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UGC POINT ACADEMY

Statistical Mechanics

$$\underset{\text{macro}}{(a+b)^2} \rightarrow \underbrace{a^2+2ab+b^2}_{\text{micro}}$$

→ One coin ; Two macrostate, Two microstate

→ 3 coin : 4-macrostate & 8-micro

Toss of n-coins;

$(a+b)^n \rightarrow (n+1) \rightarrow$ macrostate

$2^n \rightarrow$ microstate.

Pro. of equally LPE is $\rightarrow \frac{1}{2^n}$

→ If degeneracy of state are exist then No. of ways of distri.

$$\Omega = \frac{N!}{\prod_i n_i!} \prod_i g_i^{n_i}$$

Q. 10 similar coins are tossed together. wht is the prob. of getting 4 heads.

$$^{10}C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^6$$

$$\rightarrow \frac{10!}{6!\,4!} \cdot \frac{1}{2^{10}}$$

Q. A random walker may take a step towards right or left at random, wht is the prob. that the walker comes to its initial pos. after taking 6 steps $\Rightarrow {}^6C_3 \left(\tfrac{1}{2}\right)^6$

Statistical Mechanics :

(Based on distribution of particle)

Branch of mechanics which deals with the prob. of system involv large no. of molecule/particles.

How?

→ S. M. solve the prob. of macroscopic system by microscopic pt. of view.

→ In thermodynamics we solve the prob. of a macroscopic system by macroscopic pt. of view.

→ By S.M. we can calculate the average properties of Bulk material. As the no. of molecules involve in the calculation increases the result or calculations becomes more perfect.

→ This mechanics is based on distribution of particle. By Knowing the distribution of particles in system we calculate the physical properties of the system.

→ Initially this mechanics was developed by Maxwell, Baltzman, Gibbs etc.

→ This branch of S. M. is K/as M-B statistical mechanics or classical statistical mechanics. By it we can calculate the average energy ($\bar{E}$) Ave ($\bar{P}$), Ave. temp ($\bar{T}$), Av. value of chemical potential ($\bar{\mu}$) etc.

→ But laterly fails to explain BBR, specific heat of solids at low temp., BEC (Bose einstein condensation)

Then another branch of S.M. is developed by Fermi Dirac, Bose, Einstein etc. which is based on Planck's Quantum idea (discrete exchange of energy b/w radiation & matter).

→ This new branch is Klas Q.S.M.

S.M.

- C.S.M (M-B st.M.) → Application in case of system having identical but distinguishable particles of any spin
  - Ex: Gas molecule
- Q.S.M. → Applicable when system contain particle which are identical & indistinguishable
  - F-D statistics → particle has spin $\frac{1}{2}, \frac{3}{2}, \ldots$
    - Ex: e^-, P, ν, n (neutrino ferm)
  - B-E statistics → particle has spin $0, 1, 2 \ldots$
    - Ex: $_1H^3$, $_2He^4$

→ Classical particle (A, B, C, etc) Name can be given.

→ Quantum particles (. . . . . .) represented by dots of same size or same letter.

→ Total No. of particles (e^-, p, n)
$= Z + A$ (atom)
$= A$ (nucleus)

Even → Bosons
odd → fermions

if $_3Li^7$ is atom Then Boson
if $_3Li^7$ is nucleus Then it is fermions.

Ω^- → fermions

→ A photon is boson, An e- is fermion.

C.S.M. :

Distribution :

Prob: find the no. of ways of distribution, when 3 distinguishable particles are to be distributed in a box such that a) A box, contain 1 particle only $r=1$

Ans : 3

$\boxed{A}$ or $\boxed{B}$ or $\boxed{C}$

b) Box contains two particles.

Ans : 3

$\boxed{AB}$, $\boxed{BC}$, $\boxed{CA}$

c) Box contains 3 particles.

Ans : 1

$\boxed{ABC}$

→ Thus the no. of ways of distribution of N particle such that box contain r out of N distinguishable particle $= {}^{N}C_{r} = \Omega$

Macrostate / or Macroscopic state :

The state which represents the observable property of the system, is called macrostate.

Observable properties : N (No. of particles), Volume (V) Energy (E), momentum (p) etc.

→ A macrostate has large no. of microstates. (Accessible under given restriction)

Microstate: The states which represents the unobservable property of the system.

→ No. of μ-state in a macrostate is called Thermodynamical probability (Ω) > 1. explain or represent No. of ways of distribution.

$\boxed{S = K_B \ln \Omega}$ → As we increase no. of particles, more no. of microstate, so the system move toward most prob. state.

Prob:

N particles are to be distributed into two boxes such that box 1 contain n_1 particles and 2 box contain n_2 particles & $N = n_1 + n_2$. find the no. of ways of distributions? or Thermodynamical probability / no. of μ-state

$$\Omega = {}^N C_{n_1} \text{ or } {}^N C_{n_2}$$

$$\text{or } \frac{\lfloor n}{\lfloor n_1 \lfloor n_2}$$

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N particles are to be distributed in large No. of boxes having n_1, n_2, n_3 ------- particles

$$N = n_1 + n_2 + \text{-----}$$

No. of ways of distribution:

$$= {}^N C_{n_1} \times {}^N C_{n_2} \times {}^{N-n_1-n_2} C_{n_3} \times \text{-----}$$

$$\Omega = \frac{\lfloor N}{\lfloor n_1 \lfloor n_2 \lfloor n_3 \ldots} = \frac{\lfloor N}{\prod_i n_i!}$$

Prob: 1000 particles are to distributed in 3 boxes having (500, 400 & 100) particles. find the entropy of the system.

$$\boxed{\ln \lfloor N = N \ln N - N}$$ Stirling formula

(provided N large)

$$\ln \lfloor N = \ln (1, 2, 3 \cdots N)$$

$$= \sum_{x_i=1}^{N} \ln x_i$$

$$= \int_{x=1}^{N} \ln x \, dx$$

$$= x \ln x - x \Big|_1^N$$

$$= N \ln N - N + \cancel{1}$$

Prob: Two spin $\frac{1}{2}$ particles of mag. moment (μ) are placed in a mag. field in z-dir^n. find the total no. of ways of distribution.

U U

U D

D U

D D

$$\Omega = (2S+1)^N$$

		En. of particle		En of system
U	U	$-\mu B$	$-\mu B$	$-2\mu B$
U	D	$-\mu B$	μB	0
D	U	μB	$-\mu B$	0
D	D	μB	μB	$2\mu B$

→ Total No. of μ-state = 4

→ The No. of μ-state corresponding to $E = 0$

$= 2$

→ In all μ-state are equally probable find the probabi of finding the system in most probable macrostate.

$= \frac{1}{2}$

Prob: Consider a system of 3 spin 1/2 particles. The z-component of spin of system is given by

$$S_z = S_z(1) + S_z(2) + S_z(3)$$

each of $S_z(1)$, $S_z(2)$, $S_z(3)$ can take value $\pm 1/2$

find

(i) Total no. of μ-state of the system → 8

(ii) No. of μ-states in macrostate $S_z = 1/2$ → 3

(iii) If all μ-states are equally probable of finding the system in $S_z = -1/2$ → 3/8

Prob: Consider the case of 10 unbiased coins, tossed simultaneously.

find the no. of ways of distribution

i) 3 H uppermost (3H, 7T) $^{10}C_3$

Probability →

The quantative value of qualitative statements.

Spin 1/2 particles

Coins related prob.

dice related "

Random walker

Particle distr. in two boxes

baby births

Particle in given volume u of a box of vol. V.

