E.

if m_1, m_2 are real and equ
then -

$$y_1 = e^{mx}$$

$$y_2 = x e^{mx}$$

Two linearly independent sol^n of D.E.

$$\boxed{P.I. = uy_1 + vy_2}$$

Here u and v are function of x or Constants.

$$\boxed{u = \int \frac{-y_2 R(x)}{y_1 y_2' - y_2 y_1'}\,dx}$$

$$\boxed{v = \int \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'}\,dx}$$

A-7
Q.11

(iii) $\frac{d^2y}{dx^2} + y = \sec x \tan x$

Sol^n $m^2 + 1 = 0$

$m = \pm i$

So $C.F. = (A \underbrace{\cos x}_{y_1} + B \underbrace{\sin x}_{y_2})$

$\because$ $P.I. = uy_1 + vy_2$

$$y_1 y_2' - y_2 y_1' = \cos x \cdot \cos x - \sin x(-\sin x)$$
$$= \cos^2 x + \sin^2 x = 1$$

So $\boxed{y_1 y_2' - y_2 y_1' = 1}$

$$u = \int -\sin x \cdot \sec x \cdot \tan x \, dx$$
$$= -\int \frac{\sin^2 x}{\cos^2 x} dx = -\int \tan^2 x \, dx$$
$$= -\int (\sec^2 x - 1) dx$$
$$= -[\tan x - x]$$
$$\boxed{u = -\tan x + x}$$

$$V = \int \cos x \cdot \sec x \cdot \tan x \, dx$$
$$= \int \tan x \, dx$$
$$\boxed{V = \ln(\sec x)}$$

So $\boxed{P.I. = (-\tan x + x)\cos x + \ln(\sec x) \cdot \sin x}$

* Cauchy Euler Eqn :-

$$\boxed{a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = R(x)}$$

It is n^{th} order differential equation.

Second Order Cauchy - Euler Equation -

$$\boxed{a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = R(x)}$$

* Solution of C - E Equation :-

For the solution of second order C - E equation we take some standard substitution.

Put $\boxed{x = e^z}$

$$\Rightarrow z = \log_e x = \ln x$$

$$\Rightarrow \boxed{\frac{dz}{dx} = \frac{1}{x}}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

$$\Rightarrow \boxed{x \frac{dy}{dx} = \frac{dy}{dz}}$$

So

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x} \frac{dy}{dz}\right)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx}\left(\frac{dy}{dz}\right) \leftarrow F$$

$$\frac{dF}{dx} = \frac{dF}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dF}{dz}$$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dz}\left(\frac{dy}{dz}\right) \frac{1}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow \boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}}$$

$$\Rightarrow a_2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + a_1\frac{dy}{dz} + a_0y = R(e^z)$$

$$\Rightarrow \boxed{a_2\frac{d^2y}{dz^2} + (a_2-a_1)\frac{dy}{dz} + a_0y = R(e^z)}$$

linear D.E.

So, Here C-E Equation transform to second order linear differential Equation.

Q.12 Total radial displacement 'u' in a rotating disc at a distance 'r' from the axis is given by,

$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + kr^3 = 0$. Find the displacement 'u' as function of r Subject to the conditions $u(0) = 0$, $u(a) = 0$.

Sol^n

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0 \quad (K \text{ is Constant})$$

Substitution :- $r = e^z$

$$\Rightarrow z = \ln r$$

$$\Rightarrow \frac{dz}{dr} = \frac{1}{r}$$

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = kr^3$$

$$\Rightarrow \left(\frac{d^2y}{dz^2} - \cancel{\frac{dy}{dz}}\right) + \cancel{\frac{du}{dz}} - u = -ke^{3z}$$

$$\Rightarrow \boxed{\frac{d^2y}{dz^2} - u = -Ke^{3z}}$$

A.E. ~~C.F.~~ $D^2 - 1 = 0$

$D = \pm 1$

So $C.F. = C_1 e^{z} + C_2 e^{-z}$

$$\boxed{C.F. = C_1 r + \frac{C_2}{r}}$$

$P.I. = \frac{1}{(D^2-1)} k e^{-3z}$

$$\boxed{P.I. = \frac{k}{8} e^{-3z}}$$

So $\boxed{P.I. = \frac{-k}{8} r^3}$

$u(r) = C_1 r + \frac{C_2}{r} - \frac{k}{8} r^3$

$\because u(0) = 0 \Rightarrow 0 = C_1(0) + \frac{C_2}{0} \Rightarrow 0 - \frac{k}{8}(0)$

$u(r) = C_1 r - \frac{k}{8} r^3$

$u(a) = 0 \Rightarrow 0 = C_1 a - \frac{k}{8} a^3$

$\Rightarrow \boxed{C_1 = \frac{k a^2}{8}}$

So $u(r) = \frac{k}{8} a^2 . r + 0 - \frac{k}{8} r^3$

So $\boxed{u(r) = \frac{k}{8} r (a^2 - r^2)}$ Ans

* Coupled Defferential Equation :-

No. of dependent Variable $\Rightarrow$ Can be any thing
No. of independent variable = 1.

Let there is 2 dependent variable and one independent variable then -

$$\frac{dx}{dt} = ax + by + f_1(t)$$
$$\frac{dy}{dt} = cx + dy + f_2(t)$$

These are two standared representation of Couble differential equation.
"It is called Couple differential equation because rate of change of x depends upon y and rate of change of y depends upon x. Both are not independent! So it is called Coupled Differential Equation."
To eliminate one variable from two dependent variable, We transform above eq^{ns} as -

$\Rightarrow$ $Dx = ax + by + f_1(t)$

$\Rightarrow$ $Dy = Cx + dy + f_2(t)$

$$(D-a)x - by = f_1(t) \quad \text{—— (1)}$$
$$-Cx + (D-d)y = f_2(t) \quad \text{—— (2)}$$

Now eq^n (1) $\times (D-d)$ and eq^n (2) $\times b$ -

$$(D-d)(D-a)x - \cancel{(D-d)by} = (D-d)f_1(t)$$

$$-bcx + \cancel{(D-d)by} = bf_2(t)$$

$$\Rightarrow [(D-d)(D-a)x - bcx] = \underbrace{(D-d)f_1(t) + bf_2(t)}$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} - (a+d)\frac{dx}{dx} + (ad-bc)x = R(t)} \quad \left\{ \begin{array}{l} -dDx - a \\ -(a+d)D \end{array} \right.$$

It is second order linear non homogeneous Differential Equation.

$$\Rightarrow \boxed{x = x(t)}$$

$$\boxed{\frac{dx}{dt} = ax + by + f_1(t)}$$

---

* Short Method :-

say $\frac{dx}{dt} = ax + by + f_1(t)$ and

$\frac{dy}{dt} = cx + dy + f_2(t)$ are given.

then :-

$$\frac{d}{dx}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Transformation matrix
or Coupling matrix.

$$\boxed{\frac{d^2x}{dt^2} - \mathrm{Tr}[C]\frac{dx}{dt} + \mathrm{Det}[C]x = R(t)}$$

Where $R(t) = (D-d)f_1(t) + bf_2(t)$

H-+

Q.11 (vii) ⇒ $\frac{dx}{dt} + 2y + \sin t = 0$, $\frac{dy}{dt} - 2x - \cos t = 0$

Sol^n Here rate of change of x with t depends upon 'y' and rate of change of y with t depends upon 'x' So these are Coupled equation.

$$[C] = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$f_1(t) = -\sin t$, $f_2(t) = \cos t$

So $\frac{d^2x}{dt^2} + 4x = \left[\left(\frac{d}{dt} - 0\right)(-\sin t) - 2\cos t\right]$

⇒ $\boxed{\frac{d^2x}{dt^2} + 4x = -3\cos t}$

Linear second order non-homogeneous.

$D^2 + 4 = 0$

$D = \pm 2i$

$\boxed{C.F. = C_1 \cos 2t + C_2 \sin 2t}$

$P.I. = \frac{1}{(D^2+4)} - 3\cos t$

$\boxed{P.I. = -\cos t}$

So $\boxed{x = C_1 \cos 2t + C_2 \sin 2t - \cos t}$

the value of x puting in ①

$-2C_1 \sin 2t + 2C_2 \cos 2t + \sin t \qquad = -2y - \sin t$

$\Rightarrow -2y = -2C_1 \sin 2t + 2C_2 \cos 2t + 2\sin t$

$$\boxed{y = C_1 \sin 2t + C_2 \cos 2t - \sin t}$$

Ans

10(ix) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x \sin x$

P.I. $= \text{I.P.} \left(\dfrac{1}{D^2 - 2D + 1}\right)(x \cdot e^{ix})$

$= \text{I.P.} \left[e^{ix} \dfrac{1}{(D+i)^2 - 2(D+i) + 1} x \right]$

$= \text{I.P.} \left[\dfrac{1}{D^2 + 2iD + \cancel{i^2} - 2D + 2i + \cancel{1}} x \right]$

$= \text{I.P.} \left[\dfrac{1}{D^2 + 2D(i+1) - 2i} x \right]$

$= \text{I.P.} \left[\left(\dfrac{1}{-2i}\right) \dfrac{1}{1 - \dfrac{D^2 + 2D(i-1)}{2i}} \right] x$

$= \text{I.P.} \left[\dfrac{1}{(-2i)} \left[1 + \dfrac{D^2 + 2D(i-1)}{2i} + \cdots \right] x \right.$

$= \text{I.P.} \left[\dfrac{1}{(-2i)} \left[x + \dfrac{1}{\cancel{2}i} \cancel{2}(i-1) \right] \right] = -\dfrac{x}{2i} + \dfrac{i-1}{2}$

$= \text{I.P.} \left[e^{ix} \left\{ -\dfrac{x}{2i} + \dfrac{i-1}{2} \right\} \right]$

$= \text{I.P.} \left[(\cos x + i \sin x) \left(-\dfrac{x}{2i} + \dfrac{i-1}{2} \right) \right]$

$$= I.P.\left[i\frac{x}{2}\cos x - \frac{x}{2}\sin x + \frac{i}{2}\cos x - \frac{1}{2}\cos x - \frac{1}{2}\sin x - \frac{i}{2}\sin x \right]$$

$$\boxed{P.I. = \frac{x}{2}\cos x + \frac{1}{2}\cos x - \frac{1}{2}\sin x}$$

(iv) $$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

$$C.F. = c_1 \underbrace{e^x}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}$$

$$P.I. = u e^x + v e^{-x}$$

$$u = \int \cancel{-}\frac{e^{-x}}{\cancel{-2}} \cdot \frac{\cancel{2}}{1+e^x}\, dx \qquad \left\{ y_1 y_2' - y_2 y_1' = -2 \right.$$

$$= \int \frac{e^{-x}}{1+e^x}\, dx$$

$$= \int \frac{1}{e^x(e^x+1)}\, dx$$

$$= \int \frac{(e^x+1)-1}{e^x(e^x+1)}\, dx$$

$$= \int \frac{dx}{e^x} - \int \frac{dx}{e^x+1}$$

$$= -e^{-x} - \int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$\boxed{u = -e^{-x} + \ln(1+e^{-x})}$$

59

Sol^n

$$\sin x \frac{d^2y}{dx^2} + \cos x \frac{dy}{dx} = \delta\left(x - \frac{\pi}{2}\right), \quad \delta\left(x-\frac{\pi}{2}\right) = 0 \quad x \neq \frac{\pi}{2}$$

= Very high at $x = \frac{\pi}{2}$

$$\Rightarrow \frac{d}{dx}\left[\sin x \frac{dy}{dx}\right] = \delta\left(x - \frac{\pi}{2}\right)$$

$$\Rightarrow \int d\left[\sin x \frac{dy}{dx}\right] = \int \delta\left(x - \pi/2\right) dx$$

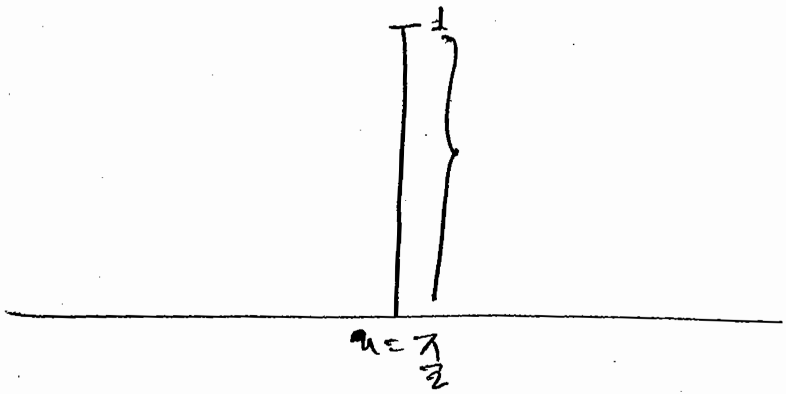
$$\Rightarrow \sin x \frac{dy}{dx} = \int \delta\left(x - \frac{\pi}{2}\right) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x} \int \delta\left(x - \frac{\pi}{2}\right) dx$$

$\Rightarrow$ at $x = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 1$ $\left\{ \int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1 \right.$

$\Rightarrow$ at $x \neq \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 0$

So function is discontinuous at $x = \frac{\pi}{2}$ having ammount of discontinuity is 1. So option (d) is correct.



*⇒ $f(x) = [x]$ → Box function

Where Box function is greatest integer not exceeding x.

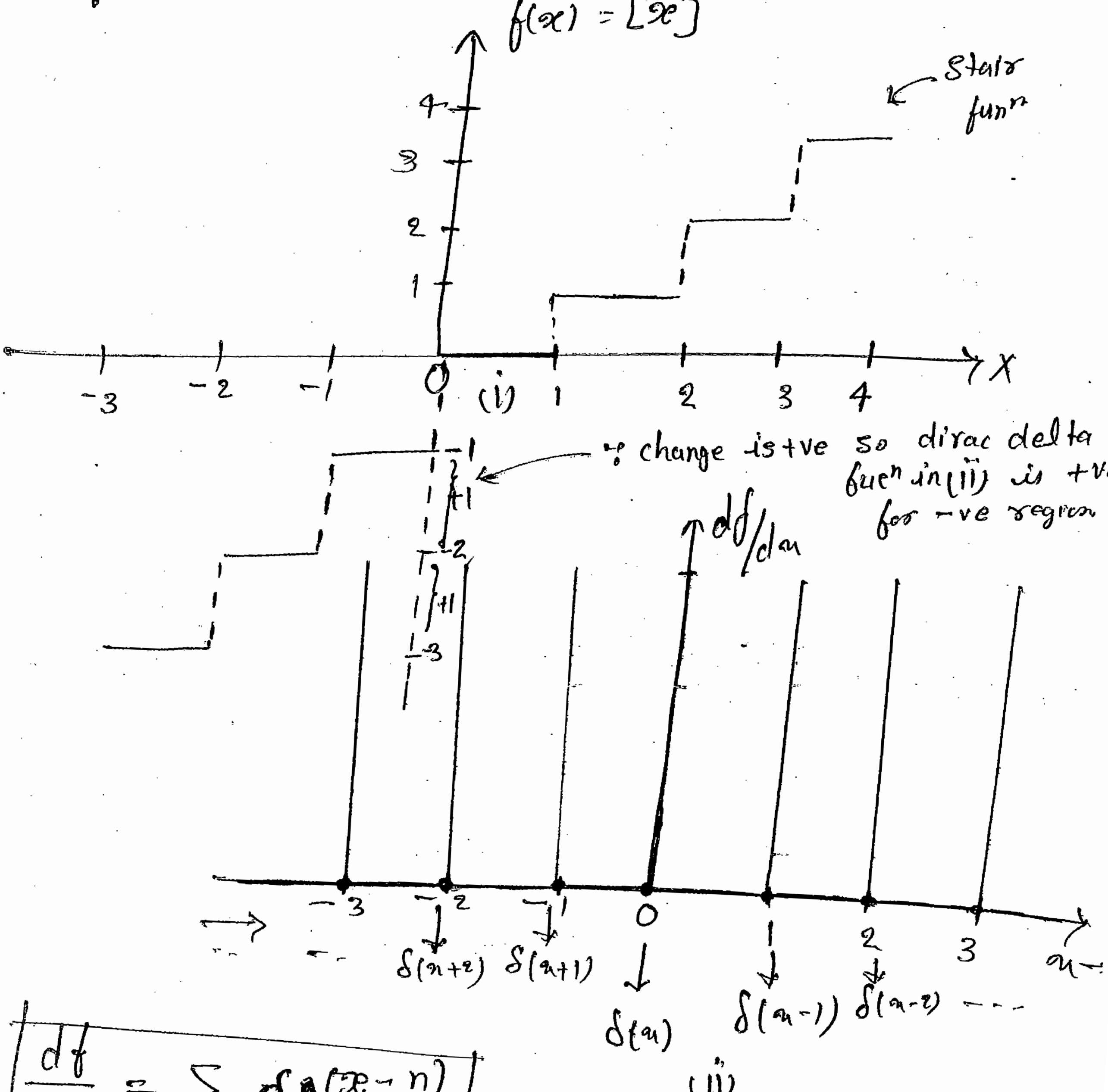
Ex- $[3.51] = 3$

$0 \le x \le 1 \Rightarrow [x] = 0$

$1 \le x < 2 \Rightarrow [x] = 1$

$-1 \le x < 0 \Rightarrow [x] = -1$

* Graph of Box function:-



$$\boxed{\frac{df}{dx} = \sum_{n = \text{integer}} \delta(x - n)}$$

If a function is discontinuous at $x = a$ then its derivative $f'(x)$ will be related to a dirac delta function at $x = a$.

So in previous question -

$$\sin x \frac{d^2y}{dx^2} + \cos x \frac{dy}{dx} = \delta\left(x - \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} \Rightarrow \delta\left(x - \frac{\pi}{2}\right)$$

$$\frac{dy}{dx} \Rightarrow \text{discontinuous at } x = \frac{\pi}{2}$$

highest order derivative

Q. $$\frac{dG}{dx} + k\,G(x) = \delta(x)$$

Related to

$$\frac{dG}{dx} \Rightarrow \delta(x)$$

So $G(x) \Rightarrow$ discontinuous at $x = 0$

In Quantum Mechanics :-

In general $\psi(x)$ and $\frac{d\psi}{dx} \Rightarrow$ Continuous

$\frac{d\psi}{dx} \longrightarrow$ discontinuous

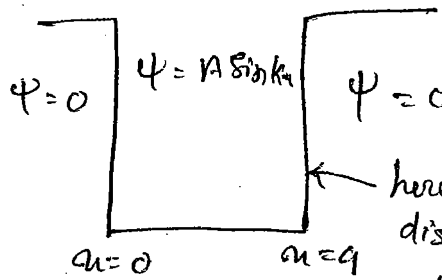
$V(x) \longrightarrow$ infinite discontinuity

or

$\delta(x-a)$

∴ Schrodinger eqⁿ:-

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\psi = 0$$



here infinite discontinuity in potential So at boundary $\frac{d\psi}{dx}$ is not continuous.

$$\therefore \rightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V(x) - E)\psi$$

dirac delta potential

$$\therefore \frac{d^2\psi}{dx^2} \Rightarrow \delta(x-a)$$

$$\therefore \frac{d\psi}{dx} \Rightarrow \text{discontinuous at } x = a$$

* If we have a second order differential equation then solution of this eqⁿ is 2 say it is y_1 and y_2.

⇒ If y_1 and y_2 is two linearly Independent solutions of a second order D.E. Then their linear combination $C_1y_1 + C_2y_2$ will also be a solution of same differential equation.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \begin{array}{l} = 0 \Rightarrow L.D \\ \neq 0 \Rightarrow L.I. \end{array}$$

Ex- $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

$$y = ce^{mx}$$

$$m^2 + Pm + Q = 0$$

$\Rightarrow \quad m = m_1, m_2$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\downarrow \qquad\qquad \downarrow$$

$$y_1 \qquad\qquad y_2$$

Here y_1 and y_2 are L.I. and both are sol^n of above eq^n. So their combination (y) is also a solution of same D.E.

If $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ is given and we calculate it's corresponding solution y_1 and y_2 and also we calculate W (Wronskian) which is not zero ($W \neq 0$) so y_1 and y_2 are Linearly independent then -

$$\boxed{W(x=b) = W(x=a)\, e^{-\int_a^b P\,dx}}$$

A-6
Q.58

Sol^n
$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + f(t)\,x = 0$$

$$\downarrow$$

$$x_1, x_2$$

$$W = x_1 \frac{dx_2}{dt} - x_2\frac{dx_1}{dt}$$

$$W = x_1 x_2' - x_2 x_1' \leftarrow \text{Wronskian}$$

$$W(0) = 1, \quad W(1) = ?$$

$$W(1) = W(0)\, e^{-\int_0^1 2\,dt}$$

$$= 1\, e^{-2\int_0^1 dt}$$

$$\boxed{W(1) = \frac{1}{e^2}}$$

* Differential Equations With Variable Coefficients

1. Legendre
2. Bessel
3. Hermite
4. Laguarre

$\rightarrow$ ① Power Series Solution Method

② Frobene'us Method

* Standard form of Second order Differential Eqn:-

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

Where $a_0, a_1, a_2 \Rightarrow$ function of x or Constant.

