$C_n =$

* Integral Transform:-

This transformation converts a function into another function through the process of integration.

$$f(x) \xrightarrow{\int dx} f(s)$$

$$f(s) = \int_a^b f(x)\, \underbrace{a(s,x)\, dx}_{\text{Kernel of the transform.}}$$

* Fourier Transform:- of $K(s,x) = e^{-isx}$

$$\boxed{f(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{isx}\, dx}$$

kernel

Fourier Transform

$$\boxed{f(x) = F^{-1}[f(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(s)\, e^{-isx}\, ds}$$

Inverse Fourier Transform.

Physical Significance:-

After applying fourier transform we convert ~~into~~ ~~space~~ position space wave function into momentum space wave function.

(1)

$$f(x) \longrightarrow f(s)$$

$$\downarrow$$

$$\psi(x) \xrightarrow{\text{F. T.}} \phi(p) / \phi(k).$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{tikzcd}
f(x) \arrow[r] \arrow[d] & f(s) \arrow[d] \\
\psi(x) \arrow[r, "\text{Fourier Trans.}"] \arrow[d] & \phi(p)/\phi(k). \arrow[d] \\
\text{Position Space Wave function} \arrow[d] & \text{Momentum Space or K-space wave function.} \arrow[d] \arrow[l, "\text{Invese Fourier Transform}"] \\
\text{Position basis eigen function} & \text{Momentum basis eigen function.}
\end{tikzcd}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\phi}$$

$\Rightarrow$ Only Component and unit vectors of the state vectors are changes. But state vector Does not effect.

$$\hat{i}\cdot\hat{i} = 1$$
$$\hat{i}\cdot\hat{j} = 0$$

Similarly

$$\langle\phi_1|\phi_1\rangle = 1$$
$$\langle\phi_1|\phi_2\rangle = 0$$

When any wave function of position space is normalised then when it transform in momentum space then it is also normalise only normalisation constant change

$\phi_1(x), \phi_2(x)$ ------ form a orthonormal basis

$$\psi(x) = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

Basis eigen function

$$\phi(p) = c_{1p}\phi_{1p} + c_{2p}\phi_{2p} + \cdots$$

* Particle in 1-D Box:-

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

$$\int_0^a \phi_m^* \phi_n \, dx = \delta_{mn}$$

$$\begin{array}{ccc} \psi(x) & = & \sum_{n=1}^{\infty} C_n \phi_n(x) \\ \big\downarrow \text{F.T.} & & \big\downarrow \text{F.T.} \\ \phi(p) & = & \sum_{n=1}^{\infty} C_{np} \phi_{np}(p) \end{array}$$

(2)

$$\begin{array}{ccc} f(x) & \xrightarrow{\text{F. T.}} & f(s) \\ \downarrow & & \downarrow \\ f(t) & & f(\omega) \end{array}$$

Time domain $\underset{\text{I.F.T.}}{\overset{\text{F.T.}}{\rightleftarrows}}$ frequency domin.

"By observing the function in the frequency domain we can calculate we can calculate the frequency components present in the time varring signal."

e.g.:- $f(t) = \cos at$

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos at \, e^{i\omega t} dt$$

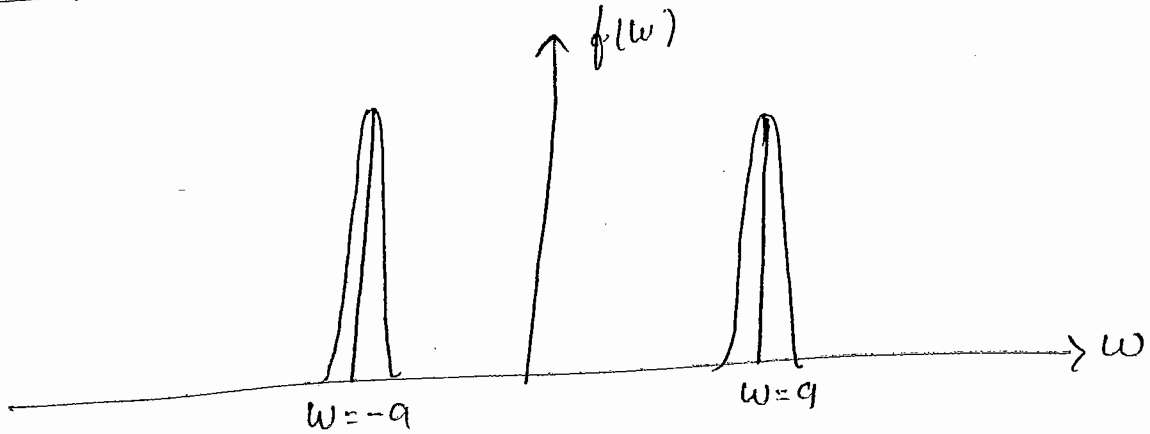
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{e^{iat} + e^{-iat}}{2} \right] e^{i\omega t} dt$$

$$= \frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} e^{i(\omega+a)t}\,dt + \int_{-\infty}^{+\infty} e^{i(\omega-a)t}\,dt\right]$$

$$= \frac{1}{2\sqrt{2\pi}}\left[2\pi\,\delta(\omega+a) + 2\pi\,\delta(\omega-a)\right]$$

$$\boxed{f(\omega) = \sqrt{\frac{\pi}{2}}\left[\delta(\omega+a) + \delta(\omega-a)\right]}$$

Plot of $f(\omega)$:-



So we can say $f(t) = \cos at$ contains only two frequency first at $\omega = +a$ and other is $\omega = -a$.

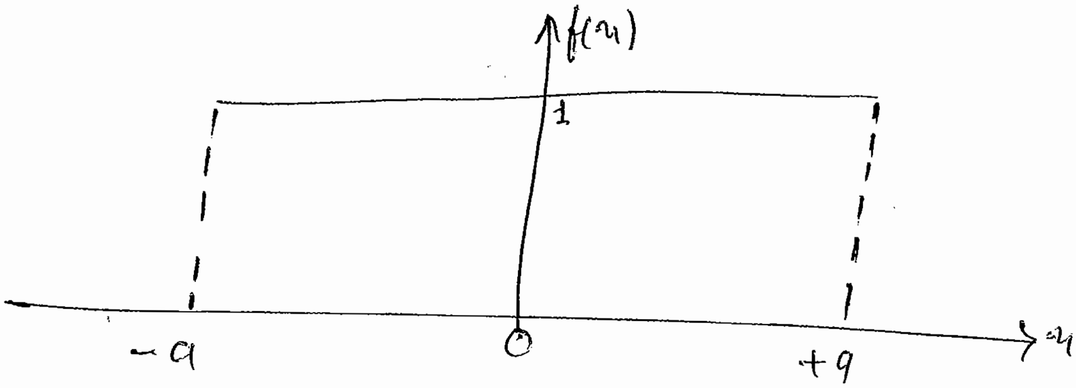
When a is a positive number then $\omega = +a$ have positive freq. and $\omega = -a$ have negative frequency

2. What is mean by negative frequency?

Ans When rotation is clockwise then we called freq. +ve freq., when rotation is anticlockwise then we called -ve freq. But freq. is always positive.

Q. (1)

Sol^n



(i) $f(s) = ??$

(ii) $f(s)$ graphical form

(iii) $f(s) \rightarrow$ freq. Comp.