→ Prob. of equally likely probable even $= \frac{1}{2^n}$

Distribution of probability:

Binomial distribution: Is discrete distribution and applicable when p & $q(=1-p)$ are not too low. The prob. p of one trial is known.

$$\underset{\text{Total Prob. in n-trial}}{(p+q)^N} = 1 = {}^N C_0 p^0 q^N + {}^N C_1 p^1 q^{N-1} + \cdots + {}^N C_r p^r q^{N-r} + {}^N C_N p^N q^0$$

wt. of event

$$P(r, N-r) = {}^N C_r p^r q^{N-r}$$

Prob: A unbiased coin is tossed 10 times. find the prob. of getting

(i) 1 H uppermost $r=1$ $\frac{{}^n C_r p^r q^{N-r}}{p^N q^1}$

(ii) 9 T uppermost

(iii) Atleast 9H uppermost ${}^N C_r - N \cdot r = 1$ ${}^{10}C_1$ 9H or 10H

(iv) 5 T uppermost $(1/2)^{10}$ max. prob.

$q = 1-p$

(v) (5T, 5H) distribution prob.

(vi) Prob. of least probable distribution All H, All T ${}^{10}C_5 (\frac{1}{2})^5 (\frac{1}{2})^5 m$

(vii) Probability of most probable distri

(viii) Prob. At least 1 T uppermost $= {}^N C_r (\frac{1}{2})^N$ $P(1) + P(2) + \cdots + P(10) = 1 - P(0)$

$p = \frac{1}{2}$, $q = \frac{1}{2}$, $N = 10$

(ix) At most 1H uppermost $P(0) + P(1)$

$$\Rightarrow {}^{10}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^9$$

$${}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

least $\Rightarrow {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$

Q. 10 unbiased coin is tossed . . . times.

coin → baby
toss → birth
H → Girl
Tail → Boy

coin → Dice
toss → Throw

p (prob. of coming a no. uppermost 1, 2, 3, 4, 5, 6) $= \frac{1}{6}$

$q = 1 - p = \frac{5}{6}$

Q. A dice is thrown 10 times find probability that

(i) no. 1 comes uppermost all times $^{10}C_{10}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^{0}$

(ii) least probable case

(iii) No. 4 comes uppermost atleast 8 times.

$P(8) + P(9) + P(10) = {}^{10}C_8\left(\frac{1}{6}\right)^8\left(\frac{5}{6}\right)^2 + {}^{10}C_9\left(\frac{1}{6}\right)^9\left(\frac{5}{6}\right) + {}^{10}C_{10}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^{0}$

(iv) No. 6 does'nt appear uppermost

$^{10}C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10}$

Q. 10 dice are thrown simultaneous.

Q. 10 balls are distributed into 2 boxes with equal prob. find prob. of

(a.) all balls occupy 1st box $(10,0) = {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$

(b.) Prob. of distribution $(8,2)$ $\quad P(r, N-r) = {}^{10}C_{8}\left(\frac{1}{2}\right)^{10}$

(c.) Most probable distri.

Random walker problem: One who dont care of taking steps

Child   Drunkard

Prob: A one dim. random walker takes 10 steps in left right with equal prob.

4 step. $^4C_2$

find prob. of

(i) Starting 4rm origin, it will be at origin after all steps.

$$^{10}C_5 \left(\tfrac{1}{2}\right)^{10}$$

(ii) Starting 4rm 0, He will be one step away from 0, after all steps. Prob $\to 0$ (Not happing)

$$\left.\begin{aligned} n_1 + n_2 &= 10 \\ n_1 - n_2 &= 1 \end{aligned}\right\} \quad n_1 = 5.5$$

(iii) Most probable distribution $\;{}^{10}C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^5$

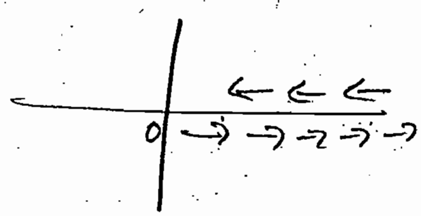
(iv) Least prob. distribution. $\;{}^{10}C_{10} \left(\tfrac{1}{2}\right)^{10}$

(v) Start 4rm 0 will be 2 steps 4rm 0 after all steps.

$$n_1 + n_2 = 10 \qquad (6,4) \text{ or } (4,6)$$
$$n_1 - n_2 = 2 \qquad {}^{10}C_6 + {}^{10}C_4$$

$^{10}C_4 \;\; \frac{1}{210}$

Ans: $2 \times {}^{10}C_6 \left(\tfrac{1}{2}\right)^{10}$



Prob: A 1-d random walker takes steps to right or left with equal prob. find the prob. that the random walker starting from the origin is back to origin after takin N even no. of steps. $^{N}c_{\frac{N}{2}} \left(\frac{1}{2}\right)^{N} = \frac{\lfloor N}{\lfloor \frac{N}{2} \lfloor N/2} \left(\frac{1}{2}\right)^{N}$

Solⁿ:

$$^{N}C_{N/2} \left(\frac{1}{2}\right)^{N}$$

$$= \frac{\lfloor N}{\lfloor N/2 \ \lfloor N/2} \left(\frac{1}{2}\right)^{N}$$

Prob: A drunkard start from a certain lamp-post on the street and he is trying to get his destination at a distance x. Some of steps he take in forward dirⁿ with prob. (p) & some steps in backword dirⁿ with prob. ($q = 1-p$), and each step is of equal length. wht is the prob. that the drunkar will reach its destination after N-steps..

Solⁿ:

$$n_f + n_b = N$$

$$n_f L - n_b L = x$$

Prob. $P(n_f) = {}^{N}C_{n_f}\, p^{n_f}\, q^{N-n}$

$$= {}^{N}C_{n_f}\, p^{n_f} (1-p)^{n_b}$$

$\rightarrow$ $N = 10$

$L = 1m$

$x = 2m$

$p = \frac{2}{3}$

$q = \frac{1}{3}$

$n_f + n_b = 10$

$n_f - n_b = 2$

$n_f = 6$

$n_b = 4$

$$= {}^{10}C_6 \left(\frac{2}{3}\right)^{6} \left(\frac{1}{3}\right)^{4}$$

$n_f + n_b = N$

$n_f l_f - n_b l_b =$

$\longrightarrow N = 10$ [which step is unequal]

$l_f = 2m, \quad l_b = 1m$

$x = 2m$ $\qquad$ $n_f + n_b = 10$

$p = \frac{2}{3} \quad q = \frac{1}{3}$ $\qquad$ $2n_f - n_b = 2$

$n_f = 4$

$n_b = 6$

$= {}^{10}C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^6$

Q. A random walker takes a step of unit length in +ve dir^n with prob. 2/3 & a step of unit length in the -ve dir^n with prob. 1/3. The mean displacement of walker after n step is

$n/3, \quad n/8, \quad 2n/3, \quad 0$

Sol^n: Mean displacement

$$\langle x \rangle = \frac{P_0 x_0 + P_1 x_1 + P_2 x_2 + \cdots + P_n x_n}{\underbrace{P_0 + P_1 + P_2 + \cdots + P_n}_{1}}$$

$$= 0 + {}^nC_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{n-1} \cdot 1 + {}^nC_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{n-2} \cdot 2 + \cdots + {}^nC_n \left(\frac{2}{3}\right)^n \left(\frac{1}{3}\right)^0 \cdot n$$

OR

$$\langle x \rangle = \frac{P_f x_f - P_b x_b}{P_f + P_b}$$

$$= \frac{\frac{2}{3}(n \times 1) - \frac{1}{3}(n \times 1)}{\frac{2}{3} + \frac{1}{3}}$$

$$\langle x \rangle = \frac{n}{3}$$

Ave. distance travelled by the walker in forward dir^n in n step with prob. $\frac{2}{3}$ of each step of length unit

$= \frac{2n}{3}$

In Backward; mean D.