→ Ordinary Point { at $x = a$; $\frac{a_1}{a_2}$, $\frac{a_0}{a_2}$ are both finite

→ Singular Point { at $x = a$, eithe $\frac{a_1}{a_2}$ or $\frac{a_0}{a_2}$ or both are infinite

Regular Singular Point

at $x = a$, $(x-a)\frac{a_1}{a_2}$, $(x-a)^2 \frac{a_0}{a_2}$ are finite

Irregular Singular Poi

at $x = a$, either $(x-a)\frac{a_1}{a_2}$, or $(x-a)^2\frac{a_0}{a_2}$ or both are infinite.

Q.14(ii) $\underbrace{x(x-2)^2}_{a_2}y'' + \underbrace{2(x-2)}_{a_1}y' + \underbrace{(x+3)}_{a_0}y = 0$

$$\frac{a_1}{a_2} = \frac{2\cancel{(x-2)}}{x(x-2)^{\cancel{2}}} = \frac{2}{x(x-2)}$$

$$\frac{a_0}{a_2} = \frac{(x+3)}{x(x-2)^2}$$

So $x=0$ and $x=2$, $\frac{a_1}{a_2}$ and $\frac{a_0}{a_2}$ are infinite
so $x=0, 2$ are two singular points.

At $x=0 \Rightarrow$ $$\cancel{(x-0)}\frac{2}{\cancel{x}(x-2)}\Big|_{x=0} = -1$$

$$(x-0)^2\frac{(x+3)}{x(x-2)^2}\Big|_{x=0} = 0$$

-1 and 0 are finite so both are regular singular point of given differential equation.

* Case I :- $x=a$ is a ordinary point of D.E.

Then for solution of D.E. we choose Power series solution method.

Solution Assume :-

$$y = \sum_{k=0}^{\infty} a_k (x-a)^k$$

↳ Power series about $x=a$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$

* Case II :- $x = a$ is regular singular Point of D.E.

Then we choose Frobenius Method for Solution of D.E.

Solution Assume :-

$$y = \sum_{K=0}^{\infty} a_K (x-a)^{m+K} \quad \text{where } m = \text{integer}$$

If $x = a$ is irregular singular point then we can not use these method { irregular S.P. is out of Syllabus

* Legendre Differential Equation :-

Standared form :-

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{Where } n = +ve \text{ integer}$$

→ $x = \pm 1$ are two regular singular point.

→ $x = 0$ are ordinary point.

* Legendre function of First Kind :-

$$y = \frac{1.3.5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{(2n-1).2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3).2.4}x^{n-1} - \ldots\right]$$

terminating Series

$P_n(x)$ = Legendre Polynomial of order 'n'.

* Rodrigue's formula :-

$$\boxed{P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

* Properties regarding of Legendere Polynomial :

1. The value of Legendere polynomial of order ~~one~~ n at $x=1$ is equal to 1. $\{P_n(1) = 1\}$

2. $P_n(-1) = (-1)^n$

3. $P_n(x) \longrightarrow$ Even function if $n =$ even
   $P_n(x) \longrightarrow$ odd function if $n =$ odd

4. No. of Nodes $= n$
   $P_n(x)$ has n nodes.

5. $P_n'(1) = \frac{1}{2} n(n+1)$

6. $P_n'(-1) = (-1)^n \frac{1}{2} n(n+1)$

Every algebraic polynomial ~~it~~ can be written as a linear combination of a Legendres Polynomial. And the heighest order ~~Legendre~~ of the Legendre polynomial present in the linear combination will be the order of the algebraic polynomial.

Ex - $ax^3 + bx^2 + cx + d = \sum_{n=0}^{3} K_n P_n(x)$

A-7

Q.15

(ii) $4x^3 - 2x^2 - 3x + 8$

$4x^3 - 2x^2 - 3x + 8 = a_3 P_3(x) + a_2 P_2(x) + a_1 P_1(x) + a_0 P_0(x)$

$\Rightarrow a_3 \frac{1}{2}(5x^3 - 3x) + a_2 \frac{1}{2}(3x^2 - 1) + a_1 x + a_0$

$\Rightarrow \frac{5}{2} a_3 x^3 - \frac{3a_3 x}{2} + \frac{3}{2} a_2 x^2 - \frac{a_2}{2} + a_1 x + a_0$

$\Rightarrow \frac{5a_3}{2} = 4$

$\boxed{a_3 = \frac{8}{5}}$

$\frac{3}{2} a_2 = -2$

$a_2 = -\frac{4}{3}$

$-\frac{3a_3}{2} + a_1 = -3$

$-\frac{3 \times \cancel{8}4}{5 \times \cancel{2}} + a_1 = -3$

$a_1 = -3 + \frac{12}{5}$

$a_1 = \frac{-15+12}{5}$

$\boxed{a_1 = \frac{-3}{5}}$

for $a_0 \rightarrow$

$8 = a_0 - \frac{a_2}{2}$

$\Rightarrow 8 = a_0 + \frac{\cancel{4}2}{\cancel{6}3}$

$\Rightarrow \boxed{a_0 = \frac{22}{3}}$

21/Oct./2014

* Generating function of Legendre Polynomials :-

$$\Rightarrow (1-2xz+z^2)^{-1/2}$$

$$= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\cdots a_n z^n$$

$$\begin{array}{cccccc} \downarrow & \downarrow & \downarrow & \vdots & & \downarrow \\ P_0(x) & P_1(x) & P_2(x) & \vdots & & P_n(x) \end{array}$$

The coefficient of z^n is the expansion of $(1-2xz+z^2)^{-1/2}$ will be the Legendre Polynomial of order 'n'.

$$\Rightarrow \boxed{(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, z^n}$$

* Orthonormality of Legendre Polynomial :-

$$\boxed{\int_{-1}^{+1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\, \delta_{mn}}$$

$$\boxed{P_n^s(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x)}$$

Shifted L.P.

$$\boxed{\int_{-1}^{+1} P_n^s(x)\, P_m^s(x)\, dx = \delta_{mn}}$$

Q.16 Prove that

(ii) $$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)\,z^n$$

or

If $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$ then find $\frac{1-z^2}{(1-2xz+z^2)^{3/2}} =$

(a) $\sum(2n+1)P_n(x)\,z^n$ (b) $\sum(2n-1)P_n(x)\,z^n$

×(c) $\sum 2n\,P_n(x)\,z^n$ ×(d) None of these.

Sol^n $$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1}P_n(x)$$

$$\Rightarrow \frac{x-z}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} n z^{n-1}P_n(x)$$ ← from option we can see here we have to multiply both side by '2z'.

So $$\frac{2zx-2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} 2nz^n P_n(x)$$

Now let's see -

$$\sum 2nz^nP_n(x) \pm \sum z^n P_n(x)$$

So $$\frac{2zx-2z^2}{(1-2xz+z^2)^{3/2}} + \frac{1}{(1-2xz+z^2)^{1/2}} = \frac{\cancel{2zx}-2z^2+1-\cancel{2xz}+z^2}{(1-2xz+z^2)^{3/2}}$$

$$= \frac{1-z^2}{(1-2xz+z^2)^{3/2}}$$ Ans

(iv) If ~~$(2n+1)xP_n = (n+1)P_{n+1}+nP_{n-1}$~~ ~~find~~ If $(2n+1)P_n = P'_{n+1}-P'_{n-1}$

$P_0+3P_1+5P_2+ \cdots + (2n+1)P_n = ??$

Sol^n put $n = 1, 2, 3, \ldots$

when $n=1$ $3P_1 = P_2' - P_0'$ Cancel

$n=2 \Rightarrow 5P_2 = P_3' - P_1'$ Cancel

$n=3 \Rightarrow 7P_3 = P_4' - P_2'$ Cancel

$\vdots$

Cancel

$n=n-1 \Rightarrow (2n-1)P_{n-1} = P_n' - P_{n-2}'$ Cancel

$n=n \Rightarrow (2n+1)P_n = P_{n+1}' - P_{n-1}'$

On adding we get-

$$3P_1 + 5P_2 + 7P_3 + \cdots + (2n+1)P_n = P_{n+1}' + P_n' - \underset{\downarrow\, 1}{P_1'} - \underset{\downarrow\, 0}{P_0'}$$

adding 1 both side,

So $$\boxed{P_0 + 3P_1 + \cdots + (2n+1)P_n = P_{n+1}' + P_n'}$$ $\boxed{\because P_0 = 1}$

(v) Given $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$ evaluate -

$$\int_{-1}^{+1} x^2 P_{n+1} P_{n-1}\, dx = ??$$

Sol^n:- $x^2 P_{n+1} P_{n-1} = [xP_{n+1}][xP_{n-1}]$

In the eq^n

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

Put $n=n+1$

$$(2(n+1)+1)xP_{n+1} = (n+1+1)P_{n+1+1} + (n+1)P_{n+\cancel{1}-\cancel{1}}$$

$$(2n+3)xP_{n+1} = (n+2)P_{n+2} + (n+1)P_n$$

$$xP_{n+1} = \frac{(n+2)}{(2n+3)}P_{n+2} + \frac{(n+1)}{(2n+3)}P_n$$

Put $n = n-1$

$$x P_{n-1} = \frac{n}{(2n-1)} P_n + \frac{(n-1)}{(2n-1)} P_{n-2}$$

So $\int_{-1}^{+1} [x P_{n+1}][x P_{n-1}]\, dx = \int_{-1}^{+1} \left[\frac{(n+2)}{(2n+3)} P_{n+2} + \frac{(n+1)}{(2n+3)} P_n\right] \times$

$$\left[\frac{n}{(2n-1)} P_n + \frac{(n-1)}{(2n-1)} P_{n-2}\right] dx$$

$$= \frac{(n+2)}{(2n+3)} \frac{n}{(2n-1)} \int_{-1}^{+1} P_{n+2} P_n \, dx + \frac{(n+2)(n-1)}{(2n+3)(2n-1)} \times \int_{-1}^{+1} P_{n+2} P_{n-2} dx + \frac{n(n+1)}{(2n-1)(2n+3)} \int_{-1}^{+1} P_n P_n \, dx + \int_{-1}^{+1} \frac{(n+1)}{(2n+3)} \times \frac{(n-1)}{(2n-1)} \int_{-1}^{+1} P_n P_{n-2} \, dx$$

$$= \frac{n(n+1)}{(2n+3)(2n-1)} \int_{-1}^{+1} P_n P_n \, dx$$

$$= \frac{2n(n+1)}{(2n+3)(2n-1)(2n+1)}$$ Ans

* BESSEL D.E. :-

$$\boxed{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0}$$ where n = real Constant

→ $x = 0 \Rightarrow$ regular singular Point.

→ first Kind ⇒ Bessel function of order 'n' $J_n(x)$.

$$\boxed{y = J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n-r)!}\left(\frac{x}{2}\right)^{n+2r}}$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots$$

$$J_1(x) = x - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} - \cdots$$

$$J_{1/2}(x) = \frac{(-1)^r}{r!\left(\frac{1}{2}+r\right)!}\left(\frac{x}{2}\right)^{\frac{1}{2}+2r}$$

← constant term.

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\left(\frac{1}{2}\right)!} - \frac{1}{\left(\frac{3}{2}\right)!}\frac{x^2}{2^2} + \frac{1}{2!\left(\frac{5}{2}\right)!}\frac{x^4}{2^4} + \cdots\right]$$

$\because \Gamma(n+1) = n\Gamma(n) = n!$

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\left(\frac{3}{2}\right)! = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$\left(\frac{5}{2}\right)! = \frac{5}{2}\cdot\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

So

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\frac{1}{(\sqrt{\pi}/2)}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \sqrt{\frac{x}{2}} \frac{2}{\sqrt{\pi}} \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$\boxed{J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$$

||

$$\boxed{J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x}$$

"Bessel functions are of oscillatery nature but with decreasing amplitude and verying time period."

At $x = 0$ $J_0(x) = 1$

$J_1(x) = 0 = J_2(x) \ldots\ldots$

* Nature of $J_n(x)$:- Where n = integer (not fractional).

(i) $J_n(x)$, will be even if n = even
will be odd if n = odd

(ii) $J_{-n}(x) = (-1)^n J_n(x)$

(iii) $J_n(x)$ has infinite no. of nodes

(iv) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

(V) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{n} J_{n+1}(x)$

Given

Q.17 $2n J_n = x(J_{n+1} + J_{n-1})$

$2J'_n = J_{n-1} - J_{n+1}$

(i) $J_{3/2}(x) = ?$

from first reacurrance formula-
put $n = \frac{1}{2}$

So $2 \times \frac{1}{2} J_{1/2} = x\left(J_{\frac{1}{2}+1} + J_{\frac{1}{2}-1}\right)$

$$J_{\frac{1}{2}} = x\left(J_{\frac{3}{2}} + J_{-\frac{1}{2}}\right)$$

$$J_{\frac{3}{2}} = \frac{1}{x}\left(J_{\frac{1}{2}} - J_{-1/2}\right)$$

$$= \frac{1}{x}\left(\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x\right)$$

$$\boxed{J_{3/2} = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \frac{\cos x}{x}\right)}$$

P.T.
(ii) $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2\left[\frac{n}{x} J_n^2(x) - \frac{n+1}{x} J_{n+1}^2(x)\right]$

Sol^n

or

Find $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = ?$

(a) $2\left[\frac{n+1}{x} J_n^2 - \frac{n}{x} J_{n+1}^2\right]$

(b) $2\left[\frac{n}{x} J_n^2 - \frac{n+1}{x} J_{n+1}^2\right]$

(c) $2\left[\frac{n}{x} J_n^2 - \frac{n-1}{x} J_{n+1}^2\right]$

(d) $2\left[\frac{n+1}{2} J_n^2 - \frac{n-1}{x} J_{n+1}^2\right]$

<u>Solⁿ</u>

$$2J_n J_n' + 2J_{n+1} J_{n+1}' = J_n[J_{n-1} - J_{n+1}] + J_{n+1}[J_n - J_{n+2}] -$$

$\Rightarrow$ Here we have to write -

$J_{n-1} \Rightarrow J_n, J_{n+1}$

$J_{n+2} \Rightarrow J_n, J_{n+1}$ To find the final ans.

So Now from ① st relation

$$\because 2n J_n = x(J_{n+1} + J_{n-1})$$

$$\frac{2n}{x} J_n = J_{n+1} + J_{n-1}$$

$$\boxed{J_{n-1} = \frac{2n}{x} J_n - J_{n+1}}$$

put $n = n+1$

$$2(n+1) J_{n+1} = x(J_{n+2} + J_n)$$

$$\boxed{J_{n+2} = \frac{2(n+1)}{x} J_{n+1} - J_n}$$

Putting these values in eqⁿ ①

$$= J_n\left[\frac{2n}{x} J_n - J_{n+1} - J_{n+1}\right] + J_{n+1}\left[J_n - \frac{2(n+1)}{x} J_{n+1} - J_n\right.$$

$$=$$

* Generating function of Bessel function:

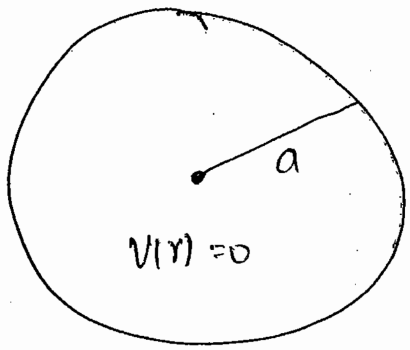
→ Diffraction through circular Aperture.

→ Infinite spherical potential well

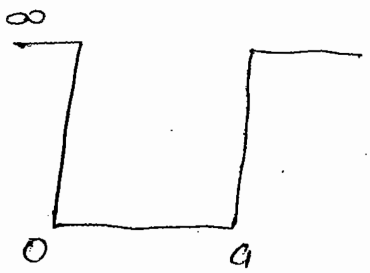
$$V(r) = 0 \qquad 0 < r < a$$
$$= \infty \qquad \text{otherwise}$$

$\downarrow$

$\varphi_n(x) \rightarrow$ Bessel function



$\infty = V(r)$



Ex - $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$

$$= \underbrace{(a_0 + a_1 z + a_2 z^2 + \cdots)}_{J_0(x)} + \underbrace{(a_{-1} z^{-1} + a_{-2} z^{-2} + \cdots)}_{J_{-1}(x)}$$

Coefficient of z^n in the expention of $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$ will be a Bessel function of order n.

$$\text{Coeff.}(z^n) = J_n(x) = a_n$$

So, $$e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{+\infty} z^n J_n(x)$$

* Orthonormality Condition:

$$\boxed{\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{\delta_{\alpha\beta}}{2}\left[J_{n+1}(\alpha)\right]^2}$$

A-7

Q. 17 (iii) $J_0 + 2J_2 \cos 2\theta + 2J_4 \cos 4\theta + \cdots$

(a) $\cos(x \sin\theta)$ (b) $\sin(x \sin\theta)$ (c) $\cos(x \cos\theta)$ (d) $\sin(x \cos\theta)$

Solⁿ $\because$ $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{+\infty} z^n J_n(x)$

$\therefore$ option is

(a) $\cos(x \sin\theta)$, (b) $\sin(x \sin\theta)$ } $e^{i(x\sin\theta)}$

× (c) $\cos(x\cos\theta)$, × (d) $\sin(x\cos\theta)$ } $e^{i(x\cos\theta)}$

$\because$ $z = x+iy$ or $re^{i\theta}$

Compaire $\frac{1}{2}\left(z-\frac{1}{z}\right) = i\sin\theta$

So put $z = e^{i\theta}$

$$e^{i(x\sin\theta)} = \sum_{n=-\infty}^{+\infty} e^{in\theta} J_n(x)$$

$$= \left[J_0 + e^{i\theta} J_1 + e^{2i\theta} J_2 + \cdots\right] + \left[e^{-i\theta} J_{-1} + e^{-2i\theta} J_{-2} + \cdots\right]$$

$$= J_0 + J_1\left(e^{i\theta} - e^{-i\theta}\right) + J_2\left(e^{i2\theta} + e^{-i2\theta}\right) + J_3\left(e^{i3\theta} - e^{-i3\theta}\right) + \cdots$$

$\{\because J_{-n} = (-1)^n J_n\}$

So $\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0 + 2iJ_1 \sin\theta + 2J_2\cos 2\theta + 2iJ_3 \sin 3\theta + \cdots$

Compairing real and imaginary part

R.P. :-

$$\boxed{\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4 \cos 4\theta + \cdots}$$

I.K. $\rightarrow$

$$\boxed{\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + \cdots}$$

CSIR

A-6 June - 2011

Q.56 Let $P_n(x)$ (where $n = 0, 1, 2, \ldots$) be polynomial of order n with real co-efficients defined in the interval $2 \le x \le 4$. If $\int_2^4 P_m(x) P_n(x)\, dx = \delta_{mn}$, then

(a) $P_0(x) = \frac{1}{\sqrt{2}}$ and $P_1(x) = \sqrt{\frac{3}{2}}(-3-x)$ (b) $P_0(x) = \frac{1}{\sqrt{2}}$ and $P_1(x) = \sqrt{3}(3+x)$

(c) $P_0(x) = \frac{1}{2}$ and $P_1(x) = \sqrt{\frac{3}{2}}(3-x)$ (d) $P_0(x) = \frac{1}{\sqrt{2}}$ and $P_1(x) = \sqrt{\frac{3}{2}}(3-x)$

Sol^n

$$\because \int_2^4 P_m(x) P_n(x)\, dx = \delta_{mn}$$

after solving this we can never find $P_0(x)$ and $P_1(x)$ directly with this given information so we have to find relations -

$$\int_2^4 P_0(x) P_0(x)\, dx = 1 \quad , \quad \int_2^4 P_0(x) P_1(x)\, dx = 0$$

$$\int_2^4 P_1(x) P_1(x)\, dx = 1 \quad , \quad \int_2^4 P_1(x) P_2(x)\, dx = 0$$

So check by options -

option (d) $P_0(x) = \frac{1}{\sqrt{2}}$

$P_1(x) = \sqrt{\frac{3}{2}}\,(3-x)$

$$\int_2^4 \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\, dx = \frac{1}{2}[x]_2^4 = \frac{1}{2}[4-2] = \frac{2}{2} = 1$$

$$\int_2^4 \sqrt{\frac{3}{2}}(3-x) \cdot \sqrt{\frac{3}{2}}(3-x)\, dx = \frac{3}{2}\int_2^4 (9 - 3x - 3x + x^2)\, dx$$

$$= \frac{3}{2}\left[\int_2^4 9\,dx - 6\int_2^4 x\,dx + \int_2^4 x^2\,dx\right]$$

$$= \frac{3}{2}\left\{9[x]_2^4 - 6\left[\frac{x^2}{2}\right]_2^4 + \left[\frac{x^3}{3}\right]_2^4\right\}$$

$$= \frac{3}{2}\left[9\times 2 - 6\times\frac{\cancel{12}\,6}{\cancel{2}} + \frac{56}{3}\right]$$

$$= \frac{3}{2}\left[18 - 36 + \frac{56}{3}\right] \Rightarrow \cancel{\frac{3\times 6}{2} + \frac{8\times 2}{3\;\,2}} = \cancel{9+4=13}$$

$$= 1$$

So option (d) is ~~wrong~~. correct

~~option (b) $P_0(x) = \frac{1}{\sqrt{2}}$, $P_1(x) = \sqrt{3}(3+x)$~~

A-6 CSIR-Dec-211

Q. 57

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

$$P_3(-1) = ??\;,\; P_5(-1) = ?? \qquad |t|<1$$

$$\text{Coeff.} = (t^n) = P_n(x)$$

$$\text{Coe}(t^3) = P_3(x)$$

$$\text{Coe}\,(t^5) = P_5(x)$$

$$P_n(-1) = (-1)^n$$

$$\left.\begin{matrix} P_3(-1) \\ P_5(-1)\end{matrix}\right\} = (-1)$$

$$P_3\Big|_{x=-1} = \frac{1}{2}(5x^3 - 3x)\Big|_{x=-1} = -1.$$

A-6

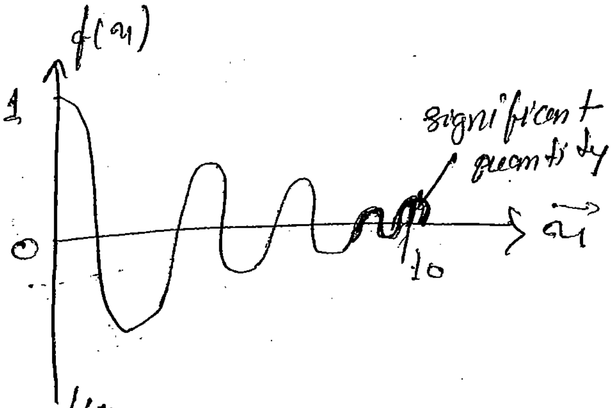
Q.60

(a) $J_0(x)$ ✓

(b) ✗ $\cos x$

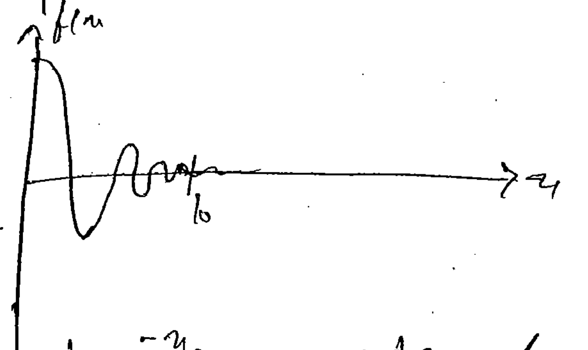
(c) $e^{-x}\cos x$ ✗

(d) ✗ $\frac{\cos x}{x}$



graph of $e^{-x} \cos x$ ✓

very small at $x = 10 = 0.00004$
$= 4 \times 10^{-5}$



So first option is correct. becoz in $e^{-x}\cos x$ rate of decay is very large.