(i) $$f(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{+a} 1 . e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{+a}$$

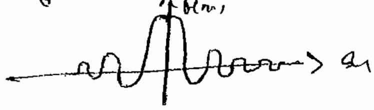
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{is} \left[e^{isa} - e^{-isa} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \frac{1}{s} \left[\frac{(e^{isa} - e^{-isa})}{2i} \right]$$

$$\boxed{f(s) = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}}$$

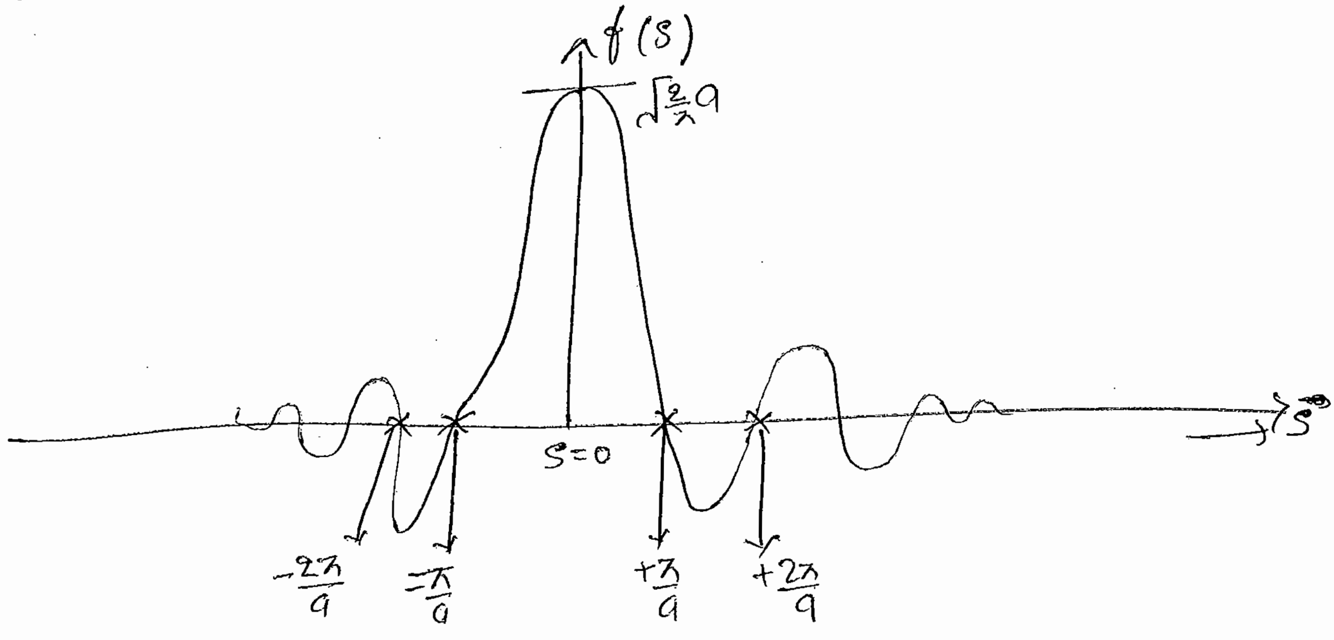
$\because$ We know $\frac{\sin x}{x} = \text{Sinc}(x) \leftarrow$ Sinc function.

graph of Sinc function



So $\boxed{f(s) = \sqrt{\frac{2}{\pi}} a \frac{\sin(sa)}{(sa)}}$

(2) $f(s) \rightarrow$ graphical form:-



$$f(x) = \sqrt{\frac{2}{\pi}}\, a \, \frac{\sin(sa)}{(sa)}$$

$$f(s) = 0 \Rightarrow \sin(sa) = 0$$

$$sa = n\pi$$

$$s = \frac{n\pi}{a} \qquad (n = \pm 1, \pm 2, \pm 3, ---)$$

(3) $f(s) \rightarrow$ frequency Components $\Rightarrow$ Since here wave function is distributed over $-\infty$ to $+\infty$ so it contains all frequencies.

Q. $f(x) = A\, e^{-x^2/2\sigma^2}$
Gaussian

$f(s) = ??$

and
Show that this result satisfies Heisenbergs uncertainity principle.

Solⁿ

$$f(s) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2} e^{+isx} dx$$

$$= \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{(is - x/2\sigma^2)x} dx$$

$$= \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2 + isx} dx$$

$$\because \int_{-\infty}^{+\infty} e^{-ax^2 + bx} dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

$$= \frac{A}{\sqrt{2\pi}} \sqrt{\frac{\pi}{\left(\frac{1}{2\sigma^2}\right)}}\, e^{-\frac{s^2}{4\frac{1}{2\sigma^2}}}$$

$$f(s) = A\sigma\, e^{-s^2\sigma^2/2}$$

It is also a Gaussian type.

So fourier transform of a gaussian function is also a Gaussian function.

$$\because \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \Delta x$$

$$\psi(x) = A\, e^{-x^2/2\sigma^2}$$

S.D. of wave function $= \sigma$ which is in position space

So $\Delta x \sim \sigma$

Similarly

$$\phi(K) = A\sigma e^{-K^2\sigma^2/2}$$

$$\phi(K) = A\sigma e^{-K^2/2(\frac{1}{\sigma^2})}$$

S.D. $= \frac{1}{\sigma}$

So $\boxed{\Delta k \sim \frac{1}{\sigma}}$

S.D represent uncertainity in corresponding wave function.

$$\Delta p = \hbar \Delta k \simeq \frac{\hbar}{\sigma}$$

So $\boxed{\begin{array}{c} \Delta x \Delta p \sim \hbar \\ \text{or} \\ \Delta x \Delta p > \frac{\hbar}{2} \end{array}}$

B. A-5

Q. 9

Sol^n

$$f(x) = e^{-x^2}$$

$$F[f(x)] = ??$$

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

$\because$ given wave function is gaussian type so fourier transform is also gussian type.

S.D. of given function :-

$$2\sigma^2 = 1$$

$$\boxed{\sigma = \frac{1}{\sqrt{2}}}$$

So after fourier transform S.D is $\frac{1}{\sigma}$

So $F[f(x)]$

$\downarrow$

$\sigma = \sqrt{2}$

so option (d) is correct.

16/oct/2014

Fourier Transform

Q. $F[f(x)] = f(s)$

$$F\left[\frac{df}{dx}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{isx} f(x)\right]_{-\infty}^{+\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} is\, e^{isx} f(x)\, dx$$

$f(x) \rightarrow 0$

∵ Fourier transform exist when -

∴ at $x = \pm\infty \Rightarrow f(x)$ will be zero / finite (small quantity)

$$= -is \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{isx}\, dx$$

$$\boxed{F\left[\frac{df}{dx}\right] = -is\, f(s).}$$

So we can say generally

$$\boxed{F\left[\frac{d^n f}{dx^n}\right] = (-is)^n f(s)}$$

* Properties :-

① change of scale property :-

$$F[f(x)] = f(s)$$

$$\boxed{F[f(ax)] = \frac{1}{a} f\left(\frac{s}{a}\right)}$$

e.g. If $f(x) = e^{-x^2/2}$ and $f(s) = e^{-s^2/2}$ is given find $F[e^{-2x^2}] = ?$