$= \frac{n}{3}$

$\langle x \rangle = Px$

$= \sum P_i x_i$

Steps	Prob.	
Left	11B	← Most pro
Right	7/B	

Q. A coin 10 tossed

Event	Prob.
0H	
1H	
2H	
⋮	
10H	

Ave Head

$= 0H \times P_0 + 1 \times P_1$

$+ \cdots$

→ Statistical Error $\propto \frac{1}{\sqrt{N}}$

or

deviation

OR

Uncertainty $\sigma \propto \frac{1}{\sqrt{N}}$

→ Unbiased → Biased coin

$^{N}C_r \, p^r \, q^{N-r}$

$p_H = \frac{1}{3}$

$q_H = 2/3$

P_t

$N = 10$

$P(4H) = {}^{10}C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^6$

Q. Consider a box of volume V containing N identical but distinguishable particle find

a.) The prob. that a given particle is in given volume v of the box. $\frac{Nv}{V}$

$$\boxed{\frac{v}{V}}$$

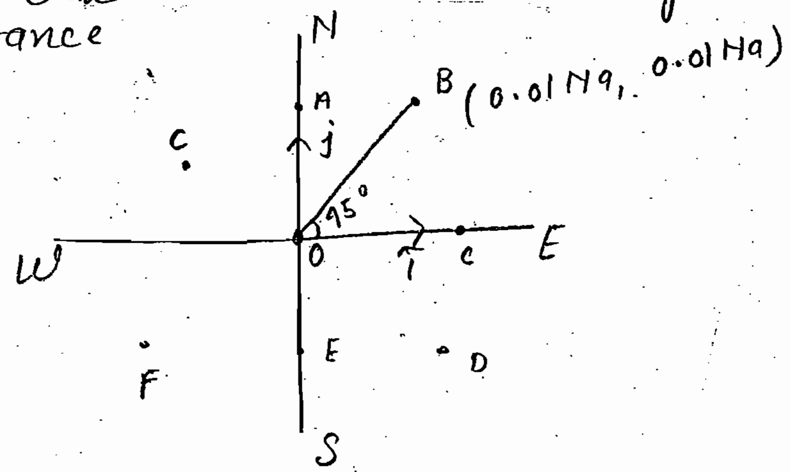
b.) The prob. that a given set of n particles are in a given volume v & all other particles are in the remaing volume. 1, 2, 3 - - n(n+1) - - N

$$\left(\frac{v}{V}\right)^n \left(\frac{V-v}{V}\right)^{N-n}$$

c.) Prob. of finding any n particles in volume v & rest (N-n) in remaing volume

$$= {}^N C_n \left(\frac{v}{V}\right)^n \left(1-\frac{v}{V}\right)^{N-n}$$

Q. A child make a random walk on a square lattice of lattice const a taking step in north, east, south & west dir^n with prob. 0.255, 0.255, 0.245, 0.245 respect. After a large No. of steps, N, the expected position of the child w.r.to the starting pt O is at a distance



$\sqrt{2} \times 10^{-2} Na$ in N-E dir^n

$\sqrt{2}N \times 10^{-2} a$ in N-E dir^n

$$\langle \vec{r}\,' \rangle = \frac{P_1\vec{r_1}' + P_2\vec{r_2}' + P_3\vec{r_3}' + P_4\vec{r_4}'}{P_1 + P_2 + \cdots}$$

$$= 0.255\, Na\hat{i} + 0.255\, Na\hat{j} - 0.245\, Na\hat{i} - 0.24$$

$$= 0.01\, Na\hat{i} + (0.01\hat{j})Na$$

Prob: spin $\frac{1}{2}$ particles are kept in a m.f find prob.

(i) Most probable distri

$${}^{N}C_{N/2}\left(\frac{1}{2}\right)^N$$

(ii) $\left(\frac{2N}{3}, \frac{N}{3}\right)$ Most prob. ${}^{N}C_{2N/3}\left(\frac{1}{2}\right)^N$

(iii) $(0, N)$ leastprob

$\rightarrow {}^{N}C_0\left(\frac{1}{2}\right)^N$

Prob: Three fair dice are thrown simultaneously find the probability of getting

a.) Sum 18 uppermost

$(6, 6, 6)$ $\frac{1}{216}$

b.) Sum of 12 uppermost

$(6, 6)$ $\frac{1}{36}$

c.) Most probable sum

m.p. 7 to 12

as 8

$(6,1), (1,6)$

$(3,4)$ $(4,3)$

$(2,5), (5,2)$

6 2

2 6

5 3

3 5

4 4

$\rightarrow \frac{1}{6}$

8 for 13.6.14

Prob: An unfair dice has prob. to appear 1, 2, 3 ---- 6 respectively as $\frac{1}{21}, \frac{2}{21}, \frac{3}{21} --- \frac{6}{21}$. In throwing such a dice 4 times find prob.

(i) No. 1 appear in uppermost 2 times

$$^4C_2 \left(\frac{1}{21}\right)^2 \left(\frac{20}{21}\right)^2 \qquad {}^nC_r\, p^r q^{n-r}$$

(ii) Least probable case

$$^4C_4 \left(\frac{1}{21}\right)^4 \left(\frac{20}{21}\right)^0 \qquad {}^NC_r\, p^N q^0$$

$$^4C_0 \left(\frac{6}{21}\right)^0 \left(\frac{15}{21}\right)^4$$

(iii) Most probable case

(iv) If this dice is thrown 2 time. find the prob. of getting sum even uppermost

$(1,1)^{1}$ $(1,3)^{2}$ $(1,5)^{2}$ $(2,2)^{1}$, $(2,4)^{2}$ $(2,6)^{2}$

$(3,3)^{1}$ $(4,4)$ $(4,6)$ $(5,3)$ $(5,5)$ $(6,6)$

1 2 2 1 1

(v) In throwing such a dice 3 times find prob. of getting sum 17 uppermost → [1 − odd sum]

(iv) $\frac{1}{441}\left[1+6+10+4+16+48+9+16+30+25+36\right]$

$$= \frac{225}{441}$$

(v.) $1 - [(1,2)(1,4)(1,6)(3,2)(3,4)(3,6)(5,2)(5,4)(5,6)]$

$$1 - \frac{2}{441}\left[2+4+6+6+12+18+10+20+30\right]$$

Q. A system has 3 particles having energy

$$E = -\epsilon_0 [J_1 J_2 + J_2 J_3 + J_3 J_1]$$

each of J_1, J_2, J_3 has values ± 1. find

1.) min^m energy of the system $\rightarrow -3\epsilon_0$ (all -ve)

2.) max. en of the system $\rightarrow \epsilon_0$ (Any 2 +ve or -ve) $^3C_2$

3.) No. of μ-state in microstate $E = \epsilon_0 = 6$

$3+3 = 6$

$E = -3\epsilon_0$ (macrostate)

having 2 μ state

Total no. of μ-state

$= 8$

Sol^n;

Q. Consider a system having 3 parameter J_A, J_B & J_C each of J_A, J_B, J_C can take value ± 1. The prob. of changing the value of these parameter is 2/3 & not changing is 1/3 at a given instant. In the let $J_A = 1$, $J_B = 1$, $J_C = -1$ at a given instant. In the next instant the prob. that $J_A + J_B + J_C$ remain unchanged sum at given instant.

is: $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{9}$, $\frac{4}{9}$

$J_A + J_B + J_C = +1$

given instant: 1 1 -1

On next, either

1 1 -1

or

-1 1 1

or

1 -1 1

$\frac{8}{27} + \frac{1}{9} \times \frac{2}{3} + \frac{1}{9} \times \frac{2}{3}$

$= \frac{12}{27}$

when it change

$\rightarrow \frac{1}{27} + \frac{4}{27} + \frac{4}{27} = \frac{9}{27}$

fundamental postulates of statistical mechanics:

→ Any gas under consideration may be considered to be composed of large no. of molecule which are constantly in motion. And Behave like very small elastic sphere

→ The configuration of the particle is completely described by six co-ordinate (x, y, z, p_x, p_y, p_z). This six dim space is Klas phase-space. This phase is divided into large no. of elementary cell called phase cell, each having equal size h^3.