* Hermite Defferential Equation :-

Standard form of Hermite defferential equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \quad (n = +ve \text{ integer})$$

→ Ist Solution :-

$$\boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})}$$

hermite polynomial of order n.

→ Used in :- 1-D. Harmonic Oscillater.

$$V(x) = \frac{1}{2}kx^2$$

$$\psi_n(x) = \underbrace{A e^{-\alpha^2 x^2/2}}\ \underbrace{H_n(\alpha x)}$$

$$\boxed{\alpha^2 = \frac{m\omega}{\hbar}}$$

(I) Parity of wave function

(ii) Node of wave function.

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

⋮

(i) $H_n(x)$ ⇒ Even if n = even

odd if n = odd

(2) $H_n(x) \longrightarrow$ 'n' nodes.

(3) $\psi_n(x) \rightarrow$ even parity $\longrightarrow$ n - even $\rightarrow$ n = even $(n=0,2,4,\ldots)$
$\rightarrow$ odd parity $\longrightarrow$ n - odd $\rightarrow$ n = odd $(n=1,3,5,\ldots)$

* Generating function of Hermite Polynomial :-

$$\boxed{e^{2xz-z^2} = \sum_{n=0}^{\infty} z^n \frac{H_n(x)}{n!}}$$

The Coefficient of z^n in the expantion of e^{2xz-z^2} will be $\frac{H_n(x)}{n!}$

* Orthonormality Condition :-

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n . n! \sqrt{\pi}\, \delta_{mn}}$$

* Shifted Hemite Polynomial :-

$$\boxed{H_n^S(x) = \frac{e^{-x^2/2}}{[2^n . n! \sqrt{\pi}]^{1/2}} H_n(x)}$$

then

$$\boxed{\int_{-\infty}^{+\infty} H_m^S(x) H_n^S(x)\, dx = \delta_{mn}}$$

$\because$ $H_3(x) = 8x^3 - 12x$

$$\therefore \boxed{H_3^S(x) = \frac{e^{-x^2/2}}{\sqrt{48\sqrt{\pi}}} (8x^3 - 12x)}$$

H-6

Q.62 Given that $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{-t^2+2tx}$, the value of $H_4(0)$ is

(a) 12 (b) 6 (c) 24 (d) -6

Sol^n

$\because H_4(x) = 16x^4 - 48x^2 + 12$

$H_4(0) = 16(0) - 48(0) + 12$

$= \cancel{32} + 12$

$$\boxed{H_4(0) = 12}$$

Second Method:-

e^{-t^2+2tx}

$= e^{-t^2}$

$= \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right)$

$= \frac{1}{2!} = \frac{H_4(0)}{4!}$

$$\Rightarrow \boxed{H_4(0) = 12}$$

$\boxed{H_{21}(0) = 0}$ Since in expansion t^{21} not Com so its coefficient is 0.

Note :- "Every Algebraic polynomial can be written as a linear combination of Hermite polynomial. And the order of the algebraic polynomial will be the heighest order hermite polynomial present in the linear combination."

* Laguarre Defferential Equation :-

Standared form:-

$$\boxed{x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0}$$

n = +ve integer

→ $x=0$ is regular singular point.

→ <u>first Solution:-</u>

The first solution of Laguarre D.E. is a terminating series is k/a Laguarre polynomial of order 'n':

$$\boxed{L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)}$$ ← Laguarre Polynomial of order one.

$L_0(x) = 1$
$L_1(x) = 1-x$
$L_2(x) = x^2 - 4x + 2$
⋮ ⋮

→ $L_n(x)$ is neither even nor odd function.

<u>Use:-</u>

Hydrogen atom's wave function

* Generating function :-

$$\boxed{e^{-xz/(1-z)} = \sum_{n=0}^{\infty} z^n L_n(x)} = \sum_{n=0}^{\infty} z^n\left(\frac{L_n(x)}{n!}\right)$$

The coeffiecient of z^n in the expansion of $e^{-2z/1-z}$ will be the laguarre polynomial of order 'n.'

* Orthonormality Condition:-

$$\boxed{\int_0^\infty e^{-x} L_m(x) L_n(x) dx = \delta_{mn}}$$

If $n!$ is not considered in L.H.S. then

$$\boxed{\int_0^\infty e^{-x} \frac{L_m(x)}{n!} \frac{L_n(x)}{n!} dx = \delta_{mn}}$$

A-7
16(iii)

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

$$1 + \frac{1}{2} P_1(\cos\alpha) + \frac{1}{3} P_2(\cos\alpha) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{P_n(\cos\alpha)}{(n+1)}$$

$$\Rightarrow \int_0^1 \frac{dz}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} \int_0^1 z^n dz \, P_n(x)$$

$$\int_0^1 \frac{dz}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} \left[\frac{z^{n+1}}{n+1}\right]_0^1 P_n(x)$$

$$\because z^2 - 2xz + 1 = (z^2 - 2xz + x^2) + 1 - x^2$$

$$= (z-x)^2 + \left(\sqrt{1-x^2}\right)^2$$

So $\displaystyle\int \frac{dz}{\sqrt{(z-x)^2+(\sqrt{1-x^2})^2}} = \ln\left[(z-x)+\sqrt{z^2-2xz+1}\right]$

$\Rightarrow \ln\left[(z-x)+\sqrt{z^2-2xz+1}\right]_0^1 = \displaystyle\sum_{n=0}^{\infty}\left[\frac{z^{n+1}}{n+1}\right]_0^1 P_n(x)$

$\Rightarrow \ln\left[(1-x)+\sqrt{2-2x}\right] - \ln\left[1-x\right] = \displaystyle\sum_{n=0}^{\infty}\frac{P_n(x)}{n+1}$

put $x = \cos\alpha$.

Q.93

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0$$

$n^2+n-6=0$

$(n+3)(n-2)=0$

$n=-3, 2$

$\because$ n is positive integer

So $\boxed{n=2}$

So Solution of diff. eq^n is $P_2(x)$

So option (c) is correct.

Q.40 The expression $P_{2n}'(0)$ can be written as

(a) 0 (b) 1 (c) $\frac{1}{2n}$ (d) $\frac{(-1)^n n!}{2^n}$

Sol^n

$P_{2n}'(0)$

$P_0'(0)$

$P_2'(0)$

$P_4'(0)$

$\vdots$

even order derivative at $x=0$ is 0.

$$\boxed{P_{2n}'(0) = 0}$$

A-7

Q.16 (vi) $\int_{-1}^{+1} (1-x^2)\left[P_n'(x)\right]^2 dx$

Sol^n

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx}} + n(n+1)y = 0$$

$$\Downarrow$$

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + n(n+1)y = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)P_n'(x)\right] = -n(n+1)P_n(x)$$

So $$\int_{-1}^{+1}(1-x^2)\left[P_n'(x)\right]^2 dx = \int_{-1}^{+1}\underbrace{(1-x^2)P_n'(x)}_{\mathrm{I}}\cdot\underbrace{P_n'(x)}_{\mathrm{II}}\,dx$$

$$= \left\{(1-x^2)P_n'(x)P_n(x)\right|_{-1}^{+1} - \int_{-1}^{+1}\left\{\frac{d}{dx}\left[(1-x^2)P_n'(x)\right]\cdot P_n(x)\right\}$$

$$= 0 + \int_{-1}^{+1} n(n+1) P_n(x) \cdot P_n(x)\, dx$$

$$= n(n+1) \int_{-1}^{+1} P_n(x) \cdot P_n(x)\, dx$$

$$= n(n+1) \cdot \left\{ \frac{2}{(2n+1)} \right\} \qquad \left\{ \text{by orthonormal property} \right\}$$

$$= \frac{2n(n+1)}{(2n+1)}$$

Ans

Q.17 (iv) $\frac{d}{dx}\left[x J_n(x) J_{n+1}(x) \right] = x\left[J_n^2(x) - J_{n+1}^2(x) \right]$

Sol^n

$$\frac{d}{dx}\left[x J_n J_{n+1} \right] = J_n J_{n+1} + x J_n' J_{n+1} + x J_n J_{n+1}'$$

$$= J_n J_{n+1} + \frac{x J_{n+1}}{2}\left[J_{n-1} - J_{n+1} \right] + \frac{x J_n}{2}\left[J_n - J_{n+2} \right]$$

$$\because \left\{ \begin{array}{l} 2J_n' = J_{n-1} - J_{n+1} \\ 2J_{n+1}' = J_n - J_{n+2} \end{array} \right\}$$

Here J_{n-1} and J_{n+2} are ~~a~~ unwanted so we reduse it in J_n and J_{n+1} by-

$$\left\{ \begin{array}{l} 2n J_n = x(J_{n-1} + J_{n+1}) \\ 2(n+1) J_{n+1} = x(J_n + J_{n+2}) \end{array} \right\}$$

$$= J_n J_{n+1} + \frac{x J_{n+1}}{2}\left[\frac{2n}{x} J_n - J_{n+1} - J_{n+1}\right]$$

$$+ x\frac{J_n}{2}\left[J_n - \frac{2(n+1)}{x} J_{n+1} + J_n\right]$$

$$= J_n J_{n+1} + \cancel{\frac{x J_{n+1}}{2} \cdot \frac{2n}{x} J_n} - \frac{x J_{n+1} \times J_{n+1}}{2} - \frac{x J_{n+1} J_n}{2}$$

$$+ \frac{x J_n}{2} \cdot J_n - \cancel{\frac{x J_n}{2} \times \frac{2(n+1)}{x} J_{n+1}} + \frac{x J_n}{2} \times J_n$$

$$=$$

Probability

* Probability :-

Probability is the numerial measurement of degree of certainity of the occurance of the event.

An event 'A' can happen in 'm' ways and can't happen in 'n' ways.

So probability -

$$P(A) = \frac{m}{m+n} = \frac{\text{no. of favourable ways}}{\text{total no. of ways}}$$

Probability that event can not happen -

$$P'(A) = \frac{n}{m+n} = \frac{\text{no. of favourable ways}}{\text{total no. of ways}}$$

$$\boxed{P(A) + P'(A) = 1}$$

Probability of an event lie $0 \leq P(A) \leq 1$

If $P(A) = 0$ ⟶ Impossible ~~event~~

If $P(A) = 1$ ⟶ Sure event.

* Permutation and Combination :-

0 – 9

Ex– □ □ □
4 7 6

Permutation ⟶ Order does matter

Combination ⟶ Order does not matter.

Cricket Team
Order doesn't matter.

20 players
↓
11 players

Team does n… change

Permutation $= {}^nP_r = \dfrac{n!}{(n-r)!}$

Combination $= {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Repetation is not allowed.

* Random Experiment :-

These are experiments in which results may be different in each case although all cases are performs under identical cases.

Ex – Tossing a coin, throughing dice etc.

* Sample Space :-

The set of all possible outcomes in a single performance of the random experiment is k/a Sample Space.

Ex –

A coin is tossed twice. ← Trial

$S = \{HH, HT, TH, TT\}$ ← events

* Trial and event :-

Performing a random experiment is a "trial" and getting outcome is an "event."

* Equal likely event :-

Two events are said to be equal likely if their corresponding probability is equal.

Ex Throughing a dice getting 1 to 6 probability is $\frac{1}{6}$

So getting a no. has probability is $\frac{1}{6}$ so equal likely event.

another example :-

When a coin is tossed twice then - $S = \{\underset{\frac{1}{4}}{HH}, \underset{\frac{1}{4}}{TH}, \underset{\frac{1}{4}}{HT}, \underset{\frac{1}{4}}{TT}\}$

event

$$\left.\begin{array}{l} HH \rightarrow 2 \Rightarrow \frac{1}{4} \\ HT, TH \rightarrow 1 \Rightarrow \frac{1}{2} \\ TT \rightarrow 2 \Rightarrow \frac{1}{4} \end{array}\right\}$$

these three are not equal likely event but HH, TH are two equal likely event.

After throughing a dice getting the even no.s probabi

$1, 2, 3, 4, 5, 6$ each $\frac{1}{6}$

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

$$P(O) = \frac{3}{6} = \frac{1}{2}$$

for Odd

* Independent Events :-

Two events are said to be independent if ~~occu~~ occurance of one event does not influence does not influence occurance of the other.

Ex- A coin is tossed twice getting a head in the first toss and getting a tail in the second toss are "independent events."

Say A, B are two independent Event then

$P(B|A)$ = Probability that B is occur such that A is already occur. $\Rightarrow P(B)$

$P(A|B)$ = Probability that A is occur such that B is already occur. $\Rightarrow P(A)$.

$$\boxed{\begin{array}{l} P(B|A) = P(B) \\ P(A|B) = PA \end{array}}$$

* Mutually Exclusive Events :-

Two events are said to be mutually exclusive if occurance of one event excludes the occurance of the other.

Ex- If a coin is tossed once getting a head and getting a tail are two mutually exclusive events.

Two mutually exclusive event can't occur simultaniously

"A Coin is tossed n times and 'n' coins tossed together ⟶ Same Sample Space."

" Suppose a random experiment has n & no. of outcome in a single trial and experiment is repeated r - times. The total no. of outcomes will be = $\boxed{n^r}$"

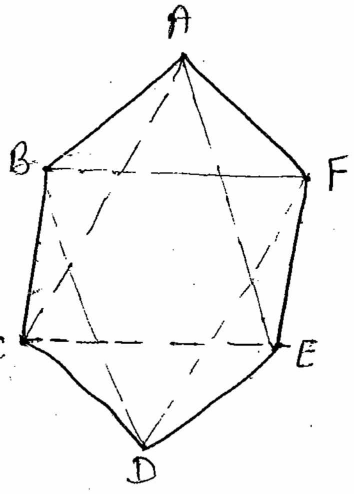
1 dice ⟶ $6 = n$

$r = 3$ ⟶ $6^3 = 216$.

Two independent event can occur simultaniously without affecting ~~one~~ another. But two mutually exclusive events can not occurs simultaniously. This is the basic difference between them.

Q. A triangle is formed by joining three points of regular hexagon. what is the probability that the triangle will be equilateral.

Soln



$$P = \frac{\text{no. of equilateral } \Delta\text{'s formed}}{\text{total no. of triangles formed}}$$

Vertices = 6

Triangle needs = 3 Vertices

So Here we use combination because order does not matter.

So total no. of Δ's = $^6C_3$

equilateral Δ's = 2

So $P = \frac{2}{^6C_3} = \frac{2}{20} = \frac{1}{10}$

A-8
Q.8 $P = \begin{bmatrix} 1 & a \\ b & c \end{bmatrix}$ where a, b, c can be chosen randomly on the set $a, b, c \in \{1, 2, 3, 4, 5\}$

What is the probability that the matrix will be singular.

Soln

$$P = \frac{\text{no. of singular matrices formed.}}{\text{Total no. of matricess formed}}$$

$$= \frac{\cancel{10}\ 2}{5 \times 5 \times \cancel{5}} = \frac{2}{25} \text{ Ans}$$

$|P| = c - ab = 0$

$c = ab$

c	a	b
1	1	1
2	1, 2	2, 1
3	1, 3	3, 1
4	1, 4, 2	4, 1, 2
5	1, 5	5, 1

= 10 Combination

H-8 6S+R Dec - ...

Q. An unbaised dice is ~~through~~ thrown three times successively what is the prob. that the no. of dots on the on the upper surface will be add to be 16.

Sol^n Unbiased / fair $p = \frac{1}{6}$

Biased / loaded P is given

$$P = \frac{\cancel{6}}{6^{\cancel{3}}}$$

$$\boxed{P = \frac{1}{36}}$$

16 : (6, 6, 4) (4, 6, 6), (6, 4, 6)
16 : (6, 5, 5), (5, 6, 6), (5, 5, 6)

order matter.

* Say A, B are two mutually exclusive events

$P(A \cup B)$ = Probability that either ~~b~~ A or B will occur.

then $\boxed{P(A \cup B) = P(A) + P(B)}$

Ex- After throughing dice for getting either 1 or 2 is ?

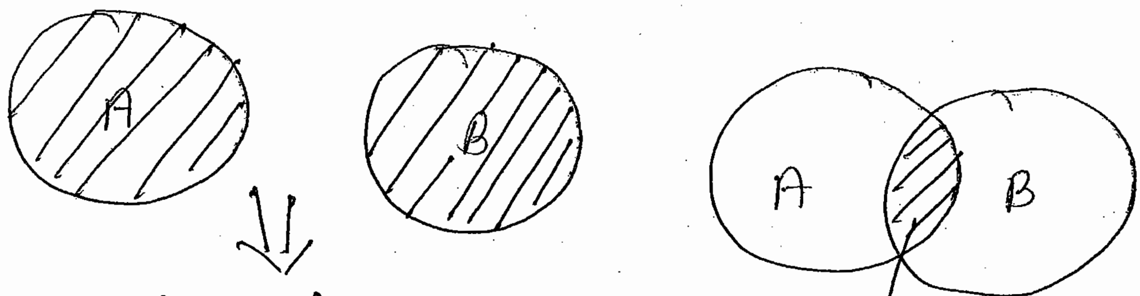
$$P(1 \cup 2) = P(1) + P(2)$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \text{ Ans}$$

Event → Either A — P(A)
+
→ or B — P(B)

* If A and B are not mutually exclusive

$P(A \cup B)$ = Probability that either A or B or both will occur.

$$\boxed{P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\text{Probability that both will occur.}}}$$



$P(A \cap B) = 0$ $\qquad$ $P(A \cap B) \neq 0$

* A, B are independent events :-

then

$P(A \cap B)$ = Probability that both A and B will occur

So $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

* A, B are not independent events then:-

$P(A \cap B)$ = Probability that

There are two cases - (1) Probability of occuring B when A is already ~~occu~~ occur

(2) Probability of occuring A when B is already occur.

$$\boxed{P(A \cap B) = P(B|A) \cdot P(A)}$$
$$\boxed{P(A \cap B) = P(A|B) \cdot P(B)}$$

$A \rightarrow B$

$B \rightarrow A$

Q. 8.

Sol^n

10 bolls are black

15 " " Red

Same Colour

either 2 B or 2 R

$P(B) + P(R)$

$$= \frac{{}^{10}C_2}{{}^{25}C_2} + \frac{{}^{15}C_2}{{}^{25}C_2}$$

$$= \frac{\frac{10!}{2!(10-2)!}}{\frac{25!}{2!(25-2)!}} + \frac{\frac{15!}{2!(15-2)!}}{\frac{25!}{2!(25-2)!}}$$

$=$

A-8
Q.10 A bag contains 6 Red and 4 green balls
A fair dice is rolled

Sol^n 6 Red, 4 Green

Events → 1 —either→ 1R → $\frac{1}{6} \times \frac{^6C_1}{^{10}C_1}$

Events → 2 —or→ 2R → $+\ \frac{1}{6} \times \frac{^6C_2}{^{10}C_2}$

Events → 3 —or→ 3R → $+\ \frac{1}{6} \times \frac{^6C_3}{^{10}C_3}$

Events → 4 —or→ 4R → $+\ \frac{1}{6} \times \frac{^6C_4}{^{10}C_4}$

Events → 5 —or→ 5R → $+\ \frac{1}{6} \times \frac{^6C_5}{^{10}C_5}$

Events → 6 —or→ 6R → $\frac{1}{6} \times \frac{^6C_6}{^{10}C_6}$

A-8 CSIR Dec-13
Q.80 A loaded dice has the probabilities $\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}, \frac{6}{21}$ of terning up 1, 2, 3, 4, 5, 6 respecti if it is thrown twice then what is the probability that the sum of the numbers that turns up is even?