Soln $F[e^{-2x^2}] = e^{-(4x^2)/2}$

$= e^{-(2x)^2/2}$

$= f(2x)$

So $F[f(2x) = e^{-2x^2}] = \frac{1}{2} e^{-(\frac{s}{2})^2/2}$

$= \frac{1}{2} e^{-s^2/8}$

Ans

(2) Shifting Property :-

$F[f(x)] = f(s)$

$F[f(x \pm a)] = e^{\mp isa} f(s)$

e.g. $f(x) = e^{-x^2+2x+1}$ $(-\infty < x < \infty)$

Given $\Rightarrow f(x) = e^{-x^2/2}$

$f(s) = e^{-s^2/2}$

Soln $f(x) = e^{-x^2+2x-1} = e^{-(x-1)^2}$

$\because F[e^{-x^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2} e^{isx} dx$

$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{1}} e^{-s^2/4.1}$

$= \frac{1}{\sqrt{2}} e^{-s^2/4}$

So $F[e^{-(x-1)}] = f(s)$

$$\boxed{F[e^{-(x-1)^2}] = e^{is} \cdot \frac{1}{\sqrt{2}} e^{-s^2/4}}$$

e.g. $F[e^{-x^2}] = ?$

$$e^{-x^2} = e^{-2x^2/2} = e^{-(\sqrt{2}x)^2/2}$$

$$F[e^{-x^2/2}] = e^{-s^2/2} = f(s)$$

$$F[e^{-x^2}] = \frac{1}{\sqrt{2}} e^{-\left(\frac{s}{\sqrt{2}}\right)^2/2}$$

$$= \frac{1}{\sqrt{2}} e^{-s^2/4}$$

Q. If the furier transform of $f(x)$ is $f(s)$ then what is the furier transform of conjugate of $f(x)$ $\Rightarrow$ $F[f^*(x)] = ??$

(a) $f^*(s)$ ✓(b) $f^*(-s)$ (c) $-f^*(s)$ (d) $-f^*(-s)$

Soln

$$F[f(x)] = f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{isx} dx$$

$$F[f^*(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f^*(x) e^{isx} dx$$

$$f^*(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f^*(x) \frac{e^{-isx}}{\to -s} dx$$

$$\boxed{f^*(-s) = F[f^*(x)]}$$ Ans

3. Modulation Property:-

$$F[f(x)] = f(s)$$

$$F[f(x)\cos ax] = \frac{1}{2}[f(s+a) + f(s-a)]$$

e.g. $F[e^{-x^2}\cos 2x] = ??$

Like modulation:-
When $[\omega]\cos ax$
$\omega - a \quad \omega \quad \omega + a$

Sol^n

$$F[\underbrace{e^{-x^2}}_{f(x)}\cos \underbrace{2}_{a} x] = \frac{1}{2}\left[\frac{1}{\sqrt{2}}e^{-(s+2)^2/4} + \frac{1}{\sqrt{2}}e^{-(s-2)^2/4}\right]$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{e^{-(s+2)^2/4} + e^{-(s-2)^2/4}}{2}\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{e^{-(s^2+4+4s)/4} + e^{-(s^2+4-4s)/4}}{2}\right\}$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{(s^2+4)}{4}}\left\{\frac{e^{-4s/4_2} + e^{+4s/4_2}}{2}\right\}$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{(s^2+4)}{4}}\left\{\frac{e^{-s} + e^{s}}{2}\right\}$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{(s^2+4)}{4}}\cosh s$$

Ans

* PERSEVAL'S IDENTITY :-

(i) $\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |f(s)|^2 ds$

(ii) $\int_{-\infty}^{+\infty} f^*(s)\, g(x) dx = \int_{-\infty}^{+\infty} f^*(s)\, g(s) ds$

Q. $f(x) = 1 \quad |x| \le a$

$= 0 \quad |x| > a$

$f(s) = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$ find $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = ??$

Sol^n Using (i) identity in given function -

$$\int_{-a}^{+a} 1^2 dx = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2 sa}{s^2} ds$$

let $sa = t$

$ds = \frac{dt}{a}$

$$\cancel{2}a = \frac{\cancel{2}}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2 t}{(t^2/a^2)} \frac{dt}{\cancel{a}}$$

$$\cancel{a} = \frac{1}{\pi} \cancel{a} \int_{-\infty}^{+\infty} \frac{\sin^2 t}{t^2} dt$$

$$\pi = 2 \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$$

So $\boxed{\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}}$ Ans

(Q.12) Fourier transform of the derivative of the dirac δ- function, namely $\delta'(x)$ is proportional to ?

(a) 0 (b) 1 (c) sin k (d) ik

Sol^n

$$F[\delta'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta'(x)\, e^{isx}\, dx$$

Property of dirac delta:-

∵ (i) $\int_{-\infty}^{+\infty} f(x)\, \delta(x-a)\, dx = f(a)$

(ii) $\int_{-\infty}^{+\infty} f(x)\, \delta'(x-a)\, dx = -f'(a)$

But here $a=0$

So $$F[\delta'(x)] = \frac{-1}{\sqrt{2\pi}}\, is\, e^{isx}\Big|_{x=0}$$

$$= \frac{-is}{\sqrt{2\pi}}$$

So $\boxed{F[\delta'(x)] \propto ik}$

If We forgot (ii) property we can also solve this -

$F[f(x)] = f(s)$

$F[f'(x)] = -is\, f(s)$

$F[\delta(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x)\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \cdot 1 = f(s)$$

$$F[\delta'(x)] = -is\frac{1}{\sqrt{2\pi}}$$

So $\boxed{F[\delta'(x)] \propto -is}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \quad \text{— (a)}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-isx}\, dx \quad \text{— (b)}$$

Both are correct only constant factor is change if e^{-isx} then $F[f'(s)] = is f(s)$

When e^{+isx} then $F[f'(s)] = -is f(s)$

LAPLACE Transform

$f(x)$ will be defined $[0, \infty]$

$K(s, x) = e^{-sx}$ ← Kernel of L.T.

$$L[f(x)] = \bar{f}(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

Note :- "Integral transforms are starts to use to solve the different kind of differential equations."

So Laplace transform is used to solve the differential equations or partial differential equations.

* Some Important Laplace Transforms:-

(1) $L(1) = \frac{1}{s}$

(2) $L(x^n) = \frac{n!}{s^{n+1}}$ where $n = +ve$

(3) $L(e^{ax}) = \frac{1}{s-a}$

(4) $L(\cos ax) = \frac{s}{s^2+a^2}$

$L(\sin ax) = \frac{s}{s^2-a^2}$

(5) $L(\cosh ax) = \frac{s}{s^2-a^2}$

$L(\sinh ax) = \frac{a}{s^2-a^2}$

* Shifting Property :-

$$L[f(x)] = f(s)$$

$$L[e^{ax} f(x)] = f(s-a)$$

$$L[\cos ax] = \frac{s}{s^2+a^2} = f(s)$$

$$L[e^{2x} \cos ax] = \frac{(s-2)}{(s-2)^2+a^2}$$

* Change of scale property :-

(1) $$L[f(x)] = f(s)$$

$$L[f(ax)] = \frac{1}{a} f\left(\frac{s}{a}\right)$$

(ii) $$L[f(x)] = f(s)$$

Polynomial

+ve integer

$$L[x^n f(x)] = (-1)^n \frac{d^n}{ds^n}[f(s)]$$

$$L[\underbrace{x}_{x^n} \underbrace{\cos ax}_{f(x)}] = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= (-1)^1 \frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}$$

$$= (-1) \frac{s^2+a^2-2s^2}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$ Ans