→ The most fundamental postulate of S.M. is "principal of equal a priori prob." ie all accessible to posible macrostate are equally probable.

→ The equilibrium state of gas corresponds to the macrostate of maxm prob

→ The total no. of molecule derived from

$$N = \sum_i n_i = \text{constt} \quad (\text{conservation of mass})$$

→ Total energy of the system remains constt:

$$E = n_1 \epsilon_1 + n_2 \epsilon_2 + \cdots = \sum_i n_i \epsilon_i = \text{constt}$$

(energy conservation)

constraints on system!

Prob: 10 particle are distributed in 3 given cells, ~~with~~ with energy per particle $0, \epsilon, 3\epsilon$ such that $n_1 = 3, \; n_2 = 5, \; n_3 = 5$

If total En of the system remains same. Then find δn_1, δn_2 when $\delta n_3 = -1$ for odds

Also find the thermodynamical prob. new distribution.

Sol^n:

$$n_1 + n_2 + n_3 = 13 = \text{constt}$$

$$\delta n_1 + \delta n_2 + \delta n_3 = 0$$

$$\delta n_1 + \delta n_2 = 1$$

$$E = n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 = \text{constt}$$

$$\Rightarrow E = n_2\epsilon + 3n_2\epsilon$$

$$0 = \epsilon\,\delta n_2 + 3\epsilon\,\underset{-1}{\underset{\shortparallel}{\delta n_3}}$$

$$\delta n_2 = 3$$

new distribution (1, 8, 4)

No. of accesible μ-state / thermodynamic prob.

$$\Omega = \frac{N!}{N_1!\,N_2!\,N_3!} = \frac{13!}{1!\,8!\,4!}$$

$$\Omega_{old} = \frac{13!}{3!\,5!\,5!}$$ most prob. distri. (equillibrium state, max^m entropy star)

Phase - space: To describe the configuration of a particle completely, six co-ordinate (x, y, z, p_x, p_y, p_z) required.

Six dim. space is called phase space

Volume element in phase space

$$d\tau = dx\,dy\,dz\,dp_x\,dp_y\,dp_z$$

$$= (dx\,dp_x)(dy\,dp_y)(dz\,dp_z)$$

$$dx\,dp_x \geq h$$

$$d\tau \geq h^3$$

$$(d\tau)_{min} = h^3$$

Thus elementary cell in phase space or phase cell has volume h^3.

In cartesian co-ordinate

$(d\tau)_{min} = 0 \rightarrow$ point element can never be possible in phase space.

→ **Volume of phase space:**

$$= \iiiint\!\!\iint dx\, dy\, dz\, dp_x\, dp_y\, dp_z$$

$$-\infty < x,\ y,\ z < +\infty$$

$$-\infty < p_x,\ p_y,\ p_z < +\infty$$

macro-state
$$\begin{cases} p_x \rightarrow p_x + dp_x \\ p_y \rightarrow p_y + dp_y \\ p_z \rightarrow p_z + dp_z \end{cases}$$

$p \rightarrow p + dp$

or

$E = E + dE$

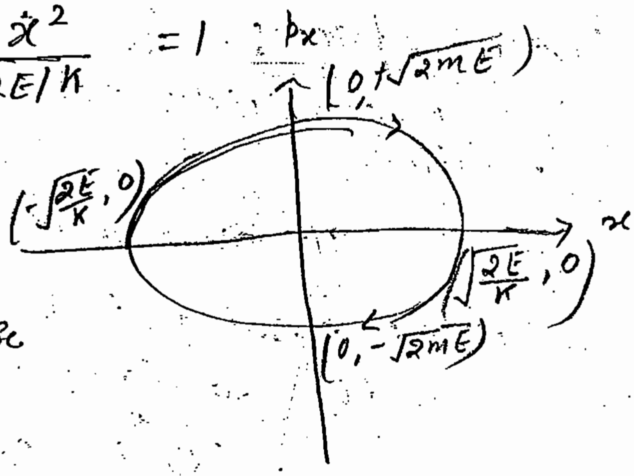
macrostate

Application of phase space:

Ex: 1-D H.O.

$$E = \frac{p_x^2}{2m} + \frac{1}{2} K x^2$$

$$\Rightarrow \frac{p_x^2}{2mE} + \frac{x^2}{2E/K} = 1$$



Accessible area to the O.sc

$$= \pi a b$$

$$= \pi \sqrt{2mE} \sqrt{\frac{2E}{K}}$$

$$= E\, 2\pi \sqrt{\frac{m}{K}} = \frac{E}{\nu}$$

No. of possible μ-state:

$$= \frac{\text{Accessible area to the Osc}}{\text{Area of 1 phase cell}}$$

$$n = \frac{E/\nu}{dx\,dp_x} = \frac{E}{h\nu}$$

$$\boxed{n = \frac{E}{h\nu}}$$

$$\boxed{E = nh\nu}$$

$n = 0, 1, 2 \ldots$

* free particle:

$P.E. = 0$

$$E = \frac{p^2}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

No. of μ-state in momentum range $p \to p+dp$
or
In energy range $E \to E+\delta E$

$$\Omega(p)\,dp = \frac{\text{Volume of phase space in mom. range } p \to p+\delta p}{\text{Volume of 1 phase cell}}$$

$$\Omega(p)\,dp = \frac{\iiiiiint dx\,dy\,dz\,dp_x\,dp_y\,dp_z}{h^3}$$

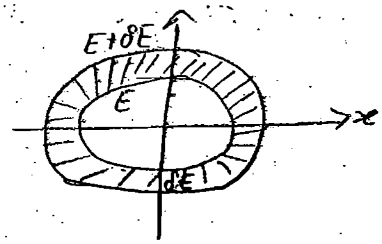
$$= \frac{\iiint dx\,dy\,dz \iiint dp_x\,dp_y\,dp_z}{h^3}$$

$$\Omega(p)\,dp = \frac{V}{h^3}\int_p^{p+dp} \underset{\substack{\downarrow \\ p^2 \sin\theta\, d\theta\, d\phi\, dp}}{d^3p}$$

$0 < r < \infty$

$0 < p < \infty$

$$= \frac{V}{h^3}\, 4\pi p^2\, dp$$



$$\delta n = \frac{\delta E}{h\nu}$$

$$\frac{\delta n}{\delta E} = \frac{1}{h\nu}$$

$$D(E) = \frac{1}{h\nu}$$

OR

$$D(E)\,\delta E = \frac{\delta E}{h\nu} = \delta n$$

$\downarrow$
Density of state i.e. no. of states per unit energy range.