Sol^n:
Sum of two numbers will be even

Either even-even or odd-odd

$$P = (P(E-E)) + P(O-O)$$
$$= P(E)\cdot P(E) + P(O)\cdot P(O)$$
$$= \left(\frac{2}{21}+\frac{4}{21}+\frac{6}{21}\right)\left(\frac{2}{21}+\frac{4}{21}+\frac{6}{21}\right) + \left(\frac{1}{21}+\frac{3}{21}+\frac{5}{21}\right)\left(\frac{1}{21}+\frac{3}{21}+\frac{5}{21}\right)$$

$$= \frac{12}{21} \times \frac{12}{21} + \frac{9}{21} \times \frac{9}{21}$$

$$= \frac{144}{21 \times 21} + \frac{81}{21 \times 21}$$

$$= \frac{144+81}{21 \times 21} = \frac{\cancel{225}\ \cancel{75}\ 25}{\cancel{441}\ \cancel{147}\ 49}$$

$$\boxed{P = \frac{25}{49}}$$

A-8

Q27 A and B plays a game of dice and A thrown first a person get 6 first is winner what is probability that A wins.

Sol^n

Either

1^{st} through $\longrightarrow \frac{1}{6}$

or $+$

3^{rd} $\longrightarrow \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

A-loose B-loose A win

or $+$

5^{th} $\longrightarrow \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

or

6^{th}

$\vdots$

$$a = \frac{1}{6}$$

$$r = \frac{5}{6} \times \frac{5}{6}$$

$$P = \frac{a}{1-r} = \frac{\frac{1}{6}}{1 - \frac{5}{6} \times \frac{5}{6}}$$

3|10|2014
A-8
Q.4

366 days

52 weeks $\times 7 = 364$ days

2 days -

$\boxed{S}$ $\boxed{M}$ - - - - - $\boxed{S}$

$$P = \frac{\text{No. of favourable days}}{\text{Total no. of days}} = \frac{2}{7}$$

$$\boxed{P = \frac{2}{7}}$$

A-8
28

A 3 Win 1 loose

M_1	M_2	M_3	M_4	
Either Win	Win	loose	Win	$\Rightarrow \frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}$
or Win	loose	Win	Win	$\Rightarrow \frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}$
or loose	Win	Win	Win	$\Rightarrow \frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$

$$= \frac{\cancel{3}\times 8}{\cancel{81}\ 27}$$

$$= \frac{8}{27}$$ Ans

Q.34

Either

First shot $\quad 0.4 \qquad = 0.4$

or

Second shot $\quad 0.6 + 0.3 \qquad = 0.18$

or

Third Shot $\quad 0.6 + 0.7 + 0.2 \qquad = 0.84$

or

Fourth Shot $\quad 0.6 + 0.8 + 0.7 + 0.8 \times 0.1 = 0.336$

$$= 0.4 + 0.18 + 0.84 + 0.336 = 0.6976 \text{ Ans}$$

Q.39

P_1 $\qquad\qquad$ P_2

2 silver, 4 copper $\qquad\qquad$ 4 silver, 3 copper

$\longrightarrow P_1 \longrightarrow 1 \text{ silver} = \frac{1}{2} \times \frac{{}^2C_1}{{}^6C_1} = \frac{1}{2} \times \frac{2!}{\frac{6!}{1! \times 5!}}$

or

$\longrightarrow P_2 \longrightarrow 1 \text{ silver} = \frac{1}{2} \times \frac{{}^4C_1}{{}^7C_1} = \frac{1}{2} \times \frac{\frac{4!}{1! \times 3!}}{\frac{7!}{1! \times 6!}}$

$$= \frac{1}{2} \times \frac{2}{6} + \frac{1}{2} \times \frac{4}{7}$$

$$= \frac{2}{12} + \frac{4}{14} = \frac{28+48}{12 \times 14} = \frac{\cancel{76}^{19}}{\cancel{12} \times 14_{3}}$$

$$= \frac{19}{42} \text{ Ans}$$

A-6

Q.43

$P(A) = x \qquad P'(A) = (1-x)$

$P(B) = y \qquad P'(B) = (1-y)$

Total probability $= xy + (1-x)(1-y)$

favourable probability $= xy$

So Probability that the statement is true is

$$= \frac{xy}{xy + (1-x)(1-y)} \quad \text{Ans}$$

A-8

Q.18

$S = \{1, 2, 3, \ldots\ldots 1001\}$

Event $A \Rightarrow$ the no. will be divisible by 7

" $B \Rightarrow$ " " " 11

events are not mutually exclusive

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{143}{1001} + \frac{91}{1001} - \frac{\cancel{143}}{\cancel{1001}_{11}} \times \frac{\cancel{91}}{\cancel{1001}_{7}}$$

$\frac{\cancel{1001}}{\cancel{7}} = 143$

$\frac{\cancel{1001}}{\cancel{11}} = 91$

${}^{1001}C_1 = 1001$

$$= \frac{143 + 91}{1001} - \frac{1}{77} = \frac{234 - 13}{1001}$$

$$= \frac{221}{1001} \quad \text{Ans}$$

or $\frac{1001}{7 \times 11} = 13$

$\frac{143}{1001} + \frac{91}{1001} - \frac{13}{1001}$

$= \frac{221}{1001}$ Ans

If Number is divisible by 6 or 8 or both

$$\frac{1001}{6} = $$

$$\frac{1001}{8} = $$

$$Pr(A \cup B) = \frac{\frac{1001}{6}}{1001} + \frac{\frac{1001}{8}}{1001} - \frac{\frac{1001}{24}}{1001}$$

$$= \frac{167}{1001} + \frac{125}{1001} - \frac{41}{1001}$$

$$= \frac{167 + 125 - 41}{1001}$$

$$=$$

A-8 CSIR June 2013

Q.79

Sol^n

	S_A	S_B	S_C	$S_A + S_B + S_C$
at $t = t_1$	$+1$	$+1$	-1	1
	$\downarrow$ No change $P = \frac{2}{3}$	$\downarrow$	$\downarrow$	
at $t = t_2$	$+1$	$+1$	-1	
	$+1$ $\downarrow$ chang $P = \frac{1}{3}$	-1	$+1$	$1.$
	-1	$+1$	$+1$	

mutually exclusive,

$$= \left(\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{2}{3}\ \frac{1}{3}\right)$$

$$= \frac{8}{27} + \frac{2}{27} + \frac{2}{27}$$

$$= \frac{\cancel{12}\ 4}{\cancel{27}\ 9}$$

$$= \frac{4}{9} \ \underline{\underline{\text{Ans}}}$$

find $P\left[\,|S_A + S_B + S_C|\,\right]$ will not change $= ?$

	S_A	S_B	S_C	$S_A + S_B + S_C$
at $t = t_1$	1	1	-1	+1
	$\downarrow$	$\downarrow$	$\downarrow$	
at $t = t_2$	1	1	-1	
	1	-1	1	+1
	-1	1	1	
	1	-1	-1	
	-1	1	-1	-1
	-1	-1	1	

$$= \left(\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{2}{3}\right)$$

$+$

BAYE'S THEOREM

"If $B_1, B_2, B_3, \ldots\ldots B_n$ are mutually exelusive event of a random experiment with non zero probability then for any arbitrary event A of the same random experiment we can have-"

$$\boxed{P(B_i|A) = \frac{P(B_i)\, P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\, P(A|B_i)}}$$

Ex -

B_1 $\qquad$ B_2

4R, 3W $\qquad$ 2R, 5W

1 Ball

$\downarrow$

W $\qquad$ (Revese Case)

A-8
Q.46

Sol^n

B_1 $\qquad$ B_2 $\qquad$ B_3

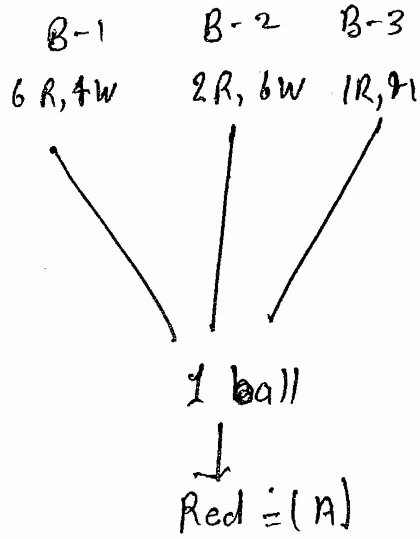
6R, 4W $\qquad$ 2R, 6W $\qquad$ 1Red, 4W

A : Selecting a red ball

B_1 : Selecting Bag 1.

B_2 : Selecting Bag 2.

B_3 : selecting Bag 3.



$$P(B_1|A) = \frac{P(B_1)\,P(A|B_1)}{\sum_{i=1}^{3} P(B_1)P(A|B_1)+P(B_2)\,P(A|B_2)+P(B_3)\,P(A|B_3)}$$

$$= \frac{\frac{1}{3}\times\frac{^6C_1}{^{10}C_1}}{\frac{1}{3}\times\frac{^6C_1}{^{10}C_1}+\frac{1}{3}\times\frac{^2C_1}{^8C_1}+\frac{1}{3}\,\frac{^1C_1}{^5C_2}}$$

$$= \frac{\cancel{\tfrac{1}{3}}\times\frac{\frac{6!}{1!\times5!}}{\frac{10!}{1!\times9!}}}{\cancel{\tfrac{1}{3}}\times\frac{\frac{6!}{1!\times5!}}{\frac{10!}{1!\times9!}}+\cancel{\tfrac{1}{3}}\times\frac{\frac{2!}{1!\times1!}}{\frac{8!}{1!\times7!}}+\cancel{\tfrac{1}{3}}\,\frac{\frac{1!}{1!(1-1)!}}{\frac{5!}{2!\,3!}}}$$

$$= \frac{\frac{6}{10}}{\frac{6}{10}+\frac{2}{8}+\frac{2}{5\times4}} = \frac{\frac{6}{10}}{\frac{6}{10}+\frac{1}{10}+\frac{1}{4}}$$

$$= \qquad = \frac{4}{7}\ \underline{\text{Ans}}$$

H-V
Q. 48

Sol^n

99 Fair 1 Double headed

↓

1 Coin

3 toss ⟶ 3 Heads

A = getting 3 Head in 3 tosses

B_1 = Selecting a fair coin

B_2 = Selecting a Double Headed coin

$$P(B_2|A) = \frac{P(B_2)\,P(A|B_2)}{P(B_1)\,P(A|B_1) + P(B_1)P(A|B_1)}$$

$$= \frac{\frac{1}{100} \times 1}{\frac{99}{100} \times \frac{1}{8} + \frac{1}{100} \times 1}$$

HHH — $\frac{1}{8}$
HHT
HTH
THH
HTT
THT
TTH
TTT

$$= \frac{\frac{1}{100}}{\frac{99}{800} + \frac{1}{100}} = \frac{\frac{1}{100}}{\frac{99+8}{800}}$$

$$= \cancel{\frac{1}{100}} \times \frac{\cancel{800}\;8}{107}$$

$= 0.074766$

$= 0.075$ Ans

* Random Variable :-

It is defined as a real number ~~x~~ connected with the outcome of a random experiments

Ex - Two types of Random Variables.

1. Discreate
2. Continuous

1. Discreate Random Variables :-

This type of variable takes finite number of discreate ~~Varia~~ values or countable infinite number of values.

Ex :- A coin is tossed twice -

HH HT TH TT

x : no. of Heads in the outcome

$x = \underbrace{0, 1, 2,}$ finite no. of discreate values.

Ex - $E_n = \frac{n^2 \pi^2 \hbar^2}{2 m a^2}$ $n = \{1, 2, 3, \ldots \infty\}$

$E_n = n^2 E_1$

$E_1, E_2, \ldots$ —

2. Continuous Random Variables :-

This type of variable takes uncountable infinite no. of continuous values.

Ex :- [1, 2]

x = real number in [1, 2] uncountable infinite number but have continuous variabl

* Probability Distribution :-

This is defined as the set of all possible outcome of a random experiment with their corresponding probability.

1. Discrete Probability Distribution :-

Ex A coin is tossed twice

$\underset{1/4}{HH}$ $\overset{1/2}{HT \ TH}$ $TT - \frac{1}{4}$

x : no. of Heads

$$\boxed{\begin{array}{l} x = 0, 1, 2 \\ P(x) = \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \end{array}}$$ ← discrete Probability Distribution.

Suppose x_i is a discete random variable with with probability $P(x_i)$, then

$$\boxed{\sum_{i=1}^{n} P(x_i) = 1}$$ ← Normalisation Condition.

$\Rightarrow P(x_1) + P(x_2) + \cdots\cdots\cdots = 1.$

(i) Mean of the Distribution :- { μ / $\langle x \rangle$ } :-

It is defined as the average value or expected value of the random variable.

$$\boxed{\mu = \langle x \rangle = \sum_{i=1}^{n} x_i P(x_i)}$$

2 Variance And Standard daviation of distribution:-

It is the measurement of the daviation of the random variable from its mean value.

Variance

$$\boxed{\sigma^2 = \sum_{i=1}^{n} [(x_i - \mu)^2] P(x_i)}$$

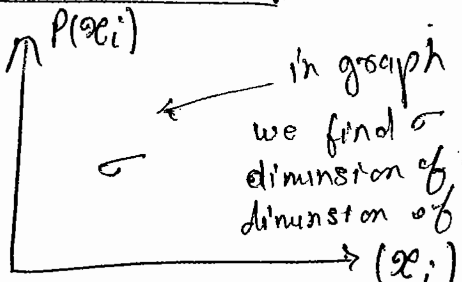
dimension of σ^2 is length^2.

Standard daviation -

$$\boxed{\sigma = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 P(x_i)}}$$

dimension of length

$$\because \sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P(x_i)$$



in graph we find σ becoz dimension of x is dimension of length

$$= \sum_{i=1}^{n} x_i^2 P(x_i) - \sum_{i=1}^{n} 2x_i\mu P(x_i) + \sum_{i=1}^{n} \mu^2 P(x_i)$$

$$= \langle x^2 \rangle - \underbrace{2\mu.\mu + \mu^2}$$

$$\boxed{\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2}$$

$$\boxed{\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \Delta x}$$

Uncertainity in x (Δx)

$\boxed{\sigma = \Delta x}$ Uncertainity of R.V.

Q.55 Suppose an probability distribution is defined as-

$$f(x) = \frac{1}{n} \quad \text{for } x = 1, 2, 3, \ldots n$$

The mean of the distribution will be-

(a) $\frac{n+1}{2}$ (b) $\frac{n}{2}$ (c) $\frac{n-1}{2}$ (d) $\frac{n}{2}-1$

Sol^n Uniform probability distribution (all outcomes are equal probability)

$$\mu = \langle x \rangle = \sum_{n=1}^{n} x_i (P(x_i))$$

$$= 1 \times \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \times \frac{1}{n} + \cdots + n \times \frac{1}{n}$$

$$= \frac{1}{n}\left[1+2+3+\cdots+n\right]$$

$$= \cancel{\frac{1}{n}} \cdot \frac{\cancel{n}(n+1)}{2}$$

$$= \frac{(n+1)}{2}$$ Ans

Q.56 The variance of the distribution will be-

$$\sigma^2 = \langle x^2 \rangle + \langle x \rangle^2$$

$$\langle x^2 \rangle = \sum_{i=1}^{n} x_i^2 \, P(x_i)$$

$$= (1^2 + 2^2 + 3^2 + \cdots)\frac{1}{n}$$

$$= \frac{\cancel{n}(n+1)(2n+1)}{6\cancel{n}} = \frac{(n+1)(2n+1)}{6}$$

$$\langle x^2 \rangle = \frac{(n+1)(2n+1)}{6}$$

$$\langle x\rangle^2 = (1+2+3+\cdots)^2 \frac{1}{n^2}$$

$$= \left(\frac{n(n+1)}{2}\right)^2 \cdot \frac{1}{n} = \frac{n^2(n+1)^2}{4}\cdot\frac{1}{\cancel{n^2}}$$

$$\langle x\rangle^2 = \frac{n(n+1)^2}{4}$$

$$\sigma^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(n+1)(2n+1) - 3[(n+1)^2]}{12}$$

$$= \frac{2[2n^2+n+2n+1] - 3[(n^2+1+2n)]}{12}$$

$$= \frac{4n^2+\cancel{6n}+2-3n^2-3-\cancel{6n}}{12}$$

$$= \frac{n^2-1}{12} \quad \text{Ans}$$

S7

$$x = \begin{cases} 1, & \text{if the die results in an even no.} \\ 0, & \text{if the die results in an odd no.} \end{cases}$$

$$\langle x\rangle = \frac{\sum_{\text{even}} x_i P(x_i)}{1\times\frac{1}{2}} - \frac{\sum_{\text{odd}} x_i P(x_i)}{\phi\times\frac{1}{2}}$$

$$=$$

* BINOMIAL DITRIBUTION:-

How we understant we apply B.D. in the given problem:- 1- A random experiment repeated many time and each trial is identical and independent of each other.

2.- In each trial there are only two outcome success and failiour.

How to Calculate probability in Binomial Dist:-

If a random experiment repeated n times then probability of r - no. of successes of event A will be-

$$\boxed{P(r) = {}^{n}C_{r}\, p^{r}\, q^{n-r}} \quad (r \leq n)$$

n = no. of trial

r = no. of successe of event A

p = probability of success of event A in a single trial

q = probability of faileour of event A in a single trial.