A-11

1 (viii) $f(x) = \sinh(ax)\cos(ax)$

$$= \left(\frac{e^{ax} - e^{-ax}}{2}\right)\cos ax$$

$$= \frac{1}{2}\left(e^{ax}\underbrace{\cos ax}_{f(x)}\right) - \frac{1}{2}\left(e^{-ax}\cos ax\right)$$

$e^{ax} \times f(x)$ $\qquad$ $e^{-ax} \times f(x)$

$$L[f(x)] = \frac{1}{2}\frac{s}{s^2+a^2}\Bigg|_{s\to s-a} - \frac{1}{2}\frac{s}{s^2+a^2}\Bigg|_{s\to s+a}$$

$$= \cancel{\frac{a(s^2\;2a^2)}{(s^4+4a^4)}}$$

$$= \frac{1}{2}\left[\frac{(s-a)}{(s-a)^2+a^2} - \frac{(s+a)}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\{(s+a)^2+a^2\} - (s+a)\{(s-a)^2+a^2\}}{\{(s-a)^2+a^2\}\{(s+a)^2+a^2\}}\right]$$

$$= \frac{1}{2}\left[\frac{s-a(s^2+a^2+2as+a^2) - (s+a)(s^2+a^2-2as+a^2)}{(s-a)^2(s+a)^2 + a^2(s-a)^2 + a^2(s+a)^2 + a^4}\right]$$

$$= \frac{1}{2}\left[\frac{(s^3+as^2 + 2as^2 + a^2s - s^2a - a^3 - 2a^2s - a^3) - (s^3 + sa^2}{\qquad}\right.$$

A-11

1(IX) $f(x) = e^{-ax}\dfrac{x^{n-1}}{(n-1)!}$

e^{-ax} $\quad$ $f(x)$

$$L[f(x)] = \frac{1}{(n-1)!}\,\frac{(n-1)!}{s^{n-1+1}}\Bigg|_{s\to s+a}$$

$$= \frac{1}{\cancel{(n-1)!}}\,\frac{\cancel{(n-1)!}}{(s+a)^n}$$

$$= \frac{1}{(s+a)^n} \quad \underline{\underline{Ans}}$$

A-13

Q.4 $\quad L[J_0(x)]$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \text{------}$$

So $L[J_0(x)] = \dfrac{1}{s} - \dfrac{1}{2^2}\dfrac{2!}{s^3} + \dfrac{1}{2^2.4^2}\cdot\dfrac{4!}{s^5} - \text{------}$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{3}{8}\,\frac{1}{s^4} - \text{------}\right]$$

$\because \quad (1+x)^{-1/2} = 1 + \left(\frac{-1}{2}\right)x + \dfrac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}x^2 + \dfrac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}x^3 + \text{------}$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \text{----}$$

So $\quad L[J_0(x)] = \dfrac{1}{s}\left[1+\dfrac{1}{s^2}\right]^{-1/2} = \dfrac{1}{s}\left[1+\dfrac{1}{s^2}\right]^{-1/2}$

$$= \frac{1}{s}\left[\frac{s^2+1}{s^2}\right]^{-1/2} = \frac{1}{s}\left\{\frac{s^2}{(s^2+1)}\right\}^{1/2}$$

$$\boxed{L[J_0(x)] = \frac{1}{\sqrt{s^2+1}}} \quad Ans$$

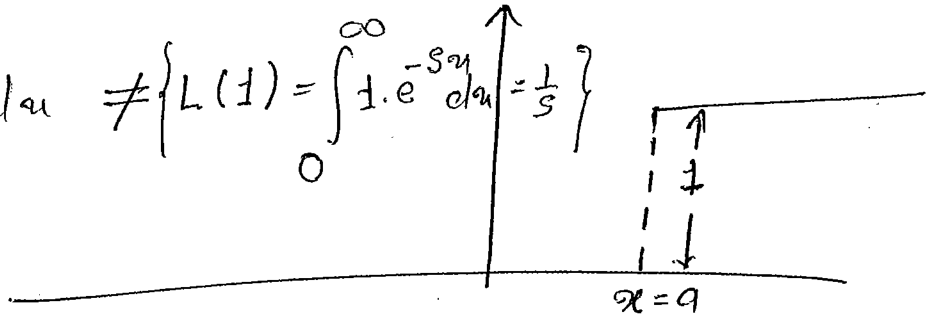
PYQ

Q.5 $u_a(x) = u(x-a) = 0 \quad x < a$

$\qquad\qquad\qquad\qquad = 1 \quad x \geq a$

unit step function.

$$L[u_a(x)] = \int_a^\infty 1 \cdot e^{-sx}\,dx \neq \left\{ L(1) = \int_0^\infty 1 \cdot e^{-sx}\,dx = \frac{1}{s} \right\}$$



So $= \left[\frac{e^{-sx}}{-s} \right]_a^\infty$

When in any Ques -

$u_{2a}(x) = 0 \quad x < 2a$

$\qquad\quad = 1 \quad x \geq 2a$

$$\boxed{L[u_a(x)] = \frac{e^{-sa}}{s}}$$ Ans

A-11

Q.6 The value of integral $\int_0^\infty x e^{-2x} \sin x\,dx$ is equal to

~~(a)~~ $\frac{4}{25}$ (b) $\frac{4}{50}$ (c) $\frac{2}{15}$ (d) $\frac{8}{25}$

Sol^n

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)\,dx$$

$$L[\underbrace{x}_{x^n} \cdot \underbrace{\sin x}_{f(x)}] = \int_0^\infty e^{-sx} x \sin x\,dx$$

$$\Rightarrow (-1)^1 \frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \int_0^\infty e^{-sx} x \sin x\,dx$$

$$\Rightarrow (-1) \frac{d}{ds} (s^2+1)^{-1} = \int_0^\infty e^{-sx} x \sin x\,dx$$

$$\Rightarrow (-1)(-1) \frac{2s}{(s^2+1)^2} = \int_0^\infty e^{-sx} \cdot x \sin x\,dx$$

put $s = 2$

$$\Rightarrow \frac{4}{(4+1)^2} = \int_0^\infty e^{-sx} \cdot x \sin x\,dx = \frac{4}{25}$$ Ans

* $L[f(x)] = f(s)$

$$\boxed{L\left[\frac{f(x)}{x}\right] = \int_s^\infty f(s)\,ds}$$

A-11

Q. 1(xiv) $f(x) = \frac{\sin 2x}{x}$

$$L\left[\frac{\sin 2x}{x}\right] = \int_s^\infty \frac{2}{s^2+4}\,ds$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{2}$$