$E \to E+\delta E$
no. of states in En. range $\delta E = \dfrac{\delta E}{h\nu}$

$$\Omega(E)\,\delta E = \frac{\delta E}{h\nu}$$

$0 \longrightarrow E$

$$\int_0^E \Omega(E)\,\delta E = \Omega = \text{Total no. of } \mu\text{-states in range } E$$

$$\Omega(p)\,dp = \frac{V}{h^3}\left[\frac{4}{3}\pi(p+dp)^3 - \frac{4}{3}\pi p^3\right]$$

$$\boxed{\Omega(p)\,dp = \frac{V\,4\pi p^2 dp}{h^3}}$$

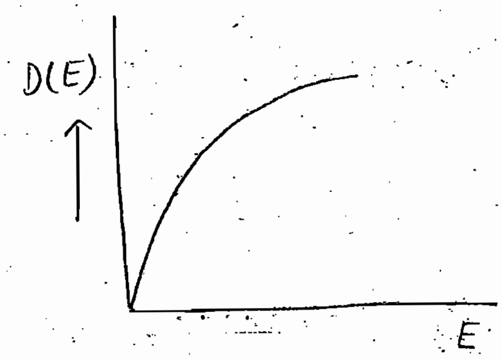
$$p = \sqrt{2mE}$$

$$dp = \sqrt{2m}\;\frac{1}{2}E^{-1/2}\,dE$$

$$dn = \Omega(E)\,dE = \frac{4\sqrt{2}\,V\pi\, m^{3/2}E^{1/2}\,dE}{h^3}$$

$$\underbrace{\Omega(E) = \frac{dn}{dE} = D(E)}_{\text{Density of state}} = \frac{4\sqrt{2}\,V\pi\, m^{3/2}E^{1/2}}{h^3}$$

$$\boxed{D(E) \propto E^{1/2}}$$



* Non- interacting photon gas :

No. of states available for photon is mom. range $p \to p+dp$ or $E \to E+dE$

$$\boxed{\Omega(p)\,dp = \frac{V\,4\pi p^2}{h^3}\,dp}$$

In terms of energy $p = E/c$

$dp = dE/c$

$$\Omega(E)\,dE = \frac{V\,4\pi E^2 dE}{c^3 h^3}$$

$$\Omega(E) = \frac{V\,4\pi E^2}{c^3 h^3}$$

$$\begin{matrix} \Omega(E) \\ \| \\ D(E) \\ \| \\ \dfrac{dN}{dE} \end{matrix}$$

In terms of frequency

$$E = h\nu$$

$$dE = h\,d\nu$$

$$\boxed{D(\nu) = \frac{V\,4\pi \nu^2 d\nu}{c^3}}$$

Phonon gas:

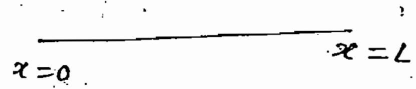
Expresion are same as photon gas..

* Density of states:

1-D: Consider a linear crystal of length L

The periodic boundary condn

$$\psi_K(x) = \psi_K(x+L)$$



$$Ae^{iKx} = Ae^{iK(x+L)}$$

$$e^{iKL} = 0 = e^{\pm i2n\pi}$$

$$K = \pm\frac{2n\pi}{L} = \cdots -\frac{4\pi}{L}, -\frac{2\pi}{L}, 0, +\frac{2\pi}{L}, +\frac{4\pi}{L}$$

No. of μ-state in range $K \to K + dK$

$$dn = \frac{\text{range of } K}{\text{length require for 1 state}}$$

$$= \frac{dK}{\left(\frac{2\pi}{L}\right)}$$

$$dn = \frac{L\, dK}{2\pi}$$

$$\frac{dn}{dK} = \frac{L}{2\pi}$$

OR

$$\boxed{\begin{array}{l} D(K) \\ DOS \end{array} = \frac{L}{2\pi}}$$

$$D(p) = \frac{L}{2\pi\hbar}$$

from

$$D(K)\, dK = \frac{L}{2\pi} dK$$

$$D(p)\, dp = \frac{L}{2\pi} \frac{dp}{\hbar}$$

$$\boxed{D(p) = \frac{L}{2\pi\hbar}}$$

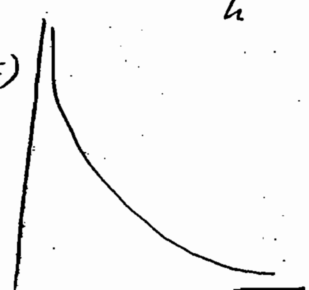
In terms of E

$$p = \sqrt{2mE} \Rightarrow dp = \sqrt{\frac{m}{2}}\, E^{-1/2} dE$$

$$D(E)dE = \sqrt{2m}\, \frac{1}{2} E^{-1/2} dE = \sqrt{\frac{m}{2}}\, E^{-1/2} dE\, \frac{L}{h}$$

$$\boxed{D(E)dE = \frac{L}{h}\sqrt{\frac{m}{2}}\, E^{-1/2}\, dE}$$

D(E)



DOS in 2-D :

No. of μ-state in range $K \to K+dK$

$$dn = \frac{\text{Area available in phase space}}{\text{Area required for 1 state}}$$

$$= \frac{2\pi K\, dK}{\left(\frac{2\pi}{L}\right)^2}$$

$$dn = \frac{A K\, dK}{2\pi}$$

$$L^2 = A$$

$$\boxed{D(K)\, dK = \frac{A K\, dK}{2\pi}}$$

$$K = \frac{p}{\hbar}$$

$$dK = \frac{dp}{\hbar}$$

$$\boxed{D(p)\, dp = \frac{A p\, dp}{2\pi \hbar^2}}$$

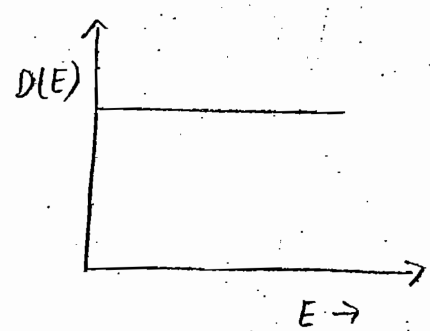
$$p^2 = 2mE$$

$$2p\, dp = 2m\, dE$$

$$p\, dp = m\, dE$$

$$\boxed{D(E)\, dE = \frac{2\pi A m\, dE}{h^2}}$$

$$D(E) \propto E^0$$



DOS in 3-D :

No. of μ-state in range

$$K \to K + dK$$

or

$$K_x \to K_y + dK_x , \quad K_y \to K_y + dK_y$$

$$K_z \to K_z + dK_z$$

$$dn = \frac{\text{Volume available in phase space}}{\text{Volume required for 1 state}}$$

$$dn = \frac{4\pi K^2 dK}{\left(\frac{2\pi}{L}\right)^3} = \frac{V K^2 dK}{2\pi^2}$$

$$L^3 = V$$

$$\boxed{D(K)\, dK = \frac{V K^2 dK}{2\pi^2}}$$

$$K = \frac{p}{\hbar}$$

$$dK = \frac{dp}{\hbar}$$

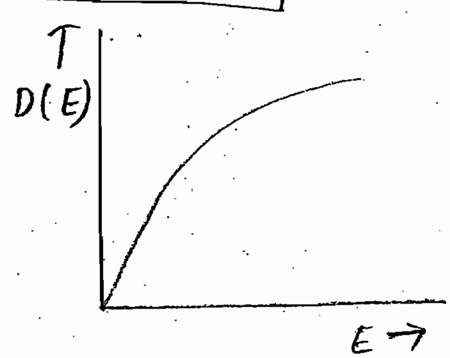
$$\boxed{D(p)\, dp = \frac{V 4\pi p^2 dp}{h^3}} \to \text{G.E.}$$