58 An unbiased coin is tossed 6 times. What is the probability of obtaining 4 heads ?

(a) $3/16$ (b) $15/64$ (c) $15/32$ (d) $21/64$

Sol^n

$n = 6$

getting head = Success

T = Failure

$r = 4$

$p = \frac{1}{2}$

$q = \frac{1}{2}$

$$P(4) = {}^6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{6-4}$$

$$= \frac{6!}{4!\,(6-4)!} \left(\frac{1}{16}\right)\left(\frac{1}{4}\right)$$

$$= \frac{\cancel{6}^{3} \times 5 \times \cancel{4!}}{\cancel{2} \times 1 \times \cancel{4!}}$$

$$= \frac{15}{64}$$ Ans

62

Sol^n Single trial → three coins are tossed together

$n = 5$

$$\underbrace{HHH,\ HHT,\ HTH,\ HTT,\ THT}_{\text{Success}},\ \underbrace{TTT}_{\text{Failure}}$$

Probability of success $(p) = \frac{7}{8}$

" " Failure $(q) = \frac{1}{8}$

no. of sucess $(r) = 5$

$$P(r) = {}^{5}C_{5} (p)^{5} (q)^{5-5}$$

$$= \frac{5!}{5!\,(5-5)!} \left(\frac{7}{8}\right)^{5} \left(\frac{1}{8}\right)^{0}$$

$$= \left(\frac{7}{8}\right)^{5} \text{ Ans}$$

Q. 64

Sol^n

$\frac{1}{3}$ is the probability of odd taste → Success

$\frac{2}{3}$ → failure

$n = 5$

$r = 4$ or 5 → mutually exclusive

$$P = P(r=4) + P(r=5)$$

$$= {}^{5}C_{4} \left(\frac{1}{3}\right)^{4} \left(\frac{2}{3}\right)^{1} + {}^{5}C_{5} \left(\frac{1}{3}\right)^{5} \left(\frac{2}{3}\right)^{0}$$

$$= 5 \left(\frac{1}{81}\right) \left(\frac{2}{3}\right) + \frac{1}{243}$$

$$= \frac{10}{243} + \frac{1}{243} = \frac{10+1}{243} = \frac{11}{243} \text{ A}$$

Q. 14

Soln

6B & 6G. ✓ $\boxed{B}$ ✓ $\boxed{B}$ ✓ $\boxed{B}$ ✓ $\boxed{B}$ ✓ $\boxed{B}$ ✓ $\boxed{B}$ ✓

So $\quad {}^{12}P_{12} = 12!$

So $\Rightarrow \quad P = \dfrac{7\times 6!\times 6!}{12!}$

Q. 19

Soln

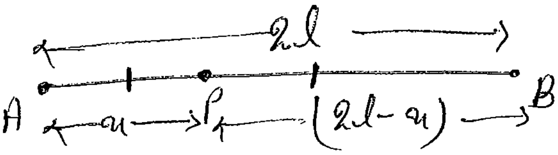
$P(AP \times BP) > \dfrac{l^2}{2}$

$P = \dfrac{\text{favourable Length}}{\text{total length}}$

So $\therefore\ x^2 < a^2 \Rightarrow -a < x < a$

$\Rightarrow \therefore\ -\dfrac{l}{\sqrt{2}} < (x-l) < \dfrac{l}{\sqrt{2}}$

$\Rightarrow\ l - \dfrac{l}{\sqrt{2}} < (x-l) < l + \dfrac{l}{\sqrt{2}}$



$x(2l-x) > \dfrac{l^2}{2}$

$\Rightarrow (2lx - x^2) > \dfrac{l^2}{2}$

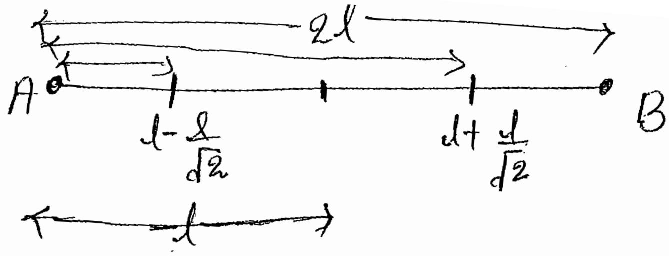
$\Rightarrow x^2 - 2lx < -\dfrac{l^2}{2}$

$\Rightarrow x^2 - 2lx + l^2 < l^2 - \dfrac{l^2}{2}$

$(x-l)^2 < \dfrac{l^2}{2}$

$\boxed{(x-l)^2 < \left(\dfrac{l}{\sqrt{2}}\right)^2}$

So,



favorable length $= \cancel{l}+\frac{l}{\sqrt{2}} - \cancel{l}+\frac{l}{\sqrt{2}}$

$= \frac{2l}{\sqrt{2}} = \sqrt{2}l$

favorable length $= \sqrt{2}l$

So $P = \frac{\sqrt{2}\cancel{l}}{2\cancel{l}} = \frac{1}{\sqrt{2}}$ Ans

Q.11

Sol^n

$P(\text{odd}) = 3P(\text{Even})$

$\because$ Probability of getting even + Probability of given even $= 1$

$P(E) + P(0) = 1$

$P(E) = \frac{1}{4}$

$P(0) = \frac{3}{4}$

$P(E-E)$ or $(0-0)$

$P = P(E-E) + P(0-0)$

$= \left(\frac{1}{4}\times\frac{1}{4}\right) + \left(\frac{3}{4}\times\frac{3}{4}\right) = \frac{10}{16} = \frac{5}{8}$ Ans

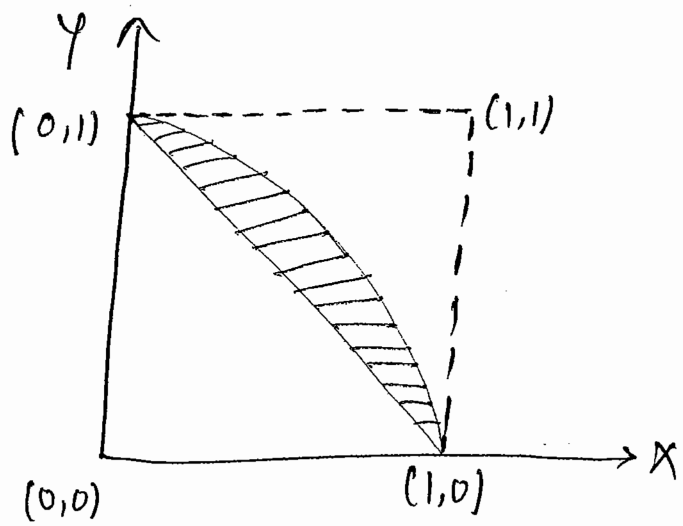
(23)

Sol^n

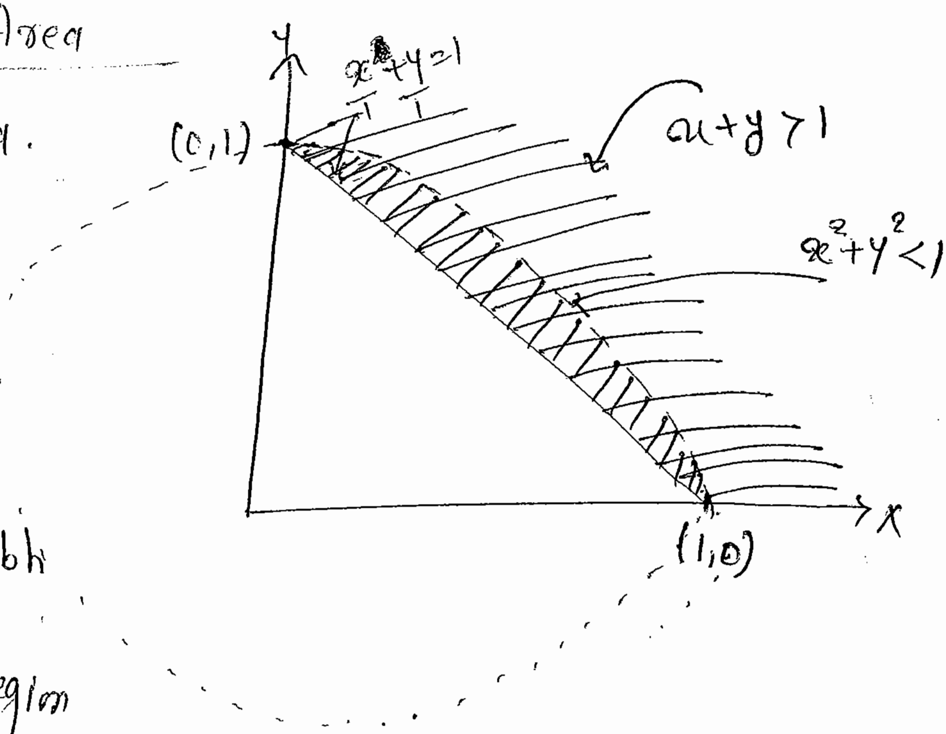
$x \in [0,1]$

$Y \in [0,1]$

$P\left[\begin{matrix} x+y>1 \\ x^2+y^2<1 \end{matrix}\right] = ?$



$P = \dfrac{\text{favourable Area}}{\text{Total area.}}$



Area of arc $= \frac{1}{4}\pi r^2$

($\because$ it is $\frac{1}{4}$ part of circle).

Area of triangle $= \frac{1}{2}bh$

So are of shaded region

$= \frac{1}{4}\pi r^2 - \frac{1}{2}bh$

$= \frac{\pi}{4} - \frac{1}{2}$

$\therefore$ Probability $= \left(\frac{\pi}{4} - \frac{1}{2}\right)$ Ans

24

Sol^n $\quad 100 \longrightarrow 999 \qquad \overset{9}{\square}\ \overset{10}{\square}\ \overset{10}{\square}$

$$P = \frac{8 \times \cancel{9} \times 9}{\cancel{9} \times 10 \times 10} = \frac{72}{100}$$

If repeatation is not allowed then -

$$P = \frac{\cancel{8} \times 8 \times 7}{9 \times 9 \times \cancel{8}} = \frac{56}{81}$$

Q.29

Sol^n

$P(H) = 0.8$

$P(W) = 0.9$

H	W
✓	✓
✓	✗
✗	✓
✗	✗

$$P = 1 - P'(H)\,P'(W)$$
$$= 1 - (0.2)(0.1)$$
$$= 0.98 \text{ Ans}$$

30

Sol^n

5 – Balls
2 Ball
2 Ball white

(i) 2W, 3 other ~~clou~~ colours $= B_1$

(ii) 3W, 2 other colour $= B_2$

(iii) 4W, 1 other colour $= B_3$

(iv) $\boxed{5W, \text{ 0 other colour}}$ $= B_4$ $P = ??$

A: Selecting 2 W Balls.

$$P(B_4 | A) = ??$$

$$\because P(B_1) = P(B_2) = P(B_3)\, P(B_4) = \frac{1}{4}$$

$$P(A|B_1) = \frac{{}^2C_2}{{}^5C_2} = \frac{\frac{2!}{2!(2-1)!}}{\frac{5!}{2!(5-2)!}} = \frac{\frac{2}{2}}{\frac{3!}{2! \times 3!}}$$

$$= \frac{2! \times 3!}{5!} = \frac{\cancel{2} \times \cancel{3} \times \cancel{2} \times 1}{5 \times \cancel{4} \times \cancel{3} \times 2 \times 1}$$

$$= \frac{1}{10}$$

$$P(A|B_2) = \frac{{}^3C_2}{{}^5C_2} = \frac{\frac{3!}{2! \times 1!}}{\frac{5!}{2! \times 3!}} = \frac{3! \times 2! \times 3!}{5! \times 2!}$$

$$= \frac{3 \times \cancel{2!} \times \cancel{2!} \times \cancel{3!}}{5 \times \cancel{4} \times 4 \times \cancel{3!} \times \cancel{2!} \times 1! \times \cancel{2!}} = \frac{3}{20}$$

$$P(A|B_3) = \frac{{}^4C_2}{{}^5C_2} = \frac{\frac{4!}{2! \times 2!}}{\frac{5!}{2! \times 3!}} = \frac{\cancel{4!}}{\cancel{2!} \times 2!} \times \frac{\cancel{2!} \times 3!}{5 \times \cancel{4!}}$$

$$= \frac{3 \times \cancel{2!}}{\cancel{2!} \times 5} = \frac{3}{5}$$

$$P(A|B_4) = \frac{{}^5C_2}{{}^5C_2} = 1$$

Q.33

Soln

1 2 3 - - - - - - n

for choosing first ball prob. $= \frac{1}{n}$

So $\frac{1}{n} \cdot \frac{1}{(n-1)} \cdot \frac{1}{(n-2)} - - - - - \frac{1}{1}$

$$= \frac{1}{n!}$$

Q.38

Soln

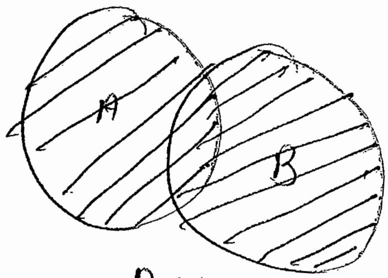
$P(A^c) = 1 - P(A)$ means ~~event A~~ probability of ~~does~~ not occuring event A.

$P(A^c \cap B^c) = \frac{1}{3}$

$P(A \cap B) = \frac{1}{2}$

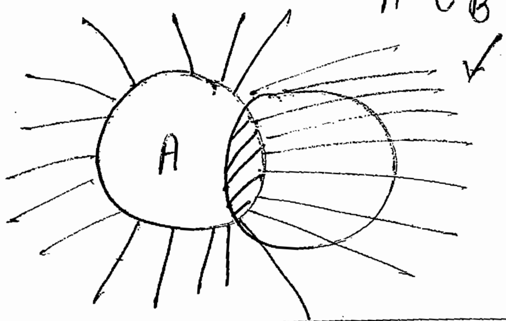
$P(B) = 2P(A)$

$P(A \cap B) = \frac{1}{9}$



A U B

$A^C \cup B^C = U - (A \cap$



$\Rightarrow P(A) + P(B) - P(A \cup B) = \frac{1}{2}$

$P(A^c \cap B^c) = \frac{1}{3}$

$\Rightarrow P(A^c) + P(B^c) - P(A^c \cup B^c) = \frac{1}{3}$

$\boxed{P(A^C \cup B^C) = 1 - P(A \cap}$

$\Rightarrow [1 - P(A)] + [1 - P(B)] - [1 - P(A \cap B)] = \frac{1}{3}$

$\Rightarrow 2 - P(A) - P(B) - 1 + \frac{1}{2} = \frac{1}{3}$

$\Rightarrow \frac{3}{2} - P(A) - 2P(A) = \frac{1}{3}$

$\Rightarrow +3P(A) = \frac{3}{2} - \frac{1}{3} = \frac{9-2}{6} = \frac{7}{6}$

So $\boxed{P(A) = \frac{7}{18}}$ Ans

Q.44

Sol^n

$$P(S) = \frac{2}{8} = \frac{1}{4}$$

$$P(F) = \frac{3}{4}$$

Either 2^{nd} $\quad \frac{3}{4} \times \frac{1}{4}$

or 4^{th} $\quad \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}$

$\vdots$

infinite G.P. Series.

$$S_n = \frac{a}{1-r} = \frac{\frac{3}{16}}{1 - 9/16}$$

$$= \frac{3}{\cancel{16}} \times \frac{\cancel{16}}{(16-9)} = \boxed{\frac{3}{7}}$$

Q.45

$\underline{Sol^n}$

Bag 1 $\xrightarrow[\text{Black / white}]{\text{1 Ball}}$ Transfer Bag 2

3W, 2B

2W, 4B

$\downarrow$ 1 Ball Selecting

White

A : Selecting a white ball from bag 2.

B_1 : transfering a white ball. ≈ Selecting a white ball from bag 1.

B_2 : " " Black "

$$P(B_1|A) = \frac{P(A|B_1)\,P(B_1)}{P(A|B_1)\,P(B_2) + P(B_2)\,P(A|B_2)}$$

$$P(B_1) = \frac{{}^3C_1}{{}^5C_1} = \frac{3}{5}$$

$$P(B_2) = \frac{{}^2C_1}{{}^5C_1} = \frac{2}{5}$$

$$P(A|B_1) = \frac{{}^3C_1}{{}^7C_1} = \frac{3}{7}$$

$$P(A|B_2) = \frac{{}^2C_1}{{}^7C_1} = \frac{2}{7}$$

$$P[B_1|A] = \frac{\frac{3}{7} \times \frac{3}{5}}{\frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{2}{7}} =$$

Q.61

Sol^n

$\because$ 10 lines $\qquad n = 10$

n = no. of trial

$p = 0.8$

p = probability of success in a single trial.

$\mu = \langle n \rangle = np$ ← Expected value.

$= 0.8 \times 10 \qquad P(r) = {}^nC_r\, p^r q^{n-r}$

r → random variable

$= 8$ Ans

Random Walk Problem

① Symmetric Random Walk
② Assymmetric Random Walk.

* When a fair coin is tossed in 3 times:-

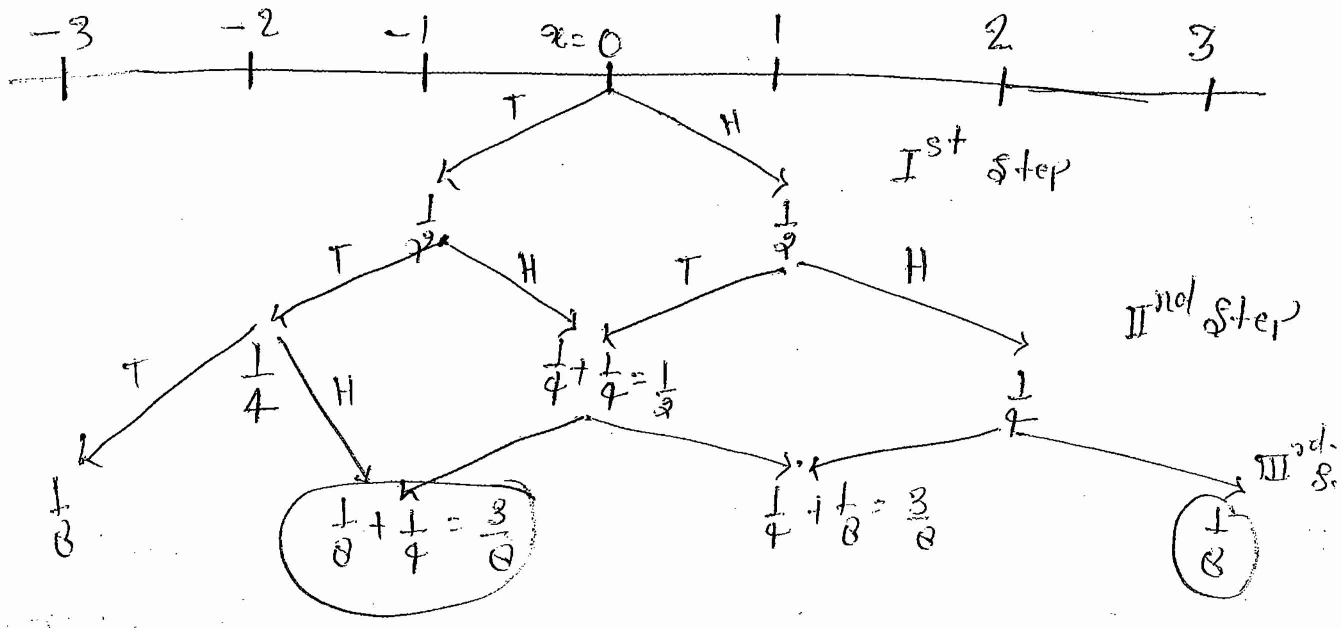
Outcomes:-

$$\left.\begin{matrix}HHH\end{matrix}\right\} 3H, 0T = \frac{1}{8}$$

$$\left.\begin{matrix}HHT\\HTH\\THH\end{matrix}\right\} 2H, 1T = \frac{3}{8}$$

$$\left.\begin{matrix}HTT\\TTH\\THT\end{matrix}\right\} 1H, 2T = \frac{3}{8}$$

$$\left.\begin{matrix}TTT\end{matrix}\right\} \text{0 Head 3-Tail} = \frac{1}{8}$$



* Assymmetric Random Walk:-

Suppose P is the probability to take a step to the right and q is the probability to take a step to the left.

P = Probability of Success in the single trial

q = Probability of failure in the single trial.

N = total no. of steps

= no. of trials

Probability of that the random walke takes m steps right and n steps left.

m steps to the right $\rightarrow$ No. of Success

n steps to the left $\longrightarrow$ No. of failure.

$$m+n = N$$

$$P = {}^N C_m \, p^m q^{N-m}$$

$$\boxed{P = {}^N C_m \, p^m q^n}$$

* What is the probability of total displacement M after N no. of steps. (step length = 1 unit).

M = displacement

N = no. of steps = $(m+n)$ $\Rightarrow m = \frac{N+M}{2}$

$M = (m-n) \cdot l$ ← step length

$\Rightarrow n = \frac{N-M}{2}$

$$P_N(M) = \frac{N!}{m!\,(N-m)!}\, p^m q^n$$

$$\boxed{P_N(M) = \frac{N!}{\left[\frac{N+M}{2}\right]!\,\left[\frac{N-M}{2}\right]!}\,(p)^{\frac{N+M}{2}}\, q^{\frac{N-M}{2}}}$$

for Symmetric Random Walk, $p = q = \frac{1}{2}$

$$\boxed{P_N(M) = \frac{N!}{\left(\frac{N+M}{2}\right)!\,\left(\frac{N-M}{2}\right)!}\left(\frac{1}{2}\right)^N}$$

Ex - 3 Steps : $N=3$, $M=-1$

$x = -1$

$$P = \frac{3!}{1!\,2!}\left(\frac{1}{2}\right)^3 = \frac{3}{8} \text{ Ans}$$

CSIR June 2017

Q. 82

Sol^n $N = 4$, $M = 0$

$$P_4(0) = \frac{4!}{2! \times 2!}\left(\frac{1}{2}\right)^4 = \frac{\cancel{4}^{2} \times 3 \times \cancel{2} \times \cancel{1}}{\cancel{2} \times 1 \times \cancel{2} \times \cancel{1}}\left(\frac{1}{2}\right)^4 = \frac{3}{8} \text{ Ans}$$

* What is the average displacement of a Assymmetric random walker after N no. of step?

$\langle M \rangle = ??$

$\because \quad M = m - n$

$= m - (N - m)$

$M = 2m - N$

$\langle M \rangle = 2\langle m \rangle - \langle N \rangle$

$\langle M \rangle = 2 \cdot \langle m \rangle - N$ $\left\{ \because N \text{ is no. of steps and it is fixed} \right\}$

$= 2(pN) - N$

$$\boxed{\langle M \rangle = (2p - 1)N}$$

We can calculate directly -

$M = m - n$

$\langle M \rangle = \langle m \rangle - \langle n \rangle$

$= Np - Nq$

$= N(p - q)$

$$\boxed{\langle m \rangle = (2p-1)N}$$

* For Symmetric Random Walk:-

$p = q = \frac{1}{2}$

So $\boxed{\langle M \rangle = 0}$

Q.13. Two drunks starts out together each having equal probability to making a steps simultanious at left an right. The probability that they will meet after n steps.