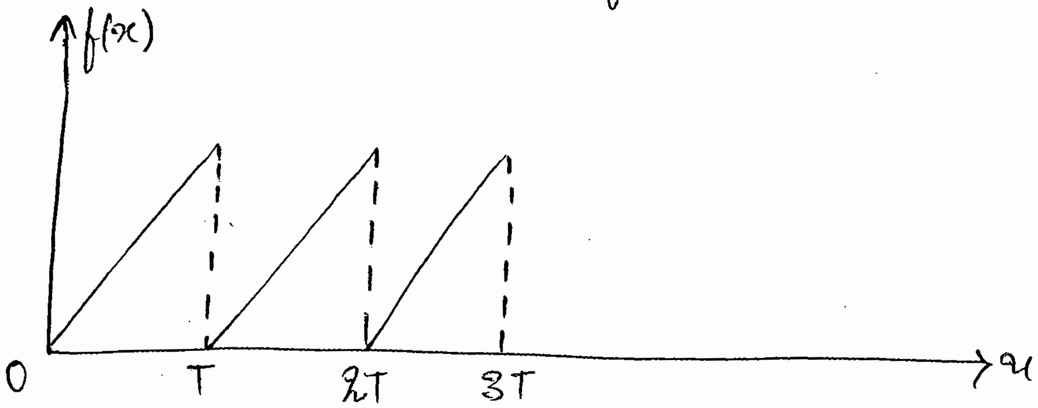
$$= \tan^{-1}\tan\frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2}$$ Ans

* Laplace Transform of a Periodic Function:-

$f(x)$ is periodic function of x of period T



$$L[f(x)] = \int_0^\infty f(x)\, e^{-sx}\, dx$$

$$= \int_0^T f(x)\, e^{-sx}\, dx$$

$$= \int_0^T f(x)\, e^{-sx}\, dx + \int_T^{2T} f(x)\, e^{-sx}\, dx + \cdots$$

So

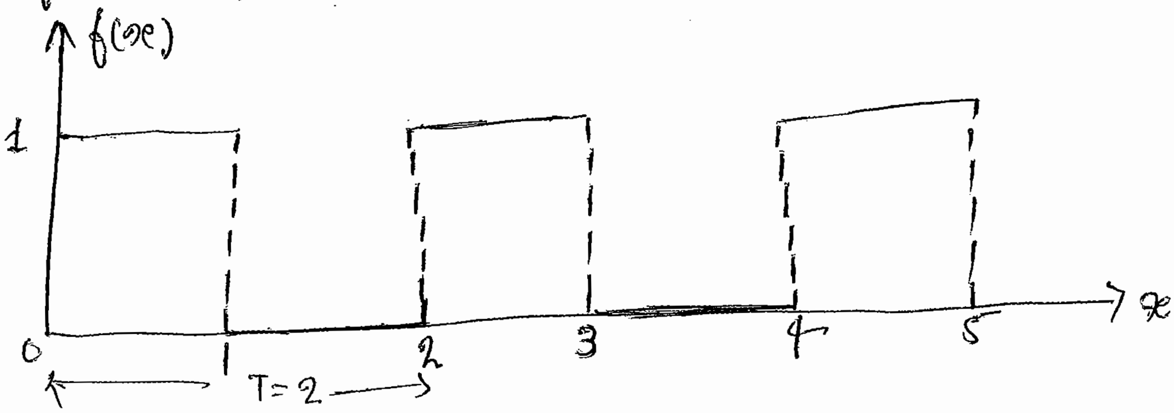
$$\boxed{L[f(x)] = \frac{1}{1-e^{-sT}} \int_0^T f(x)\, e^{-sx}\, dx}$$

A-11
Q.14

$$f(x) = 1 \quad \text{for } 2n \le x \le 2n+1$$
$$= 0 \quad \text{for } 2n+1 \le x \le 2n+2$$
$$(n = 0, 1, 2, \cdots)$$

$$f(s) = ?$$

Sol^n



$$L[f(x)] = \frac{1}{1-e^{-sT}} \int_0^T f(x)\, e^{-sx}\, dx$$

$$L[f(x)] = \frac{1}{1-e^{-2s}} \int_0^2 f(x)\, e^{-sx}\, dx$$

$$= \frac{1}{1-e^{-2s}} \int_0^1 1 \cdot e^{-sx}\, dx$$

$\left\{ \because \text{in } \int_0^1 \text{ function is 1 and } \int_1^2 \text{ function is 0.} \right.$

$$= \frac{1}{1-e^{-2s}} \left[\frac{e^{-sx}}{-s}\right]_0^1$$

$$= \frac{1}{1-e^{-2s}} \frac{[e^{-s}-1]}{(-s)} = \frac{1}{s} \frac{(1-e^{-s})}{(1-e^{-2s})}$$

$$= \frac{1}{s}\left[\frac{\cancel{(1-e^{-s})}}{\cancel{(1-e^{-s})}\,(1+e^{-s})}\right]$$

$$\boxed{L[f(x)] = \frac{1}{s(1+e^{-s})}}$$

Ans

Second Method :-

$$L[f(x)] = \int_0^\infty f(x)\, e^{-sx}\, dx$$

$$= \int_0^1 1 \cdot e^{-sx}\, dx + \int_2^3 1 \cdot e^{-sx}\, dx + \cdots$$

$$= \left.\frac{e^{-sx}}{-s}\right|_0^1 + \left.\frac{e^{-sx}}{-s}\right|_2^3 + \cdots\cdots$$

$$= \frac{1}{s} - \frac{1}{s} e^{-s} + \frac{1}{s} e^{-2x} - \frac{1}{s} e^{-3s} \cdots\cdots$$

This is G.P. Series.

$$= \frac{\frac{1}{s}}{1+e^{-s}} \qquad \left\{ a = \frac{1}{s}, \ r = -e^{-s} \right.$$

$$= \left(\frac{1}{s}\right)\left(\frac{1}{1+e^{-s}}\right)$$

$$= \frac{1}{s(1+e^{-s})} \quad \text{Ans}$$

$$L[e^{ax}] = \frac{1}{s-a}$$

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{ax}$$

← Braking partial fraction is not easy so we find L.T. of option which give inverse L.T.'s function, this way is more eaisier than proper way.

A-11 CSIR June-2013

Q.15 $f(s) = \dfrac{1}{s^2(s+1)}$

(a) $\frac{1}{2}t^2e^{-t}$ (b) $\frac{1}{2}t^2+1-e^{-t}$

✓(c) $t-1+e^{-t}$ (d) $\frac{1}{2}t^2(1-e^{-t})$

Soln $L[t-1+e^{-t}] = \dfrac{1!}{s^2} - \dfrac{1}{s} + \dfrac{1}{(s+1)}$

$$= \frac{(s+1) - s(s+1) + s^2}{s^2(s+1)}$$

$$= \frac{\cancel{s}+1 - \cancel{s^2} - \cancel{s} + \cancel{s^2}}{s^2(s+1)}$$

$$= \frac{1}{s^2(s+1)} \quad \text{Ans}$$

"Partial Differential Equation

This type of Differential equation contains partial derivatives more than one independent variable.

1. Laplace Equation :- $\nabla^2 \phi = 0$

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

2. Wave Equation :- $$\boxed{\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi}$$

C = Velocity of light in free space.

3. Heat Equation :- $$\boxed{\frac{\partial \psi}{\partial t} = -k^2 \nabla^2 \psi}$$ — 3D

K = Constant

$\psi \Rightarrow$ temperature changing w.r. to time.

4. Schrodinger Equation :-

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi}$$ — 3D .