$$p = \sqrt{2mE}$$

$$dp = \sqrt{\frac{m}{2}}\, E^{-1/2} dE$$

$$\boxed{D(E)\, dE = \frac{4\sqrt{2}\,\pi\, m^{3/2} E^{1/2} dE}{h^3}}$$

$$\boxed{D(E) \propto E^{1/2}}$$



$$\to \omega = v_0 K$$

$$\frac{\hbar\omega}{v_0} = p = \frac{E}{v_0}$$

ask $\rightarrow$ $E \propto p^s$ (given) $\qquad$ $s=2$ free particle

$s=1$ for photon or phonon

$$\boxed{D(E) \propto E^{d/s - 1}} \text{ in } d\text{- dim.}$$

$\rightarrow$ $E \propto p^2$ 3-dim

$D(E) \propto E^{1/2}$

$\rightarrow$ $E \propto \sqrt{p}$ 3-D

$D(E) \propto E^{3\times 2-1}$ or E^5

Particle in 1-D rigid box $\rightarrow$

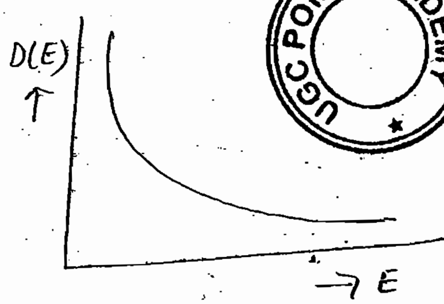
$V(x) = 0 \qquad 0<x<L$

$= \infty \qquad$ elsewhere

$D(E) \propto E^{-1/2}$

$E = \frac{p^2}{2m} + 0$

$D(E) \propto E^{\frac{1}{2}-1}$

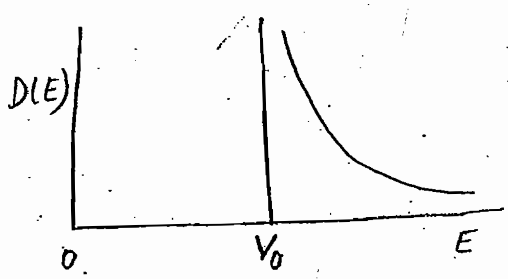


$\rightarrow$ $V(x) = V_0 \qquad 0<x<L$

$= \infty \qquad$ elsewhere

$E = \frac{p^2}{2m} + V_0$

$D(E) \propto (E-V_0)^{-1/2}$



$E \propto p^s$

$E \propto k^s$

$\omega \propto k^s$

all are same things.

$$\longrightarrow D(E) = \frac{dn}{dE}$$

$$E = \frac{n^2h^2}{8mL^2} + V_0$$

$$dE = \frac{nh^2\,dn}{4mL^2}$$

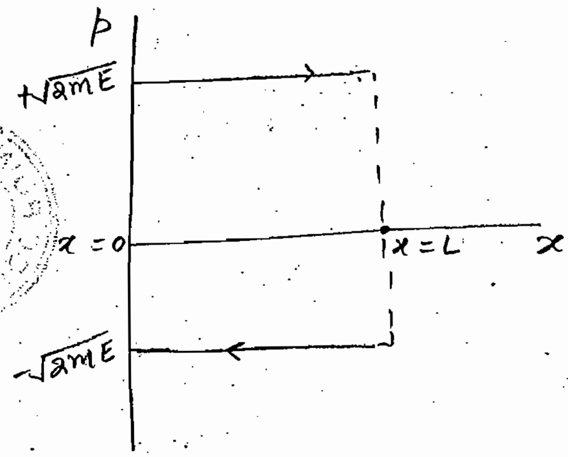
$$\frac{dn}{dE} = \frac{4mL^2}{h^2}\cdot\frac{1}{n}$$

$$\frac{dn}{dE} \propto \frac{1}{n} \propto \frac{1}{(E-V_0)^{1/2}}$$

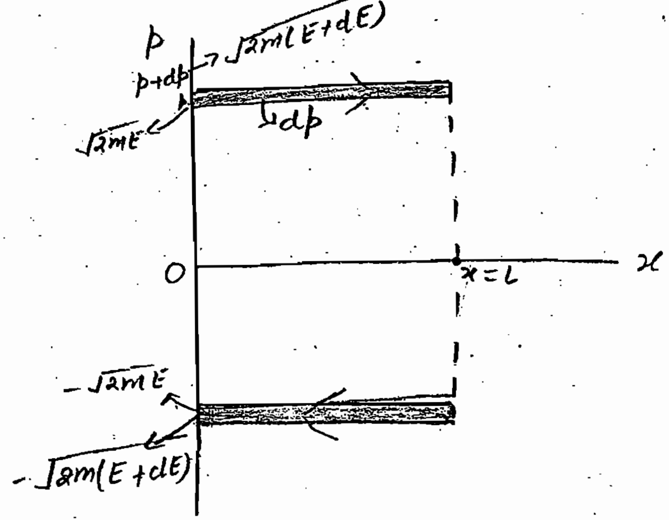
Phase-space diagram:

$$E = \frac{p^2}{2m}, \quad 0<x<L$$

$$p = \pm\sqrt{2mE}$$



$\longrightarrow$ If p-varies from $p \rightarrow p+dp$



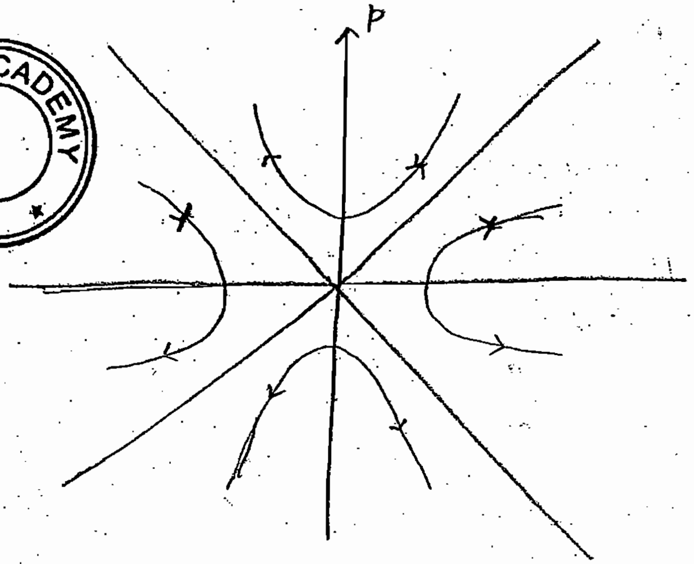
Prob: The Hamiltonian of a particle is given by

$$H = \frac{p^2}{2m} - \frac{\alpha z^2}{2}$$

which of the following fig represents the motion of the particle in phase space.