(a) $\frac{1}{4^n} \frac{(2n)!}{(n!)^2}$ (b) $\frac{1}{2^n} \frac{(2n)!}{(n!)^2}$

(c) $\frac{1}{2^n} (2n)!$ (d) $\frac{1}{4^n} (n!)$

Solⁿ

$R_1 \longrightarrow n$ steps

$R_2 \longrightarrow n$ steps

Consider Relative motion :-

R_1 fixed and R_2 is moving w.r. to R_1

$N = 2n$

Relative displacement $M = 0$

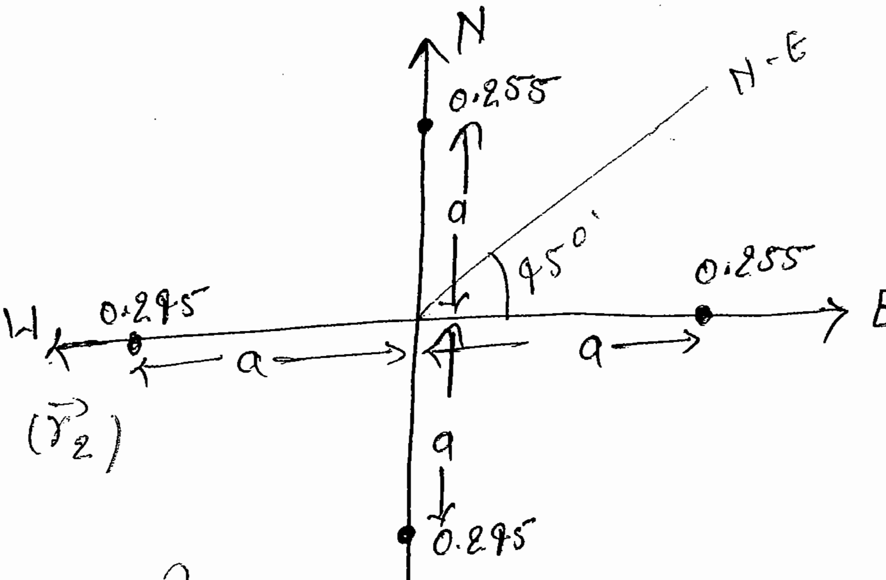
$$P_{2n}(0) = \frac{(2n)!}{\left(\frac{2n+0}{2}\right)! \left(\frac{2n-0}{2}\right)!} \left(\frac{1}{2}\right)^{2n}$$

$$= \frac{2n!}{n! \times n!} \left(\frac{1}{2}\right)^{2n}.$$

A-8
Q.21

Sol^n

After one step :-



$$\langle \vec{r}^{\,2} \rangle = \sum \vec{r}_2 \, P(\vec{r}_2)$$

$$= a\hat{i}\,(0.255) + a\hat{j}\,(0.255) + a(-\hat{i})\,0.245 + a(-\hat{j})(0.245)$$

$$= 0.01\,a\,(\hat{i}+\hat{j})$$

After N step :-

$$\langle \vec{r} \rangle = 0.01\,Na\,(\hat{i}+\hat{j})$$

$$= \sqrt{2} \times 0.01\,Na$$

direction $\theta = \tan^{-1}\frac{y}{x}$

$$= \tan^{-1} 1 = 45^\circ$$

(d) is wrong because it is not a symmetric walk

(c) " " " Prob. of N-E dirc^n is greatter so S-E dirc^n is wrong

(b) Here N is in square root so it is wrong.

* POISSON

Poisson distribution is a limiting form of binomial distribution.

(i) If the no. of trial is infinitely large $(n \to \infty)$

(ii) Probability of success in a single trial is very small $(p \to 0)$

(iii) But n.p is a finite quantity. (np = finite).

Then Bionomial distribution converts into Poisson distribution.

Ex - like

$$n = 0.001 \leftarrow \text{Very small.}$$
$$p = 1000 \leftarrow \text{very large}$$
$$\boxed{np = 14} \text{ finite}$$

⇒ If the probability of success in a single trial is calculated from average concept then we generally use poinsson distribution.

Ex- Suppose we have a book which have 100 errors in 600 p.g. book. distributed randomly.

then 100 error.

600 pages

no. of single trial = 600

$$p = \frac{100}{600} = \frac{1}{6}$$

If a random experiment is repeated 'n' times then the probability of success of event A for 'r' times will be -

$$\boxed{P(r) = \frac{e^{-np}(np)^r}{r!}}$$

n = no. of trial

p = prob. of success in a single trial

r = no. of success.

Mean of distribution $(\mu) = np$

Variance " " $(\sigma^2) = np$

Standard deviation " $(\sigma) = \sqrt{np}$

A-8

Q. 67

Sol^n

$n = 2000$

$p = 0.001$

Success is person is affected in single trial.

$np = 2.$

$P(r > 2) = ?$

$r = 0, 1, 2, \boxed{3, \ldots\ldots 2000}$

$\Rightarrow$ ~~P(r>2)~~ $P(r=3) + P(r=4) \ldots\ldots P(r=2000)$

$= 1 - [P(r=0) + P(r=1) + P(r=2)]$ ...

$\therefore P(r > 2) = 1 - (P(r \leq 2))$

$= 1 - \left[\frac{e^{-2} \cdot 2}{0!} + \frac{e^{-2}\, 2^{1}}{1!} + \frac{e^{-2} \cdot 2^{2}}{2!} \ldots\right]$

$= 1 - [\cancel{1} e^{-2} + 2e^{-2} + \frac{\cancel{4} e^{-2}}{\cancel{2}}]$

$= 1 - 5e^{-2}$

$= 1 - \frac{5}{e^2}$

Q.69

Sol^n

60 min $\longrightarrow$ 20 buses

1 min $\longrightarrow \frac{\cancel{20}}{\cancel{60}\,3} = \frac{1}{3}$

Checking in 1 min $\rightarrow$ Single trial
$\rightarrow$ Prob. $\left(\frac{1}{3}\right)$

No. of trial(n) = 5

Success(r) is the coming the Bus. in single trial

$P(r=0) = \frac{e^{-5 \times \frac{1}{3}} \left(5 \times \frac{1}{3}\right)^{0}}{0!} = e^{-\frac{5}{3}}$

$\boxed{P(r=0) = 0.19}$ Ans {given $e^{-5/3} = 0.19$.}

Q 71

Sol^n

~~No. of trial $\neq$ N~~

Here is 3 flows (errors) in = 1 sq. m (average)

We have to check 0.5 sq. m. $\rightarrow$ 4 flows

= 10000 sq. c.m.

5000 sq.cm

Single trial $\longrightarrow$ Checking 1 sq.cm.

$n = 5000$

$p = \dfrac{3}{10000}$ = Prob. of success in single trial.

$np = \dfrac{3}{2}$

$$P(r=4) = \frac{e^{-3/2}\,(3/2)^4}{4!} = \frac{0.223 \times \frac{3^4}{2^4}}{4\times3\times2\times1}$$

$$= \frac{81 \times 0.223}{16 \times 24} = \frac{18.063}{384}$$

$$\boxed{P(r=4) = 0.047}$$

* Continuous Probability distribution :-

Suppose x is a continuous random variable corresponding probability $f(x)$

$$\boxed{\int_{-\infty}^{+\infty} f(x)\, dx = 1} \Leftarrow \text{Normalisation condition.}$$

then $f(x)$ will be a probability density function or probability distribution function of a perticular continuous probability distribution

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} f(x)\, dx$$

If it is non normalised then -

$$P(x_1 < x < x_2) = \frac{\int_{x_1}^{x_2} f(x)\, dx}{\int_{-\infty}^{+\infty} f(x)\, dx}$$

* Mean or Expected valeu of x :-

$$\text{Mean } \mu = \langle x \rangle = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$\text{Variance } \sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

Standard daviation $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

Q.51

Sol^n (a) $f(x) = K e^{-|x|} \quad (-\infty < x < \infty)$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{+\infty} K e^{-|x|}\,dx = 1$$

$$\Rightarrow \boxed{K = \frac{1}{2}}$$

(b) $$\mu = \langle x \rangle = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x \cdot e^{-|x|}\,dx = 0$$

(c) $$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

(d) $P(0 < x < 4) = \int_0^4 f(x)\, dx$

qus. TIFR. - 2 times

Q.56 If the distribution function of x is
B-A-6

$$f(x) = x\, e^{-x/\lambda} \qquad (0 < x < \infty)$$

(i) mean of x is $\langle x \rangle$

(ii) $\dfrac{\langle x \rangle}{x_p} = ??$ where x_p = most probable value of 'x'.

Sol^n

$$\langle x \rangle = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$= \frac{\int_{-\infty}^{+\infty} x^2 e^{-x/\lambda}\, dx}{\int_0^{\infty} f(x)\, dx}$$

$\because$ function is non-normalise. it is distribution function not probability.

$$= \frac{\int_0^\infty x^2 e^{-x/\lambda} dx}{\int_0^\infty x e^{-x/\lambda} dx} = \frac{\frac{2!}{\left(\frac{1}{\lambda}\right)^3}}{1/\left(\frac{1}{\lambda}\right)^2}$$

$$= \frac{2\lambda^3}{\lambda^2} = 2\lambda$$

$$\boxed{\langle x \rangle = 2\lambda}$$

$$f(x) = x e^{-x/\lambda}$$

$$\Rightarrow \frac{d f(x)}{dx} = 0$$

$$\frac{d}{dx}\left(x e^{-x/\lambda}\right) = 0$$

$$\Rightarrow x\left[\frac{-1}{\lambda} e^{-x/\lambda}\right] + e^{-x/\lambda} = 0$$

$$\Rightarrow e^{-x/\lambda}\left[1 - \frac{x}{\lambda}\right] = 0$$

$$\therefore \quad 1 - \frac{x}{\lambda} = 0$$

$$\boxed{x = \lambda}$$

$$\therefore \quad x = x_p = \lambda$$

$$\therefore \quad \frac{\langle x \rangle}{x_p} = \frac{2\cancel{\lambda}}{\cancel{\lambda}} = 2$$

$$\boxed{\frac{\langle x \rangle}{x_p} = 2}$$

* Gaussian / Normal Distribution:-

Suppose x is is continuous random variable corresponding to probability $f(x)$

$$x \rightarrow f(x)$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Normalization Constant

μ = Mean

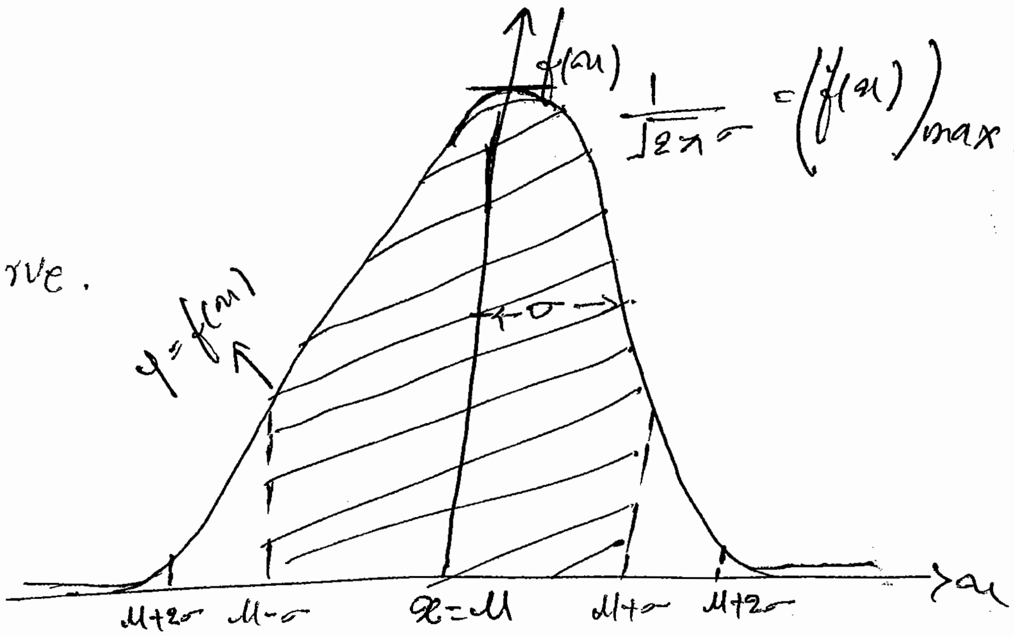
σ = S.D.

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1$$

$f(x)$ Continuous

Plot:-

$y = f(x) \rightarrow$ Curve.



$$\int_a^b f(x)\, dx = \text{area under the curve from a to b}$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = \text{area under the curve.}$$

Area under the Gaussian curve is a measurement of probability. Therefor the total area under the curve should be unity.

$\sigma \uparrow = f(x)|_{max} \downarrow$ so width $\uparrow$

$\Rightarrow$ Area under the curve between $\mu-\sigma$ to $\mu+\sigma$ is the probability of finding x ~~is~~ b/w $\mu-\sigma$ to $\mu+\sigma$. i.e. 0.687.

$$\int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx = 0.687$$

$= 68.7\%$ of total area.

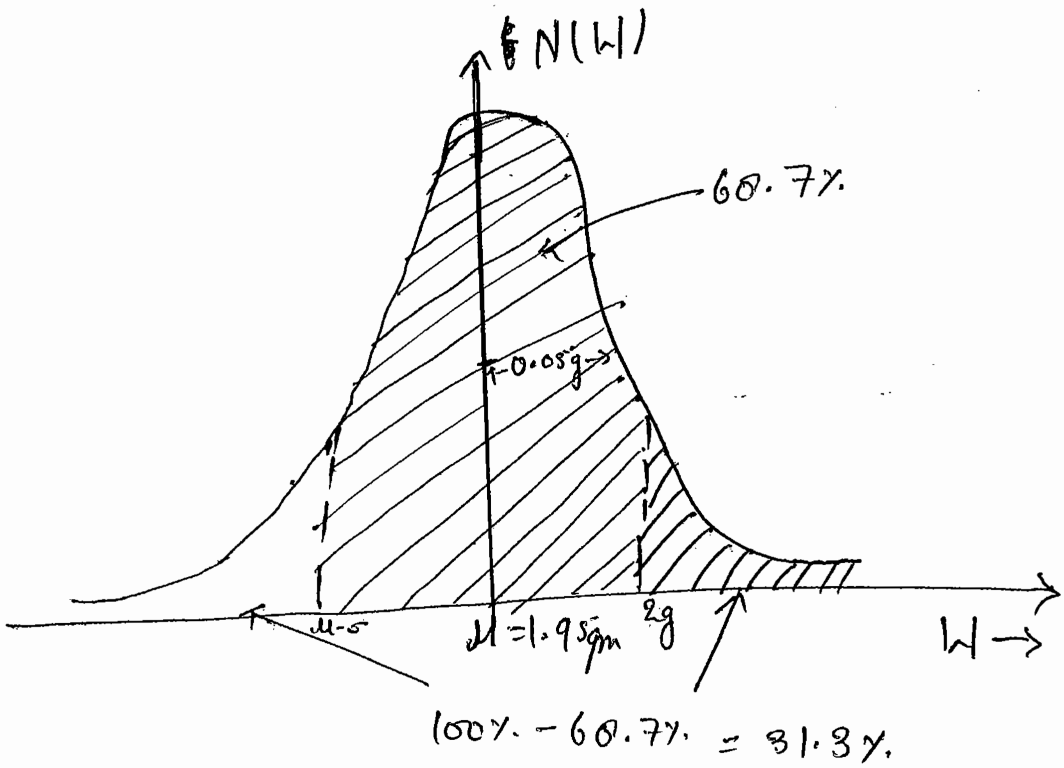
$$P(\mu-2\sigma \le x \le \mu+2\sigma)$$

$$= \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)\,dx = 0.955$$

$= 95.5\%$ of total area.

Q. 79.

Solⁿ



So area of black shaded region $= \frac{31.3\%}{2} = 15.63\%$

So $= 1000 \times 15.65\%$

$= 156.5$

≈ 155 Ans

* Normal Distribution :-

G. D. $\Rightarrow$ $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

let $\frac{x-\mu}{\sigma} = z$ (New variable)

$f(x)\,dx$ = prob. of finding x between x to $x+dx$

$f(z)\,dz$ " " z " z to $z+dz$

$f(x)\,dx \Rightarrow$ x to $x+dx$

$\downarrow \qquad \downarrow$

$f(z)\,dz \Rightarrow$ z to $z+dz$

$$f(x)\,dx = f(z)\,dz$$

$$f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

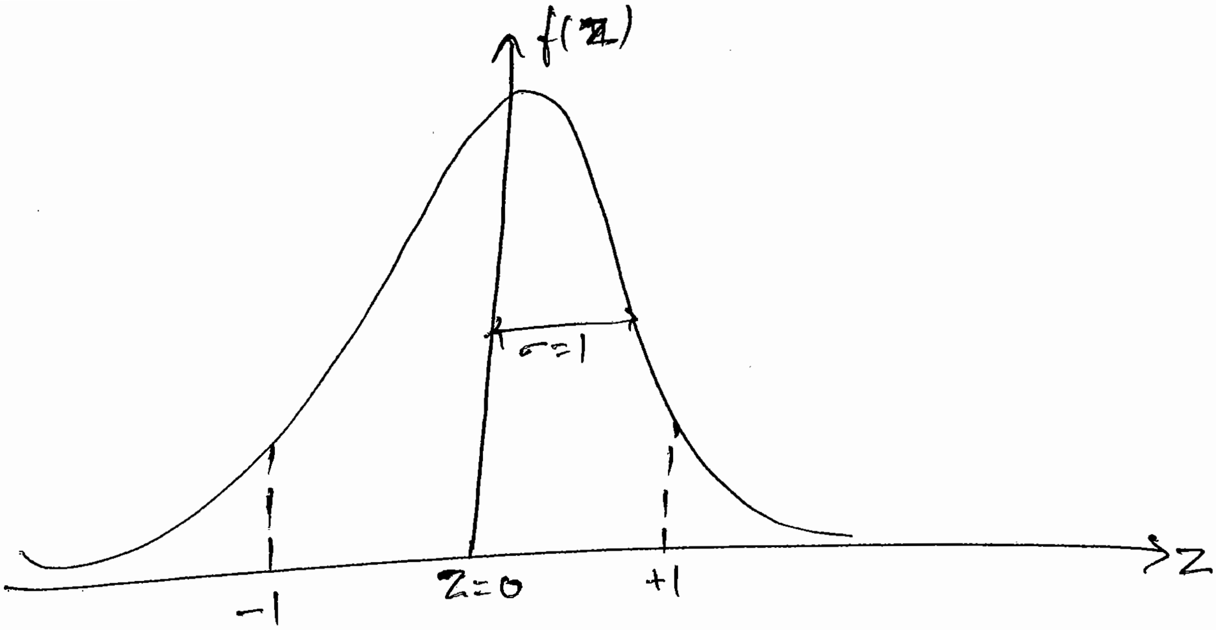
$$\frac{x-\mu}{\sigma} = z \Rightarrow \frac{dx}{\sigma} = dz$$

$$\Rightarrow f(x)\,dx = \frac{1}{\sqrt{2\pi}} - \exp\left[-\frac{z^2}{2}\right] dz = f(z)\,dz$$

$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right]$ $\rightarrow$ Probability distribution function of normal distribution.

$\mu = 0$, $\sigma = 1$

Normal distribution is Gaussian distribution of mean 0 and standard deviation = 1.