* Separation of Variables :-

Assume -

$$\boxed{\psi(x,y,z,t) = X(x) \cdot Y(y) \cdot Z(z) \cdot T(t)}$$

A-12
Q.2 $\quad \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

Assume $u(x,t) = X(x) \cdot T(t)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial X}{\partial x} \cdot T$$

$$\frac{\partial u}{\partial t} = X \frac{\partial T}{\partial t}$$

now put here values -

$$T \frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

Divided by XT both side -

$$\Rightarrow \frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1 = K$$

x- dep.	t dep
$\frac{1}{X}\frac{dX}{dx} = K$	$\frac{2}{T}\frac{dT}{dt} + 1 = K$
$\Rightarrow X(x) = Ae^{Kx}$	$\Rightarrow T(t) = Be^{(K-1)t/2}$

$$u(x,t) = X(x)T(t)$$

$$= AB e^{Kx + (K-1)t/2}$$

Given condition : $u(x,0) = 6e^{-3x}$

$\Rightarrow AB = 6$

$k = -3$

$$\boxed{u(x,t) = 6e^{-3x-2t}}$$ Ans

A-12

Q 5 The vibration of a elastic spring is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad \text{(wave eqn)}$$

The length of the spring is 'π' & both ends are fixed : the initial velocity of the spring is zero. & the initial diflection is given by - $u(x,0) = 2(\sin x + \sin 3x)$

The diflection of the vibrating spring will be-

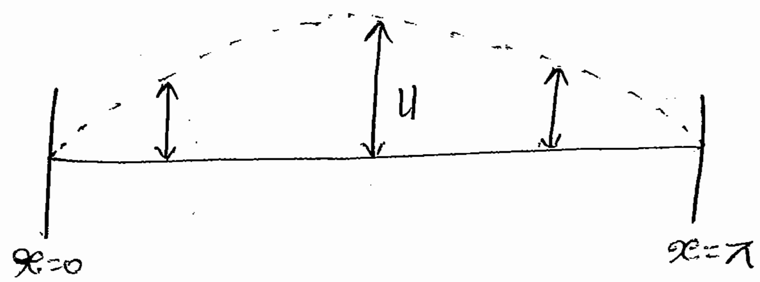
(a) $u(x,t) = 2[\cos 2t \sin x + \cos 3t \sin 3x]$

(b) $u(x,t) = 2[\cos t \sin x + \cos 3t \sin 3x]$

(c) $u(x,t) = 2[\sin t \sin x + \cos 3t \cos 3x]$

(d) $u(x,t) = 2[\sin t \sin x - \cos 3t \cos 3x]$

Soln



Assume : $u(x,t) = X(x) \cdot T(t)$

$$\Rightarrow X\frac{d^2T}{dt^2} = T\frac{d^2X}{dx^2}$$

Divided by XT on both side

$$\Rightarrow \frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \text{ (Say as const.)}$$

Here We Can not assume $+k^2$ becoz

$$\frac{1}{T}\frac{d^2T}{dt^2} = k^2$$

$$\Rightarrow T(t) = Ae^{kt} + Be^{-kt}$$

$Ae^{kt} \to \infty$ (At $t=\infty$)

So eq^n tells that at $t=\infty$ $T(t)=\infty$ So $u(x,t)=\infty$ Which means at infinite time deflection is infinite by by physics it is not physical acceptable becoz physicall after $t=\infty$ ~~time~~ ~~sp~~ string has deflection zero. So We can not choose $+k^2$ We only choose $-k^2$.

$$T(t) = A\cos kt + B\sin kt$$

$$X(t) = C\cos kx + D\sin kx$$

$$\therefore u(x,t) = (A\cos kt + B\sin kt)(C\cos kx + D\sin kx)$$

$$\begin{bmatrix} u(0,t) = 0 \\ u(\pi,t) = 0 \end{bmatrix}$$ Boundary Conditions (depends on space variable)

initial Condition

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \longrightarrow \text{(true for all values of } x\text{.)}$$

$$u(x,0) = 2(\sin x + \sin 3x) \longleftarrow$$ (depends on time variable)

① $u(0,t) = 0 \Rightarrow \therefore \quad C = 0$

② $u(\pi,t) = 0 \Rightarrow D \sin k\pi = 0$

$\because D \neq 0$

$\therefore \sin k\pi = 0$

$k\pi = n\pi$

$\boxed{k = n}$ Here ($n = 1, 2, 3, \ldots$)

mode no.

(It represents different modes of vibration)

$$\therefore \boxed{u_n(x,t) = (A\cos nt + B\sin nt)\cdot D\sin nx}$$

Deflection of the string in the n^{th} mode.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u_n(x,t) = (F_n \cos nt + G_n \sin nt)\sin nx \qquad \left\{ \begin{array}{l} \text{Say } AD = F_n \\ BD = G_n \end{array} \right.$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = (-F_n \sin nt \cdot n + G_n \cos nt \cdot n)\sin nx = 0$$

$$= \therefore G_n = 0$$

So $u_n(x,t) = (F_n \cos nt)\sin nx$

$$u(x,t) = \sum_{n=1}^{\infty} F_n \cos nt \cdot \sin nt$$

At $t = 0$

$$\Rightarrow 2(\sin x + \sin 3x) = \sum_{n=1}^{\infty} F_n \sin nx$$

$$2\sin x + 2\sin 3x = F_1 \sin x + F_2 \sin 2x + F_3 \sin 3x + \cdots$$

$F_1 = 2$

$F_2 = 0$

$F_3 = 2$

$F_4 = 0$

$F_5 = 0$

$\vdots$

So $u(x,t) = \sum_{n=1,3} F_n \cos nt \sin nx$

$\therefore$ $\boxed{u(x,t) = 2[\cos t \sin x + \cos 3t \cdot \sin 3x]}$ Ans

Short Trick :-

first apply the given condition and find which satisfing the given option if more than one option satisfing then we check coefficients of $\cos(n)t \sin(n)x$ which must be same. Because -

$$u(x,t) = \sum_{n=1,3} F_n \cos(n)x \sin(n)x$$

In the above question fistly we put $t=0$ we get option (c) & (d) are not satisfing but (a) and (b) are satisfing then we check value of n in first option $n=2$, in $\cos nx$ and 1 in $\sin nx$ so it is wrong. So correct option is (b).

A-12

Q.9 $\frac{\partial^2}{\partial t^2} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = 0$

(i) $u(0,t) = 0$

(ii) $u(L,t) = 0$

(iii) $u(x,0) = \sin \frac{\pi x}{L}$

(iv) $\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \frac{2\pi x}{L}$

(a) $\sin \frac{\pi x}{L} \cos \frac{\pi t}{L} + \frac{L}{2\pi} \sin \frac{2\pi x}{L} \cos \frac{2\pi t}{L}$

X (b) $2 \sin \frac{\pi x}{L} \cos \frac{\pi t}{L} - \sin \frac{\pi x}{L} \cos \frac{2\pi t}{L}$

(c) $\sin \frac{\pi x}{L} \cos \frac{2\pi t}{L} + \frac{L}{\pi} \sin \frac{2\pi x}{L} \sin \frac{\pi t}{L}$

(d) $\sin \frac{\pi x}{L} \cos \frac{\pi t}{L} + \frac{L}{2\pi} \sin \frac{2\pi x}{L} \sin \frac{2\pi t}{L}$

A-12

Q.6 A string is streched and fixed between two points at a distance 'l' apart. The motion is started by displacing the string into the form $y = Kx(l-x)$ where K = constant. from which it is released at $t=0$. The displacement of string at any time can be written as –