$$H = \frac{p^2}{2m} - \frac{\alpha z^2}{2}$$ $E \to$ constt

P-S-digi: Parabola $V(x) = -\frac{\alpha z^2}{2}$ ie eq^n of para



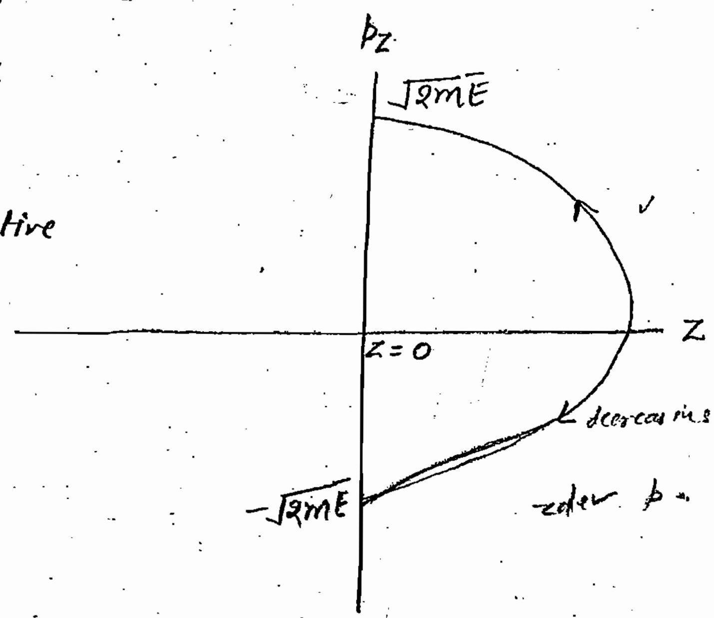
$\dot{q} = \frac{\partial H}{\partial p} = p/m$

$\dot{p} = -\frac{\partial H}{\partial q} = \alpha q$

* The trajectory (Phase space diagram) on $z-p_z$ plan of ball bouncing from a perfect elastic hard sphere.

$$H = \frac{p^2}{2m} + mgz$$

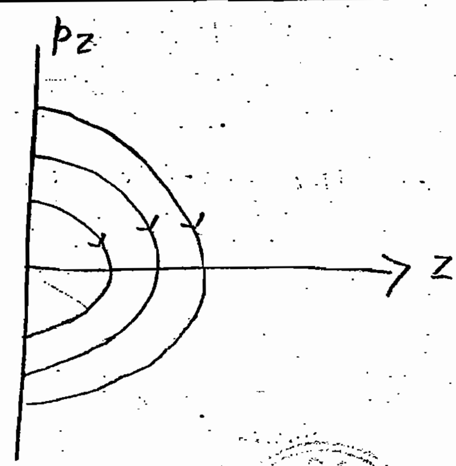
E = constt
system is conservative



→ If the particle is bouncing from inelastic sphere :

Eleastic
$P = P'$, $T = T'$
$\vec{P} \neq \vec{P'}$

$P \neq P'$ Inelastic
$\vec{P} \neq \vec{P'}$
$T = T'$

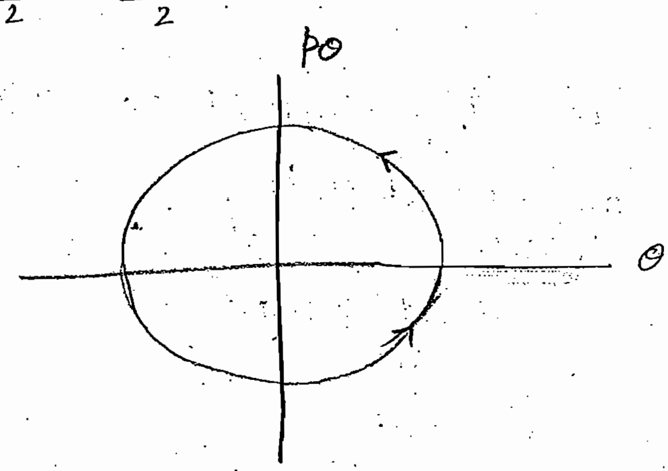


→ Phase space diagram of simple pendulum :

$$H = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$$

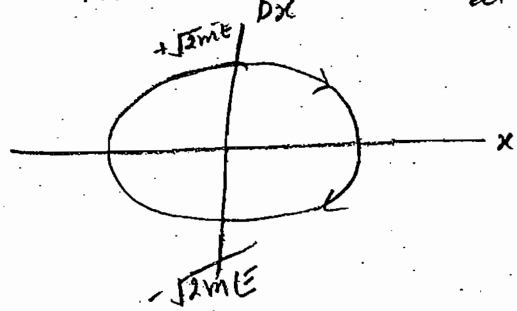
θ is small
$\cos\theta = 1 - \theta^2/2$

$$H = \frac{p_\theta^2}{2ml^2} + \frac{mgl}{2}\theta^2$$

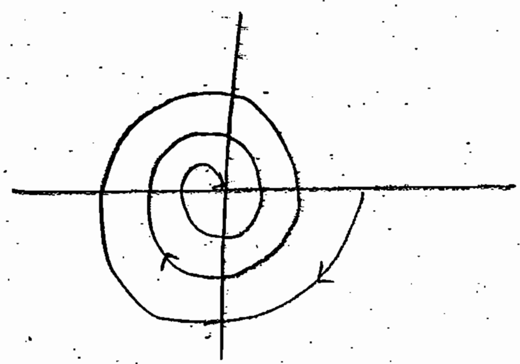


$$E = \frac{p_x^2}{2m} + \frac{1}{2}Kx^2$$

$\dot{p_x} = -Kx \Rightarrow \frac{dp_x}{dt} = -Kx$
D_x



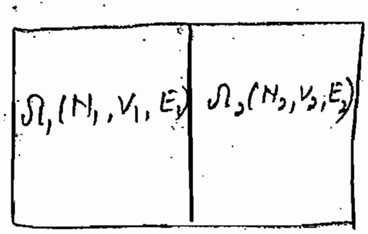
→ If bob is immersed in water then phase space diagram



* Baltzmann's Definition of Entropy →

$$S = K_B \ln \Omega$$

Consider two independent system having macrostates are described by (N_1, V_1, E_1) & (N_2, V_2, E_2) with thermodyna. probabilities $\Omega_1(N_1, V_1, E_1)$ & $\Omega_2(N_2, V_2, E_2)$ respectively



Thermodynamical probability of composition

$$\Omega = \Omega_1 \times \Omega_2 \quad \text{①}, \quad \text{En. of composition } E = E_1 + E_2 \quad \text{②}$$

If energy is allowed to exchange

$$\Omega = \Omega_1(E_1) \times \Omega(E_2)$$

Let equilibrium energy is $\bar{E}$

$$\Omega = \max$$

$$\left.\frac{\partial \Omega}{\partial E_1}\right|_{E=\bar{E}} = 0$$

$$0 = \Omega_2 \frac{\partial \Omega_1}{\partial E_1} \frac{\partial E_1}{\partial E_1}^{\,+1} + \frac{\partial \Omega_2}{\partial E_2} \frac{\partial E_2}{\partial E_1}^{\,-1} \Omega_1$$

$$\frac{1}{\Omega_1} \frac{\partial \Omega}{\partial E_1} = \frac{1}{\Omega_2} \frac{\partial \Omega_2}{\partial E_2}$$

$$\frac{1}{\Omega}\frac{\partial \Omega}{\partial E} = \text{constt} \quad -③$$

$$\frac{\partial}{\partial E}(\ln \Omega) = \text{constt} \quad -④$$

$$\beta = \frac{1}{K_B T} \qquad S \circ \text{entropy} = \frac{1}{\tau}$$

By thermodynamic

$$\left.\frac{\partial S}{\partial E}\right|_{N,V} = \frac{1}{T} \quad -④ \qquad \left[\begin{array}{l}\text{from} \\ TdS = dE + PdV - \mu dN\end{array}\right.$$

Dividing ⑤ by ④

$$\frac{\partial S}{\partial(\ln \Omega)} = \frac{1}{\beta T} = K_B$$

$$S = K_B \ln \Omega + C$$

$$C = 0 \qquad \text{at } T \to 0K$$
$$S \to 0$$

$$\boxed{S = K_B \ln \Omega}$$

$$\frac{\partial S}{\partial V} = \frac{P}{T}$$

→ If we allow volume to exchange

$$\frac{\partial}{\partial V}(\ln \Omega) = \eta = \frac{P}{K_B T} = \frac{P}{\tau}$$

If allow N to exchange

$$\frac{\partial}{\partial N}(\ln \Omega) = \gamma = -\frac{\mu}{K_B T}$$

$$\left.\frac{\partial S}{\partial E}\right|_{N,V} = \frac{1}{T}$$

$$\left.\frac{\partial S}{\partial V}\right|_{N,E} = \frac{P}{T}$$

$\sigma \to$ statistical entropy

$\tau \to$ statistical temp.