$P(\mu - \sigma < x < \mu + \sigma) = P(-1 < z < 1) = 0.607$

$P(\mu - 2\sigma < x < \mu + 2\sigma) = P(-2 < z < 2) = 0.955$

<u>15/oct/2014</u>

<u>Q.35</u>

<u>Sol</u>

$A_1, A_2, \ldots\ldots A_n$

$P(A_i) = \frac{1}{i+1} \quad (i = 1, 2, 3, \ldots n)$

$P(A_i^c) = 1 - \frac{1}{i+1} = \frac{i}{i+1}$

$P(A_1^c \cap A_2^c \cap A_3^c \cap \ldots\ldots A_n^c)$

$= P(A_1^c)\, P(A_2^c) \ldots\ldots P(A_n^c)$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \ldots\ldots \frac{\cancel{n}}{n+1}$

$= \frac{1}{n+1}$ <u>Ans</u>

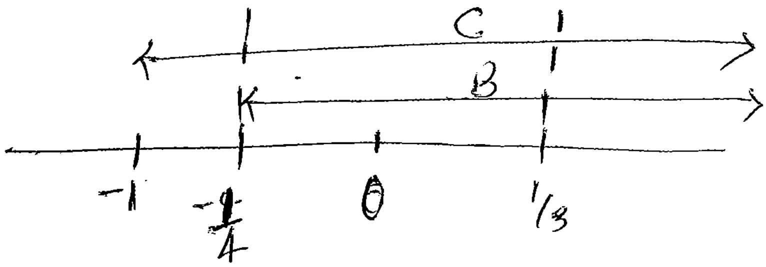
Q.40

Sol^n

$A: \frac{1-3P}{2} \geq 0 \Rightarrow 1-3P \geq 0 \Rightarrow P \leq \frac{1}{3}$

$B: \frac{1+4P}{3} \geq 0 \Rightarrow P \geq -\frac{1}{4}$

$C: \frac{1+P}{6} \geq \Rightarrow P \geq -1$



$$\boxed{P \equiv \left[-\frac{1}{4}, \frac{1}{3}\right]}$$

Q.13

Sol^n

A	B	C
✓	✓	✓
✓	×	✓
✓	✓	×
×	✓	✓
✓	×	×
×	✓	×
×	×	✓
×	×	×

$P = 1 - P(A' \cap B' \cap C')$

$= 1 - \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{4}$

$= 1 - \frac{1}{4}$

$$\boxed{P = \frac{3}{4}}$$

Q.6

Sol^n

STRANGE

$$P = \frac{\text{favorable no. of words}}{\text{Total no. of words.}}$$

$$P = \frac{{}^3P_2 \times {}^5P_5}{{}^7P_7}$$

=

Q.5

Sol^n

60000 to 70000

1 to 9

$\boxed{6}\ \square\ \square\ \square\ \boxed{7}$

$\boxed{6}\ \square\ \square\ \square\ \square$

$$P = \frac{1 \times \cancel{7} \times \cancel{6} \times \cancel{5} \times 1}{1 \times 8 \times \cancel{7} \times \cancel{6} \times \cancel{5}}$$

$\boxed{P = \frac{1}{8}}$ Ans

Second method by Permutation:-

$$\frac{{}^7P_3}{{}^8P_4} = \frac{1}{8}$$

Q.25

Sol^n

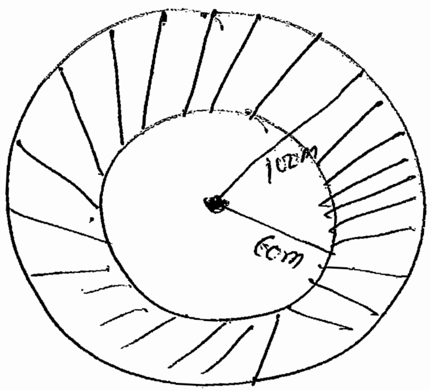
$$10 \times 10 \times 10 = 1000$$

$$000 \longrightarrow 999 \Rightarrow 1000$$

$$P = \frac{{}^{100}C_1}{{}^{1000}C_1} = \frac{1}{10} \text{ Ans}$$

Q. 26

Sol^n



$$P = \frac{\text{Favorable Area}}{\text{Total Area}}$$

$$= \frac{\pi(100)^2 - \pi(60)^2}{\pi(100)^2}$$

$$=$$

Fourier Series

If a function $f(x)$ repeats its value after equal interval of x then $f(x)$ is said to be a periodic function of 'x'.

$$\boxed{f(x) = f(x+T) = f(x+2T) = ----}$$

$f(x)$ periodic function of x of period T.

1. Sinusoidal Periodic Function ⇒ Ex- $\sin x$, $\cos x$ $T = 2\pi$
2. Non Sinusoidal Periodic Function. ⇒ Ex - $\tan x$, $\cot x$, $T = \pi$

   Square wave ⎤ T = Variab
   Rectangular wave. ⎦

* Any non sinusoidal periodic function can be written as a sum of sine or Cosine functions of Complex exponentials. This series is Known as the Fourier Series Expansion of the given non-sinusoidal periodic function.

$f(x) \longrightarrow$ $\cos nx$ / $\sin nx$ ⇒ Real Fourier Series expan

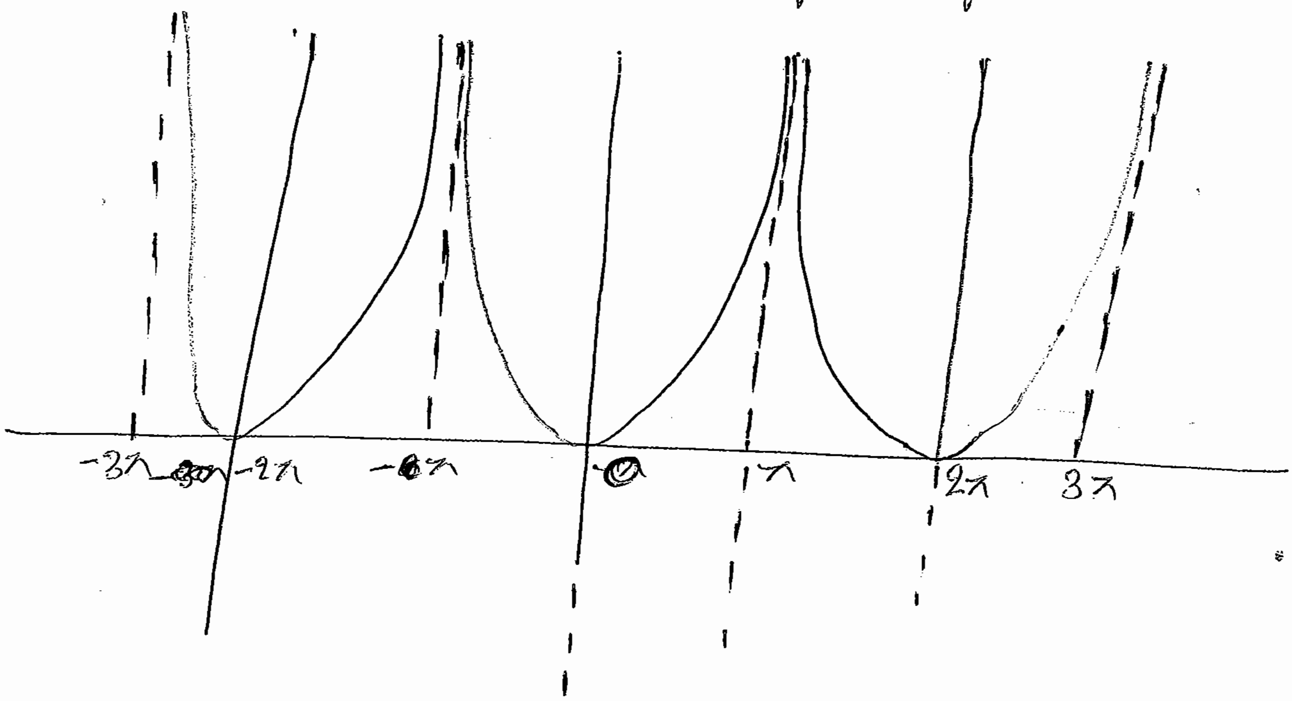
e^{inx} / e^{-inx} ⇒ Complex Fourier Series expan.

* Real Fourier Series Expansion :-
* Complex Fourier Series Expansion.

* Dirichlet's Condition :-

(i) $f(x)$ is defined in $[a,b]$

(ii) $f(x)$ is periodic function of x of period $T = |b-a|$

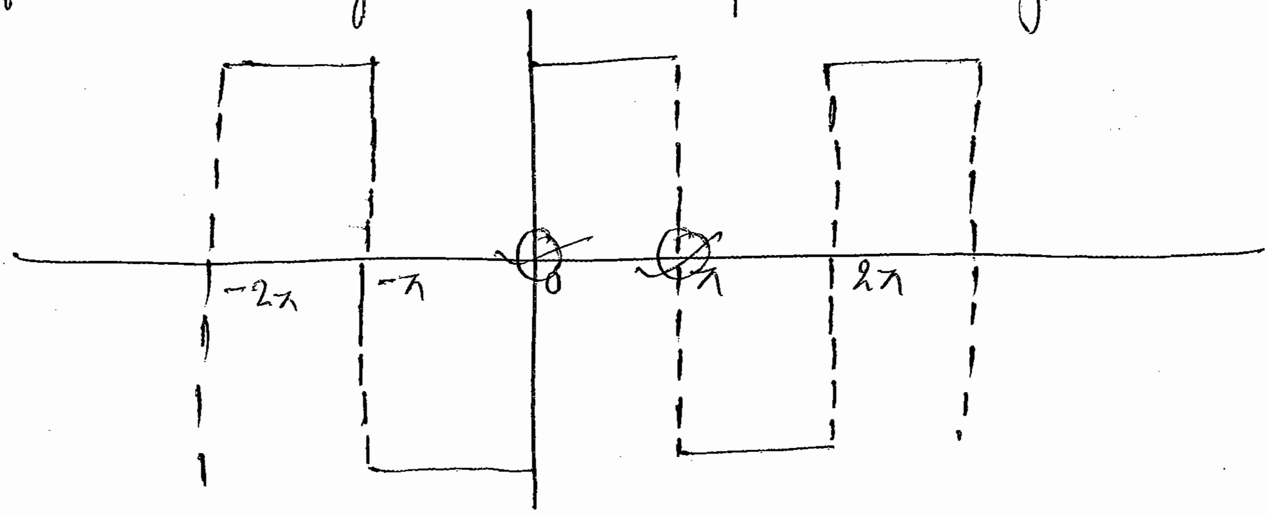


(ii) $f(x)$ will be single valued and bounded within the given interval.

$f(x) = (x^2-1)^{1/2}$ ← Let a single ~~variable~~ valued. (it contain 2 values)

Bounded when ⇒ $f(x) < \infty$ in $[a,b]$

(iii) $f(x)$ has finite no. of maxima and minima and finite no. of Discontinuity in the given interval



$$f(x) = \begin{cases} a & 0 \le x \le \pi \\ -a & -\pi \le x \le 0 \end{cases}$$

$[-\pi, \pi]$

Let $f(x)$ is defined in $[-\pi, \pi]$ or $[0, 2\pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—} \circledast$$

$a_0, a_n, b_n \Rightarrow$ Fourier Co-efficients.

$\frac{a_0}{2} \Rightarrow$ It is a constant term / DC term of the series

* $(n=1) \Rightarrow a_1 \cos x,\ b \sin x$; Fundamental frequency term

It is the minimum frequency term present in the fourier series

$T = 2\pi$

$$\boxed{f_1 = \frac{1}{T} = \frac{1}{2\pi}}$$ fundamental frequency

Principle of Harmonics:-

* $(n=2)$ to $\infty \Rightarrow$ $a_2 \cos 2x,\ a_3 \cos 3x, \ldots$; $b_2 \sin 2x,\ b_3 \sin 3x \ldots$] Harmonic

The frequency of the harmonic terms will be integer multiple of the fundamental frequency.

For $n = 2$, $T = \pi$ $\quad f_2 = \frac{1}{T} = \frac{1}{\pi} = 2 f_1$ first harmon

Second Harmonics:-

$$\boxed{f_3 = 2f_1} \leftarrow \text{Second Harmonics.}$$

Amplitude of Second cosine harmonic term $= a_3$

* Determination of Constant :-

(I) a_0 :

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^{2\pi} \cos nx\,dx}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_0^{2\pi} \sin nx\,dx}_{0}$$

$$\Rightarrow \int_0^{2\pi} f(x)\,dx = a_0 \pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx}$$

II a_n : $\rightarrow$ Multyply eq^n ⊛ with $\cos mx$

$$\int_0^{2\pi} f(x).\cos mx\,dx = \frac{a_0}{2}\underbrace{\int_0^{2\pi} \cos mx\,dx}_{0} + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos nx \cos mx\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\int_0^{2\pi} \sin nx \cos mx\,dx}_{0}$$

Formula :-

$$\left\{ \begin{array}{l} \int_0^{2\pi} \sin nx \cos mx\,dx = 0 \\ \int_0^{2\pi} \cos nx \cos mx\,dx = \pi\,\delta_{mn} \end{array} \right\}$$

$$\Rightarrow \int_0^{2\pi} f(x) \cos nx\,dx = a_n \pi$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx}$$

Similarly

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx}$$

(2) If $f(x)$ is defined in $[-l, l] / [0, 2l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\,dx$$

H-4
Q4

Soln

$$f(x) = 1 \qquad -1 \le x \le 0$$
$$= -2 \qquad 0 \le x \le 1$$

$\because$ Here $f(x) = -1$ to $+1$ So it is case (2)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n x \pi}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{L}$$

$$a_0 = \frac{1}{1}\left[\int_{-1}^{0} 1 . dx + \int_{0}^{1} -2\, dx\right]$$

$$\boxed{a_0 = -1}$$

$$a_3 = \frac{1}{l}\int f(x) \cos \frac{n\pi x}{l}\, dx = \frac{1}{1}\int_{-1}^{0} \cos(3\pi x)\, dx + \int_{0}^{1} -2 \cos(3\pi x) dx$$

$$= \left.\frac{\sin 3\pi x}{3\pi}\right|_{-1}^{0} - 2 \left.\frac{\sin 3\pi x}{3\pi}\right|_{0}^{1}$$

$$= \frac{1}{3\pi}\left[\sin 0 - \right.$$

$$\boxed{a_3 = 0}$$

$$b_3 = \frac{1}{1}\left[\int_{-1}^{0} 1 . \sin \frac{3\pi x}{1}\, dx + \int_{0}^{1} (-2) \sin \frac{3\pi x}{1}\, dx\right.$$

$$\boxed{b_3 = -\frac{2}{\pi}}$$

So option (b)

* Even function :-

$$f(-x) = f(x)$$

$f(x)$ is defined $[-l, l]$

∴ We know that -

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \quad \text{if } f(x) = \text{Even}$$

$$= 0 \qquad \text{if } f(x) = \text{odd}$$

So

$$a_0 = \frac{1}{l}\int_{-l}^{l} \overset{\text{Even}}{\boxed{f(x)}}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int \underbrace{f(x)}_{\text{Even}} \underbrace{\cos\frac{n\pi x}{l}}_{\text{Even}}\,dx$$

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{even}} \cdot \underbrace{\sin\frac{n\pi x}{l}}_{\text{odd}}\,dx = 0$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}}$$ ← Fourier Cosine Series.

* For Odd Function :-

$$f(-x) = -f(x)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\,\sin\frac{n\pi x}{l}\,dx$$

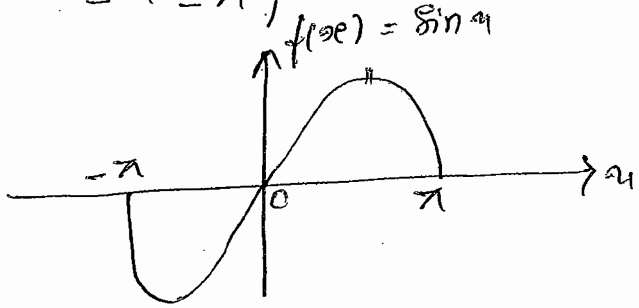
$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}} \leftarrow \text{Fourier sine series.}$$

Q.9

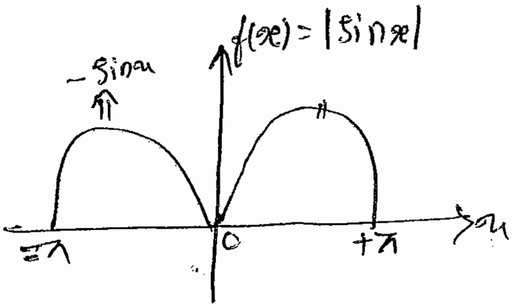
Sol^n

$$f(x) = |\sin x| \qquad (-\pi \le x \le \pi)$$

$$f(-x) = |\sin(-x)|$$
$$= |-\sin x|$$
$$= \sin x = f(x)$$

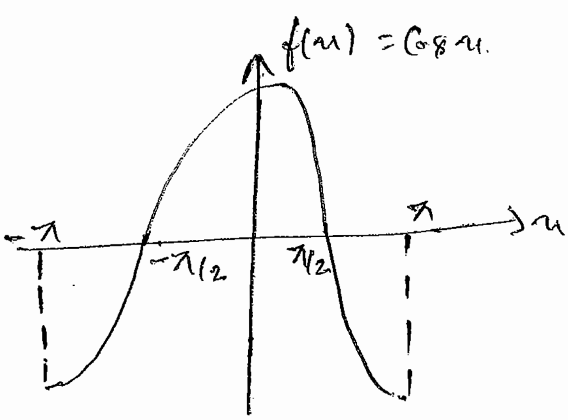


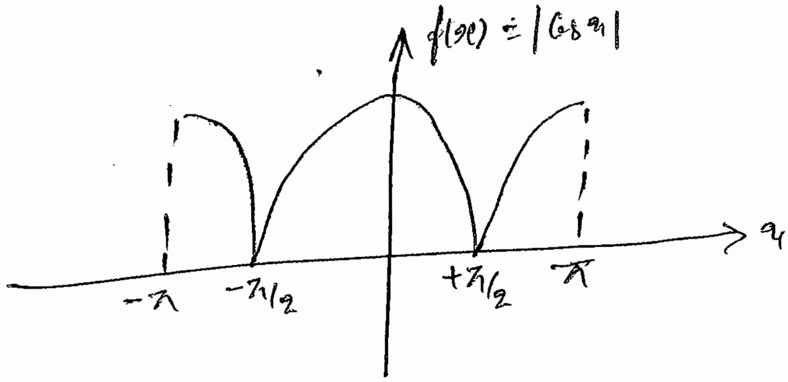
$$f(x) = \sin x \qquad 0 \le x \le \pi$$
$$= -\sin x \qquad -\pi \le x \le 0$$



$$b_n = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

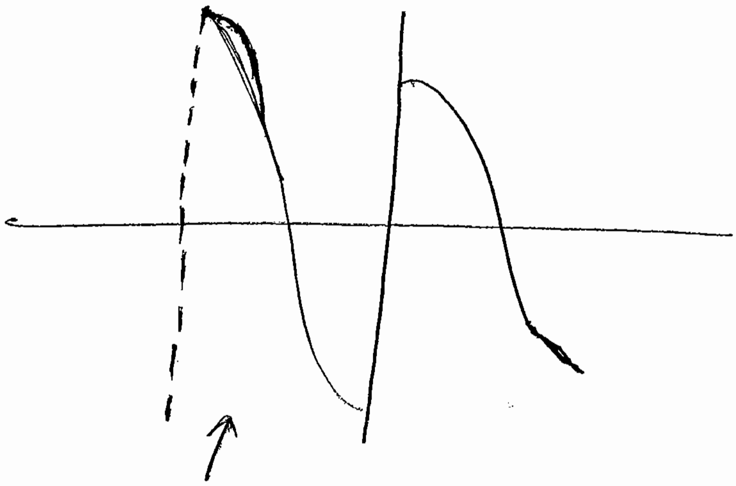




For

$$\boxed{\begin{aligned} f(x) &= \cos x \quad & x > 0 \\ &= -\cos x \quad & x < 0 \end{aligned}}$$

We can not write like this.



this is not the graph of $|\cos x|$
So We can not write $|\cos x|$ like $|\sin x|$.

So $a_n = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n+1}}{(n-1)} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad \left\{\because \cos n\pi = (-1)^n\right.$$

$a_n = 0$ for n = odd

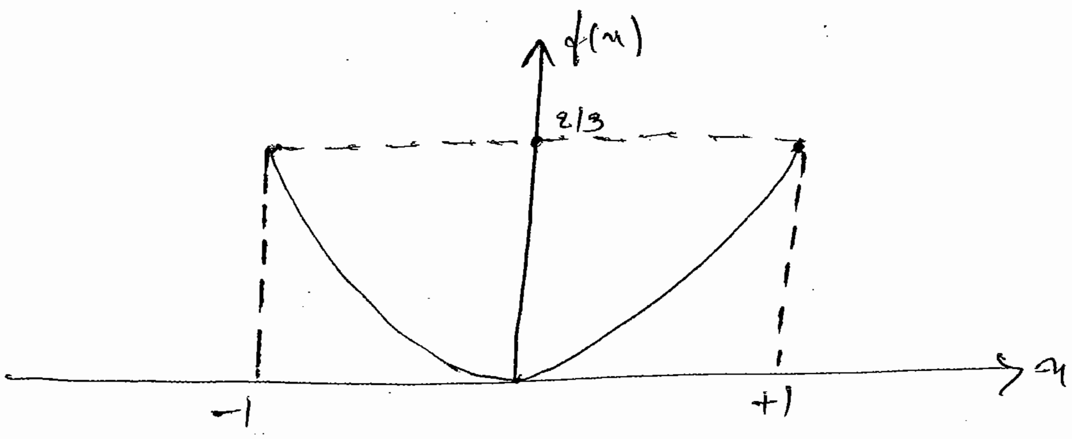
$a_n \neq 0$ for n = even

So option (b) is correct.

A-9
Q. 19

Q.2.3

Sol^n



$$f(x) \approx a_0 + a_1 \cos\frac{\pi x}{2} + a_2 \sin\frac{\pi x}{2}$$

this is a some kind of fourier series.