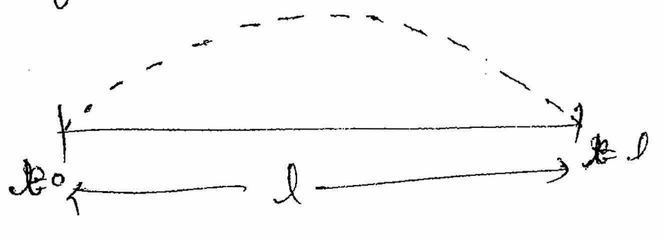
$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l} \quad \text{—} (a)$$

What is the value of $A_n = ?$

Sol^n

$y(0,t) = 0$

$y(l,t) = 0$



$$y = Kx(l-x) \longleftarrow \text{Initial displacement}$$

(Therefore here no time dependence.)

$$y(x,0) = Kx(l-x)$$

Putting $t=0$ in eqn (1)

$$y(x,0) = Kx(l-x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \int_0^l Kx(l-x)\sin\frac{m\pi x}{l}\,dx = \sum_{n=1}^{\infty} A_n \left\{\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx\right\}$$

It give non-zero solution when $n=m$

$$\therefore \int_0^l \sin^2\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_0^l \left[1 - \cos\frac{2n\pi x}{l}\right]dx$$

$$\begin{cases} A_n = 0 & \text{even } n \\ = \dfrac{8Kl^2}{n^3\pi^3} & \text{odd } n. \end{cases}$$

A-9
(11)

Sol^n $f(x) = x^2 \quad 0 < x \leq 2\pi$

$$f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \cancel{\frac{4\pi}{n}\sin nx}\right]$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = ??$$

$$\Rightarrow \quad 2\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

A-9
Q.1

Sol^n $\sin\theta + \pi\cos\theta \Rightarrow T = 2\pi$

$T = 2\pi \qquad T = 2\pi$

$\sin\pi\theta + \cos\pi\theta \longrightarrow T = 2$

$T = 2$

$\sin\theta + \cos\pi\theta \longrightarrow T = 2$

$T = 2\pi$

$T = 2\pi$

$\sin\theta + \cos(\theta + \pi) = \sin\theta - \cos\theta$

Periodic in nature but not sinusoidal.

$T = 2\pi$

Sum of any combination of sinusoidal periodic function with equal time period will also be a sinusoidal periodic function with same time period.

A-10
Q.10

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

1(xi) $$f(x) = \begin{cases} \sin \omega x & 0 < x < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < x < \frac{2\pi}{\omega} \\ \sin \omega x & \frac{2\pi}{\omega} < x < \frac{3\pi}{3} \\ \vdots & \end{cases}$$

It gives infinite g.p. series.

$$f(s) = \frac{1}{1-e^{-sT}} \int_0^T f(x)\, e^{-sx}\, dx$$

Q8. $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$ $\quad \left\{ L[x^n] = \frac{n!}{s^{n+1}} \right\}$

$L[\operatorname{erf}(\sqrt{t})] = ??$

$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - u^2 + \frac{u^4}{2!} - \frac{x^6}{3!} + \cdots\right] d$

$= \frac{2}{\sqrt{\pi}} \left[u - \frac{u^2}{2} + \frac{u^5}{10} - \cdots \right]_0^{\sqrt{t}}$

(9)

Sol^n $f(x) = K[u_0(x) - 2u_a(x) + 2u_{2a}(x) \;---]$

$$f(x) = K\left[\frac{1}{s} - 2\frac{e^{-as}}{s} + 2\frac{e^{-2as}}{s} \;----\right]$$

$$= \frac{K}{s}\left[1 - 2e^{-as} + 2e^{-2as} \;----\right]$$

$$= \frac{K}{s}\left[1 - 2\{e^{-as} - e^{-2as} + ----\}\right]$$

$$= \frac{K}{s}\left[1 - 2\frac{e^{-as}}{(1+e^{-as})}\right]$$

$$= \frac{K}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{K}{s}\left[\frac{e^{-as/2}\{e^{as/2} - e^{-as/2}\}}{e^{-as/2}\{e^{as/2} + e^{-as/2}\}}\right]$$

$$\boxed{f(x) = \frac{K}{s}\tanh\left(\frac{as}{2}\right)}$$

Numerical Techniques

Significant figure :-

(i) All non-zero digits are significant.
(ii) Zeros appearing b/w two non zero digits are significant.
(iii) Leaving zeros are insignificant.

Ex – 0.0025 ⇒ 2.5×10^{-4} here only 2 significant figures.

(iv) Trailing zeros are significant.

Ex –
12.3400 ⟶ 6 significant figures.
↓
because 12.34
we measure correctly in 4 places of decimal. Least count of appratus is 0.0001

12.3423 ← Result change.

$2000 = 2 \times 10^3$ ⟶ significant figure is 1.

2000.0 = 5 significant figure becoz least count = 0.1kg.

* Rounding - off Numbers :- {upto n^{th} decimal place}

(i) If the digit

Ex – 3.186 | 68 ≈ 3.187

(ii) If the digit in the $(n+1)^{th}$ decimal place is equal to 5 followed by a zero then the digit in the n^{th} place will be increacrease by 1 if it is odd. and it remains unchange if it is ~~zer~~ even.

Ex- $3.186|50 \simeq 3.186$

$3.187|50 \simeq 3.188$

If 5 followed by non zero then increase 1.

Ex- $3.186|52 \simeq 3.187$.

* Round-off error :-

$$\boxed{E_{ROE} = X - X_R}$$

(-ve/+ve)

X = Original Number

X_R = Rounded-off Number

If a no. is rounded-off upto n^{th} decimal place then the round-off error is less than or equal to $0.5 \times 10^{-n+1}$

$$\boxed{|E_{ROE}| \leq 0.5 \times 10^{-n+1}}$$

$$\boxed{|E_{ROE}|_{max} = 0.5 \times 10^{-n+1}}$$

A-13

Q.3 $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}$

Sol^n $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}$

$\because$ ~~one of each~~ each is rounded-off upto 2 decimal places So one term -

$$\sqrt{1}$$

$$\downarrow$$

$$0.5 \times 10^{-2+1} = 0.05$$

but here total terms are 100

So $0.05 \times 100 = 5$ Ans

EQUATION

① Polynomial Equation

② Transcedental Equation

① Polynomial Equation :- It contains only algebraic function of x.

② Transcedental Equation :- It contains all terms of function of x.

⇒ Let any function $f(x) = 0$ bounded by $[a, b]$
{How to find how many roots exist.}

(i) $f(a)$, $f(b)$ are of opposite sign $f(x) = 0$ has odd number of roots in $[a, b]$

(ii) $f(a)$, $f(b)$ are of same sign $f(x) = 0$ has even number / zero number of roots in $[a, b]$

A-13
Q.5

Soln

$x^3 - 3x + K = 0$ real root in $[-1, 1]$

$f(-1) = K + 2$

$f(1) = K - 2$

$-2 < K < 2$

$\Rightarrow K > -2$ $\Rightarrow K < 2$

$K + 2 > 0$, $K - 2 < 0$

or

$K + 2 < 0$, $K - 2 > 0$

$\Rightarrow$ $K < -2$, $K > 2$

It can not satisfies simultaniously.

So range is -2 to $+2$.

Ans

1. BISECTION METHOD :-

$$f(x) = 0$$

Assume , $x = a$, $x = b$

Such that $f(a) \cdot f(b) < 0$

$\because$ product is $-ve$ so ~~an~~ one is positive and another is negative.

So $[a, b] \longrightarrow$ Min one root will exist.

a — b

$f(a) = +ve$ $\qquad$ $f(b) = -ve$

I^{st} guess value : $x_1 = \frac{a+b}{2}$

$$f\left(\frac{a+b}{2}\right) = +ve / -ve$$

If $f\left(\frac{a+b}{2}\right) = +ve \Rightarrow f\left(\frac{a+b}{2}\right) f(b) < 0$

root lies between $x = \frac{a+b}{2}$ to $x = b$

2^{nd} guess value $x_2 = \frac{a+3b}{4}$

If $f\left(\frac{a+b}{2}\right) = -ve \Rightarrow$

then II^{nd} guess value $x_2 = \frac{3a+b}{4}$

⋮

n^{th} guess value : $x_n \Rightarrow f(x_n) = 0$

This method is not so accurate.

(2.) Newton - Raphson Method :-

$$f(x) = 0$$

Assume , $x = a$, $x = b$

Such that $f(a) \cdot f(b) < 0$

$[a, b] \longrightarrow$ min one root.

Out of $f(a)$ and $f(b)$ which one is closer to zero the corresponding to the value of x will be the first guess ~~valt~~ value.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

x_n = nth guess value

x_{n+1} = (n+1)th guess value

This is the Newton - Raphson formula for calculating guess value.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots \qquad \vdots$$

After (n+1) itteration -

When $x_{n+1} \approx x_n$ root of the eqn.

if $f(x_n) = 0$.

Limitations :-

It is not applicable when derivative of function is zero.

(I) Newton - Raphson method will not work when $f'(x_n) = 0$ that mean function $f(x)$ have maxima or minima at $x = x_n$ therefore the guess value used in the Newton - Raphson method should not be the maxima or minima of the corresponding $f(x)$ of the equation.

$$y = f(x) \longrightarrow \text{Curve}$$

$$\frac{dy}{dx} \longrightarrow \text{slope of curve.}$$

$$\boxed{f'(x_n) = 0} \longleftarrow f(x) \text{ has maxima/minima at } x = x_n.$$

(II) This method will give accurate result if the slope of the curve at the guess values i.e. $f'(x_n)$ is very high.

A-13
Q.7

Soln say $x = \sqrt[3]{N}$

$\Rightarrow x^3 = N$

$\Rightarrow x^3 - N = 0 \Rightarrow f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 - N}{3x_n^2}$$

$$= \frac{2}{3} x_n + \frac{N}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2}{3}\left(x_n + \frac{N}{2x_n^2}\right)}$$ Ans

(8)

Sol^n

$$f(x) = x^4 - x - 10 = 0$$

$$f(0) = -10$$

$$f(1) = -10$$

$$f\left(\frac{3}{2}\right) = \frac{81}{16} - \frac{3}{2} - 10 = -ve$$

$f(2) = 4$ much closer to zero

So $x_1 = 2$

$f'(x) = 4x^3 - 1$

$f'\left(\frac{3}{2}\right) =$

$f'(2) =$

$f'(2) > f'\left(\frac{3}{2}\right)$

Interpolation

The method of finding the value of y for any value of x between x_0 to x_n is known as Interpolation.

If. $y = f(x)_x$

Where x :	$\overset{x}{\hat{x}_0}$	$\overset{x+h}{\hat{x}_1}$	$\overset{x+2h}{\hat{x}_2}$	------	$\overset{x+nh}{\hat{x}_n}$
y :	y_0	y_1	y_2	----	y_n
	$\downarrow$ $f(x)$	$\downarrow$ $f(x+h)$	$\downarrow$ $f(x+2h)$		$\downarrow$ $f(x+nh)$

$x_1 - x_0 = x_2 - x_1 = \cdots\cdots = h$ (say)

1. Forward Difference Operator :-

I^{st} Order :-

$$\Delta f(x) = \underset{\text{Next Value}}{f(x+h)} - \underset{\text{Present Value}}{f(x)}$$

$$\Rightarrow \boxed{\Delta y_0 = y_1 - y_0}$$

$$\Delta f(x+h) = f(x+2h) - f(x+h)$$

$$\Rightarrow \boxed{\Delta y_1 = y_2 - y_1}$$

II^{nd} Order :-

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$

$$= \Delta[f(x+h) - f(x)]$$

$$= [f(x+2h) - f(x+h)] - [f(x+h) - f(x)]$$

$$\Rightarrow \Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

Similarly

$$\Delta^2 y_1 = y_3 - 2y_2 + y_1 \quad \left\{ \text{type of } (a-b)^2 = a^2 - 2ab + b^2 \right\}$$

IIIrd Order :-

$$\boxed{\Delta^3 y_0 = y_0 - 3y_2 + 3y_2 - y_0} \quad \left\{ \text{type of } (a-b)^3 \; a^3 - 3a^2b + 3ab^2 - b^3 \right.$$

2. Backward Difference Operator :-

Ist order :-

$$\nabla f(x) = \underset{\substack{\uparrow \\ \text{Present} \\ \text{Value}}}{f(x)} - \underset{\substack{\downarrow \\ \text{Previous} \\ \text{Value.}}}{f(x-h)}$$

$$\boxed{\nabla y_1 = y_1 - y_0}$$

IInd order :-

$$\boxed{\nabla^2 y_2 = y_2 - 2y_1 + y_0}$$

(3) Shift Operator :-

$$E\, f(x) = f(x+h)$$
$$E\, f(x+h) = f(x+2h)$$
$$E^2 f(x) = f(x+2h)$$
$$\vdots$$
$$E^n f(x) = f(x+nh)$$

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Sol^n

$$E[f(x)] = f(x+h)$$
$$D[f(x)] = \frac{df}{dx}$$

$\because$ Taylor series -

$$f(x) = f(a) + f'(a)(x-a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

at x about $x=a$

$x \rightarrow x+h$

$a \rightarrow x$

$$E[f(x)] = f(x+h)$$
$$= f(x) + f'(x)\, h + \frac{h^2}{2!} f''(x) + \cdots$$

$$= f(x) + D[f(x)]h + D^2[f(x)]\frac{h^2}{2!} + \cdots$$

$$= \left[I + hD + \frac{(hD)^2}{2!} + \cdots\right] f(x)$$

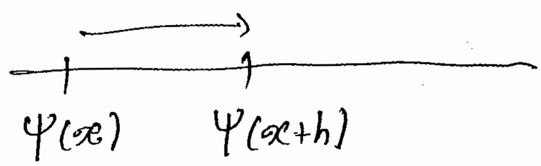
$$= \left[e^{hD}\right] f(x)$$

So $\boxed{E \equiv e^{hD}}$ Ans

$$\Rightarrow \boxed{E \equiv e^{h\frac{d}{dx}}}$$

Amount of displacement

Translation along x-axis



$$E[f(x)] = f(x+h)$$

$$E[\psi(x)] = \psi(x+h)$$

Translation operator. (in Q. M.)

$\because \hat{p}_x = -i\hbar\frac{d}{dx}$

$$\boxed{E \equiv e^{ia p_x/\hbar}}$$

a = amount of translation ($a = h$)

$d \leftarrow D$

In 3-D :-

$$\boxed{E \equiv e^{i\vec{a}\cdot\vec{p}/\hbar}}$$

$\vec{a}\cdot\vec{p} = a_x p_x + a_y p_y + a_z p_z$