$\because$ it is even function and we know in even function contian no ~~cos~~ sine term so $a_2 = 0$.

$$f(x) = a_0 + a_1 \cos\frac{\pi x}{2}$$

When $x = \cancel{2}$ $\quad f(x) = \frac{2}{3}$

$$\frac{2}{3} = a_0 + \underbrace{a_1 \cos\frac{\pi}{2}}_{=0}$$

$$\boxed{a_0 = \frac{2}{3}}$$

When $x = 0$ $\quad f(x) = 0$

$$0 = a_0 + a_1 \cos 0$$

$$0 = a_0 + a_1$$

$$a_0 = -a_1$$

$$\boxed{a_1 = -\frac{2}{3}}$$

So option (b) is Correct.

A-9

Q. 10

Sol^n

$$f(x) = -\frac{\pi^2}{3} + \sum \frac{4}{n^2}(-1)^{n+1}\cos nx + \sum \frac{2}{n}(-1)^{n+1}\sin nx$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = ?$$

∵ We see that Here given series is connected in the cosine term becoz when we put $\frac{1}{n^2}$ $n=1$ and 2 and 3 .... we get series so we have to put a selected value of x in which third term not survive and second term survive.

Generally we put in the place of x is either 0 or $\pm\pi$ how we deside when we put 0 or where put $\pm\pi$ in plafce of x!

But we see when we put $x=0$ then $\frac{1}{n^2}(-1)^{n+1}$ gives alternate + and - sign ~~but~~ so we don't need to change the sign so here we put $x=0$

When we need to change the sign then we have to put $(x=\pm\pi)$ or either $+\pi$ or $-\pi$.

So in this ques put $\boxed{x=0}$

$$f(x) = -\frac{\pi^2}{3} + \sum \frac{4}{n^2}(-1)^{n+1} \quad C_0\ 0$$

$$0 = -\frac{\pi^2}{3} + 4\sum \frac{1}{n^2}(-1)^{n+1}$$

$$0 = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Ans

B.A. 5

Q.8

Soln

$$f(x) = x \qquad (-\pi < x < \pi)$$

$$f(x) = \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)(-1)^n \sin nx \quad \text{---} \ (1)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = ??$$

Whether function is different or fourier series is different but sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does not change. which is $\frac{\pi^2}{6}$. Let's we calculate now -

$$f(x) = \left(-\frac{2}{m}\right)(-1)^m \sin(mx) \quad \text{— (11)}$$

just by replacing n by m.

(1) × (2)

$$[f(x)]^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{mn} (-1)^{m+n} \sin nx \sin mx$$

Integrating

$$\int_{-\pi}^{+\pi} [f(x)]^2 dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{mn} (-1)^{m+n} \underbrace{\int_{-\pi}^{+\pi} \sin nx \sin mx \, dx}_{\pi \delta_{mn}}$$

so $$\int_{-\pi}^{+\pi} [f(x)]^2 dx = \sum_{n=1}^{\infty} \frac{4}{n^2} \cdot \pi$$

$$\int_{-\pi}^{+\pi} x^2 dx = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^3}{3} = 4\pi \sum \frac{1}{n^2}$$

so $$\boxed{\sum \frac{1}{n^2} = \frac{\pi^2}{6}}$$

When any Ques

$\sum \frac{1}{n^2}$ = given

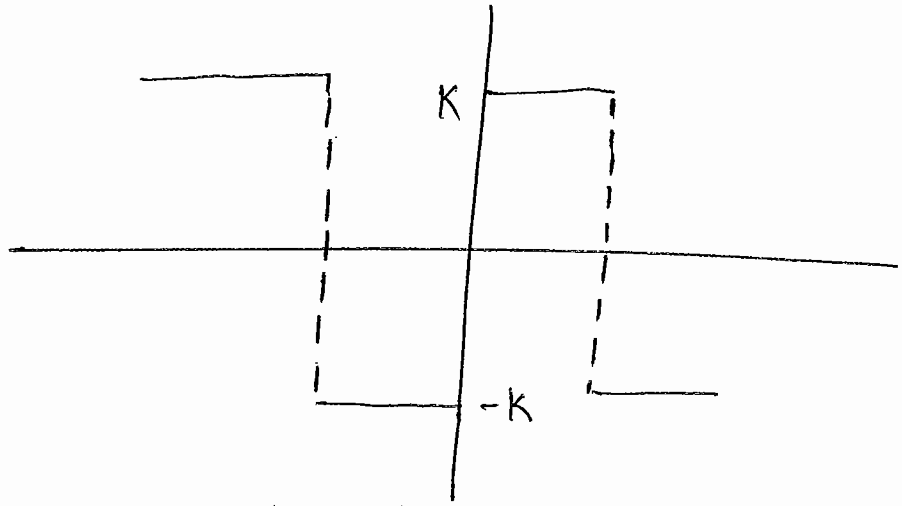
$\sum \frac{1}{n^4} = ??$

then proceed similarly.

* Square Wave :-

$$f(x) = +K \quad , \quad 0 \le x \le \pi$$
$$= -K \quad , \quad -\pi \le x \le 0$$

Graphical Nature of function:-



So from graph we can see it is inverted mirror emage so it is odd function.

$$a_0 = 0$$
$$a_n = 0$$

So $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} k \sin nx\,dx = \frac{2K}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \begin{cases} \frac{4K}{n\pi} & , \text{ for } n = \text{odd} \\ 0 & , \text{ for } n = \text{even} \end{cases}$$

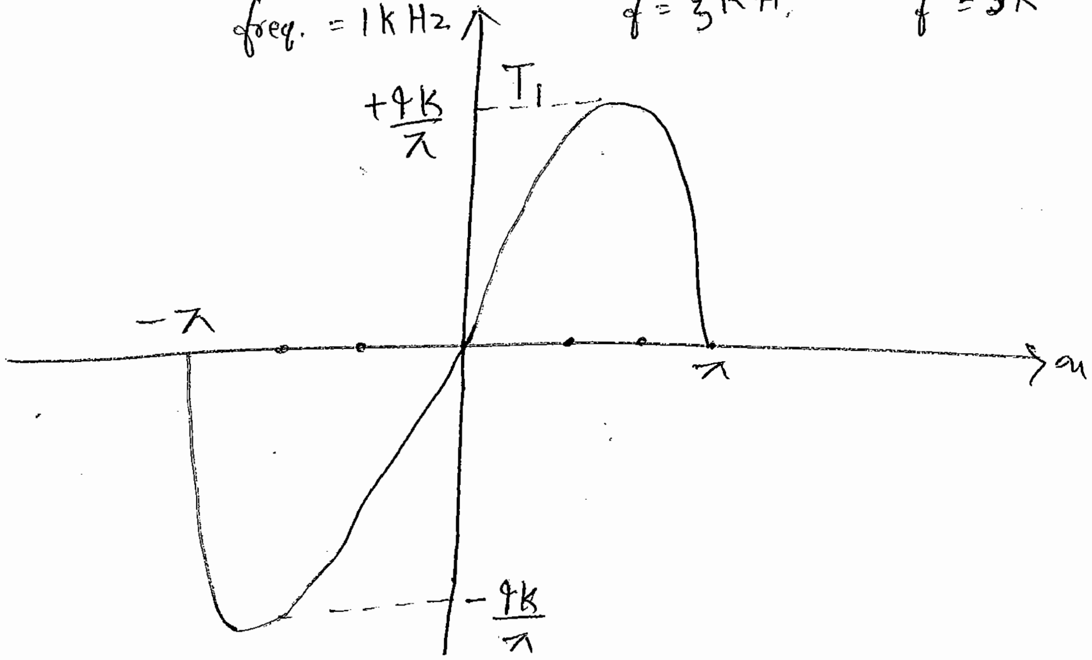
$$f(x) = \sum_{n=odd} \left(\frac{4K}{n\pi}\right) \sin nx$$

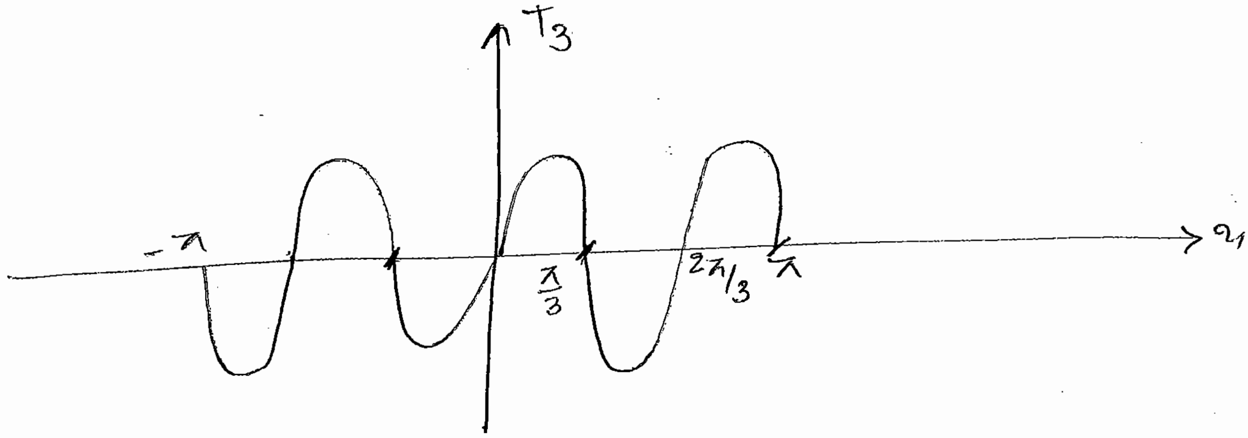
$$\boxed{f(x) = \left(\frac{4K}{n\pi}\right) \sin x + \left(\frac{4K}{n\pi}\right) \sin 3x + \ldots\ldots}$$ ← Square wave

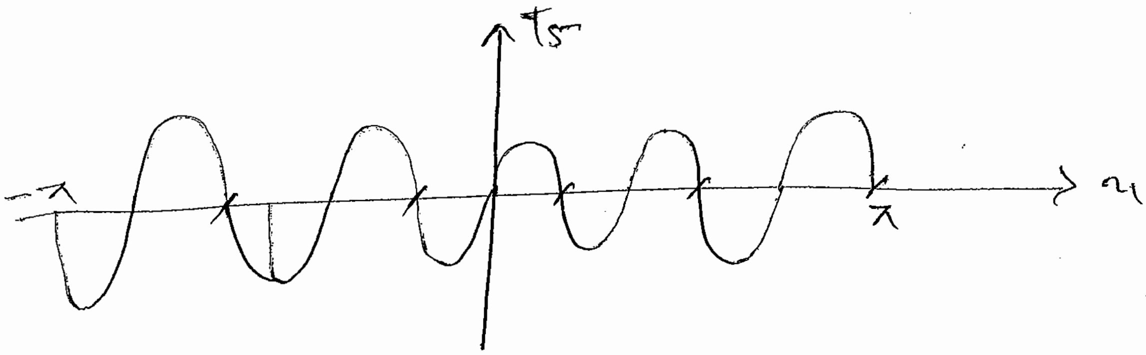
$T_1 = 2\pi$ $\qquad$ $T_3 = \frac{2\pi}{3}$

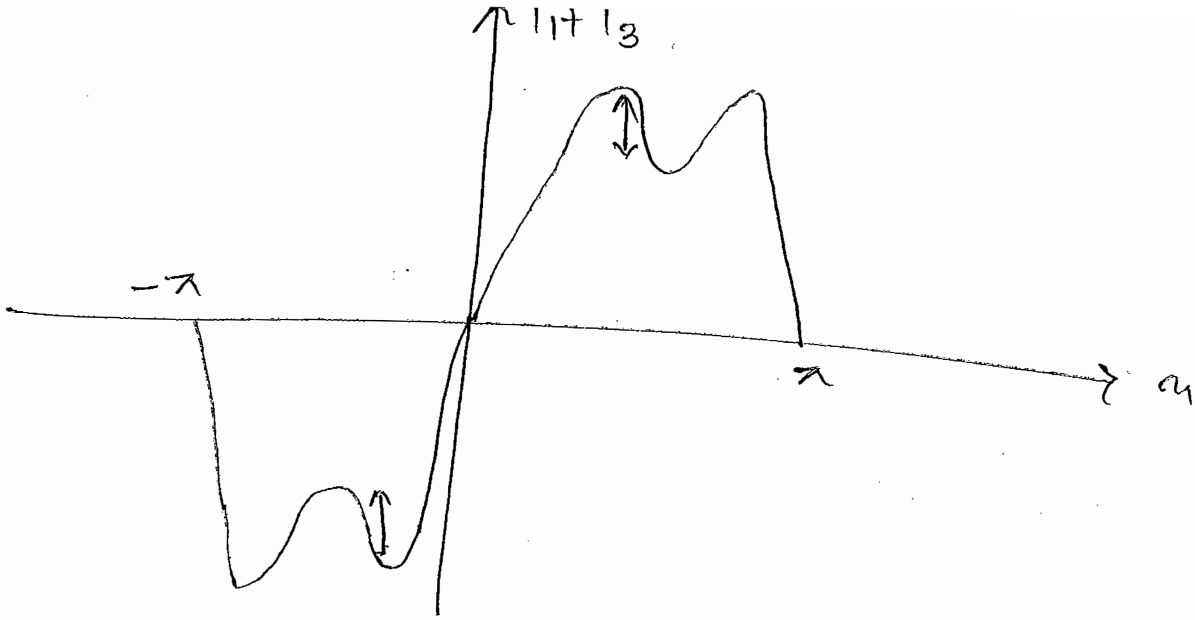
for Visualisation:

freq. = 1 KHz $\qquad$ $f = 3$ KH $\qquad$ $f = 5$K $\qquad$ $\{\because f = \frac{2\pi}{T}\}$

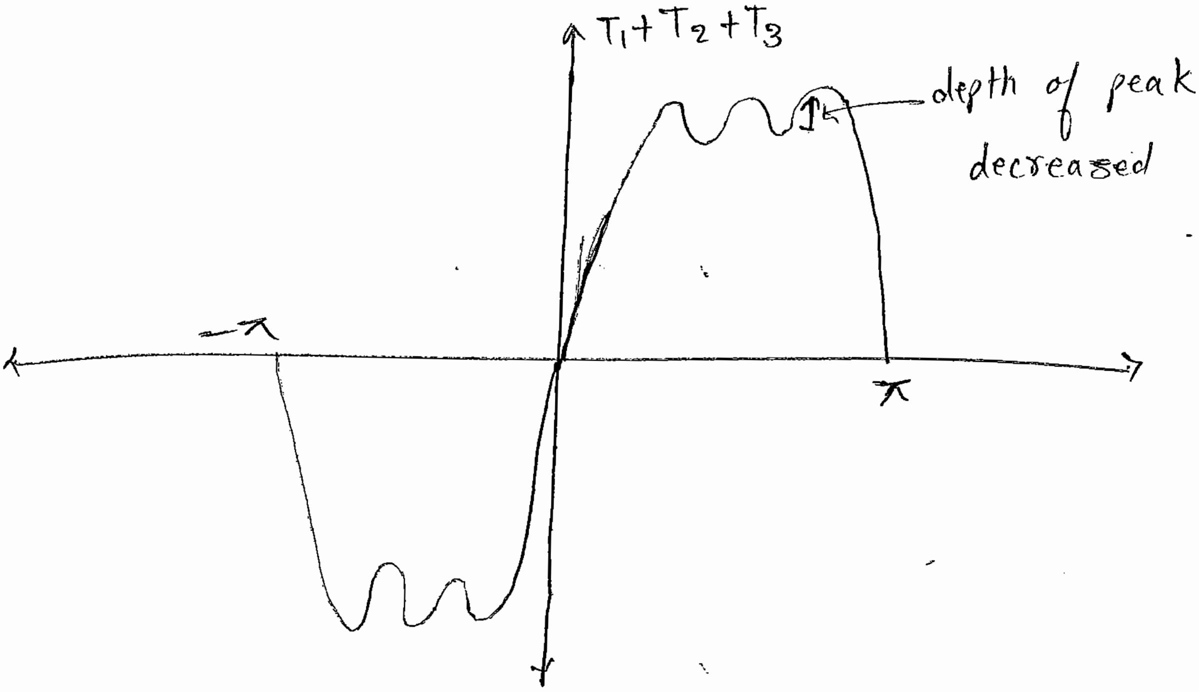




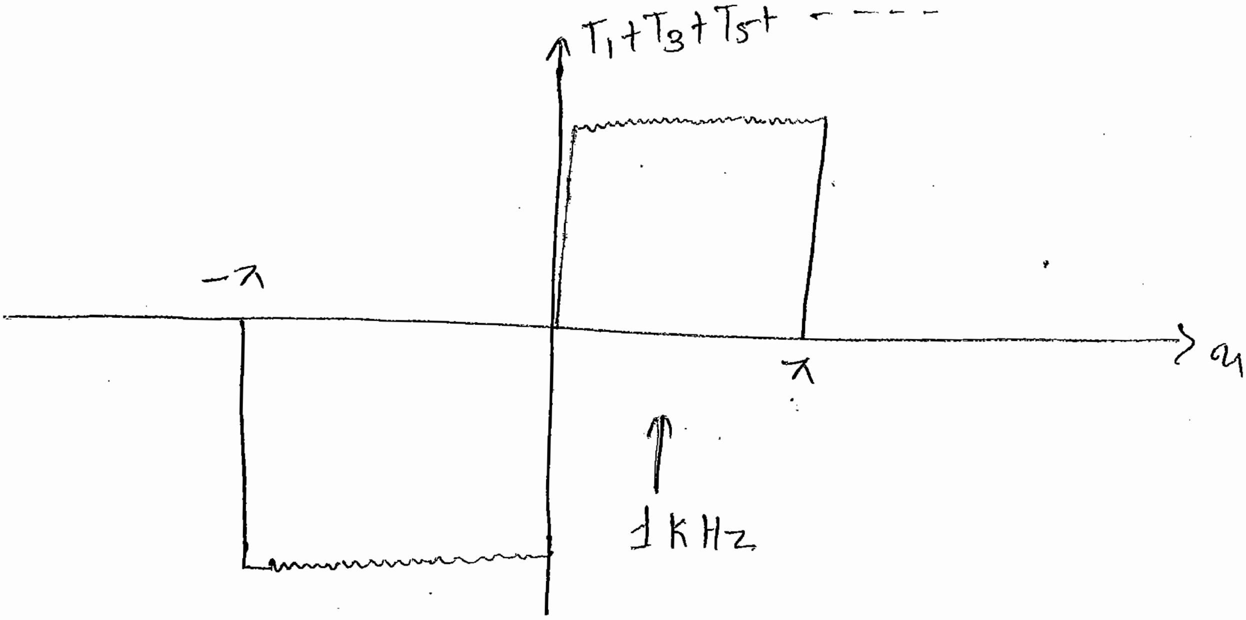




$I_1 + I_3$
$-\pi$
π
x



$T_1 + T_2 + T_3$
depth of peak decreased
$-\pi$
π



$T_1 + T_3 + T_5 + \cdots$
$-\pi$
π
x
1 KHz

$$f(x) = \underset{\downarrow \atop 1\,KHz}{\frac{4K}{\pi}\sin x} + \underset{\downarrow \atop 3\,KHz}{\frac{4K}{3\pi}\sin 3x} + \underset{\downarrow \atop 5\,KHz}{\frac{4K}{5\pi}\sin 5x} + \dots$$

$f(x) = 1\,KHz$

Input 1KHz square wave

O/P :- 1KHz sine wave $\frac{4K}{\pi}$

3 KHz sine wave $\frac{4K}{3\pi}$

⋮ ⋮

(B) Complex Fourier Series Expansion:

$f(x)$ is defined in $[-l, l] \,/\, [0, 2l]$

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{in\pi x/l} + \sum_{n=1}^{\infty} C_{-n} e^{-in\pi x/l}$$

Here l is absorbed.

$$C_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$C_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-inx\pi/l}\,dx$$

$$C_{-n} = \frac{1}{2l}\int_0^{2l} f(x)\, e^{in\pi x/l}\,dx$$

$\frac{A-9}{Q.18}$ Obtain the complex form of the fourier series of the expansion.

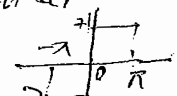
$$f(x) = 0 \qquad -\pi \le x \le 0$$
$$= 1 \qquad 0 \le x \le \pi$$

$$\cancel{C_0 = \frac{1}{2\pi}\int_0^{\pi} -1 \cdot e^{-in\pi x/2\pi}\,dx = -\frac{1}{2\pi}\int e^{-inx/2}\,dx}$$

$$C_0 = \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{1}{2\pi}\cdot\pi$$

$$\boxed{C_0 = \frac{1}{2}}$$

Note: Calculation of $f(x)$ which is discontinuous



$$f(x=0) = \frac{1}{2}[\text{R.H.L} + \text{L.H.L.}]$$
$$= \frac{1}{2}[0+1] = \frac{1}{2}$$