β = inverse temp.

Th. E. $\boxed{S = K_B \sigma \to S.E.}$

$$\left.\frac{\partial \sigma}{\partial E}\right|_{N,V} = \frac{1}{\tau}$$

$$\left.\frac{\partial \sigma}{\partial V}\right|_{E,N} = \frac{P}{\tau}$$

$$\left.\frac{\partial \sigma}{\partial N}\right|_{E,V} = -\frac{\mu}{\tau}$$

Prob: Consider a system of N magnetic atom with spin 1/2. At low temp. the system is ferromagnetic while at high temp. T it is paramagnetic. Neglecting all DOF except spins, find entropy of the system at.

(i) $T \to 0$ b.) $T \to \infty$

(i) $T \to 0K$

Distribution of spins

→ Almost all spins are aligned in same dir^n.

No. of ways of distribution

$$\Omega \to 1$$

$$S = K_B \ln \Omega$$

$$S \to 0$$

gmp

(ii) $T \to \infty$

System attain equilibrium.

$$\Omega = \max.$$

No. of distr. or Max. No. of microstate $= (2S+1)^N = \left(2 \times \frac{1}{2} + 1\right)^N = 2^N$

Entropy $S = K_B \ln(2^N)$

$$= \left(\ln 2^{NK_B}\right)$$

Q. Consider a system of N paramagnetic atoms each have mag. moment M, are placed in mag. field B. n atoms are allined $\parallel^e$ to B & (N−n) allined anti$\parallel^e$ to B. find

(i) Internal En. of the system

(ii) Entropy of the system

(iii) The thermodynamic temp. of the system.

(i) $E = nE_{\parallel e} + (N-n)E_{anti\parallel e}$

$$= n(-MB) + (N-n)MB$$

$$E = (N-2n)MB$$

(ii) $$\Omega = {}^{N}C_{n} = \frac{\lfloor N}{\lfloor n \; \lfloor N-n}$$

$$S = K_B \ln \Omega$$

$$= K_B \ln \left(\frac{\lfloor N}{\lfloor n \; \lfloor N-n} \right)$$

$$S = NK_B \ln N - NK_B - NK_B \ln n + nK_B - (N-n)K_B \ln (N-n) + (N-n)K_B$$

$$S = K_B \left[N \ln \left(\frac{N}{N-n} \right) - n \ln \left(\frac{n}{N-n} \right) \right]$$

(iii) $$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{\partial S}{\partial n} \cdot \frac{\partial n}{\partial E}$$

$$= \frac{\partial S}{\partial n} \left(-\frac{1}{2\mu B} \right)$$

$$= \frac{K_B}{2\mu B} \ln \left(\frac{n}{N-n} \right)$$

$$T = \frac{2\mu B}{K_B \ln \left(\frac{n}{N-n} \right)}$$

$$\tau = \frac{2\mu B}{\ln \left(\frac{n}{N-n} \right)}$$

Distribution Laws :-

Boltzmann's distribution law:

→ Applicable to classical particles (identical, distinguishable of any spin.)

Consider a system of N particles system has different energy states having energies ϵ_1, ϵ_2 ----
The no. of ways of distributions of particles such that ;

ϵ_1 energy state has n_1 particle.
ϵ_2 energy state has n_2 particles.
- -

$$= {}^{N}C_{n_1} \times {}^{N-n_1}C_{n_2} \times \text{------}$$

$$= \frac{\lfloor N}{\lfloor N_1 \; \lfloor N_2 \; \lfloor N_3}$$

$$= \frac{\lfloor N}{\prod_i n_i}$$ when state are Non-degenerate

If energy state are degenerate
i.e., ϵ_1 energy state has degeneracy g_1
ϵ_2 " " " g_2

No. of ways of distribution

$$\Omega = \frac{\lfloor N}{\prod_i n_i} \underbrace{g_1^{n_1} \times g_2^{n_2} \times \text{----}}_{\prod_i g_i^{n_i}} \quad \text{---} (1)$$

ϵ_1. En. states has two particles A, B
If deg. of this state is 3.

□ □ □

of degenerate ← g^n → No. of particle
→ 3^2 → [illegible]

Q. $N = 3$
No. of state ϵ_1, ϵ_2 — find No. of ways of distribution
$g_1 = 3 \quad g_2 = 1$
$n_1 = 2 \quad n_2 = 1$

$\longrightarrow$ At equilibrium
$\Omega = \max. = \ln \Omega$
with constraints

$N - \sum_i n_i = 0$

$E - \sum_i n_i \epsilon_i = 0$

$f(x, y, z) = \max$
$\phi_1(x, y, z) = 0$
$\phi_2(x, y, z) = 0$

$$\frac{\partial f}{\partial x_i} + \sum_K \lambda_K \frac{\partial \phi_K}{\partial x_i} = 0$$

$$\frac{\partial f}{\partial x} + \sum_K \lambda_K \frac{\partial \phi_K}{\partial x} = 0$$

$\lambda_K \rightarrow$ undetermined multiplier

$$\frac{\partial \Omega}{\partial n_i} + \alpha \frac{\partial \phi_1}{\partial n_i} + \beta \frac{\partial \phi_2}{\partial n_i} = 0$$

$$\boxed{\ln \frac{g_i}{n_i} - \alpha - \beta \epsilon_i = 0}$$

OR

$$\frac{\partial (\ln \Omega)}{\partial n_i} + \alpha \frac{\partial \phi_1}{\partial n_i} + \beta \frac{\partial \phi_2}{\partial n_i} = 0$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$

$$d\phi_1 = \frac{\partial \phi_1}{\partial x} dx + \frac{\partial \phi_2}{\partial y} dy + \cdots$$

$$d\phi_2 = \frac{\partial \phi_2}{\partial x} dx + \cdots$$

$$df + \lambda_1 d\phi_1 + \lambda_2 \phi_2 = 0$$

from eqn ①

$$\ln \Omega = \ln \lfloor N - \sum \ln n_i + n_i \sum \ln g_i$$

$n_i \ln n_i + n_i$ (S.F)

$$\frac{\partial (\ln \Omega)}{\partial n_i} = -\ln n_i - \frac{n_i}{n_i} + 1 + \ln g_i$$

$$\ln \frac{g_i}{n_i} = \alpha + \beta \epsilon_i$$

$$\frac{g_i}{n_i} = e^{\alpha + \beta \epsilon_i}$$

$$\boxed{n_i = \frac{g_i}{e^{\alpha + \beta \epsilon_i}}}$$

α, β are undetermined multiplier

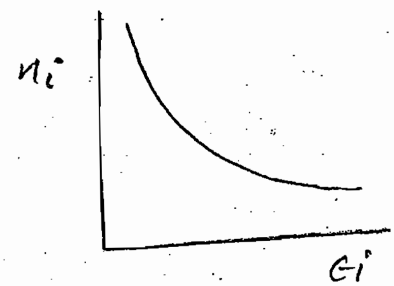
$n_i \propto g_i$

g_i = priori probability
or
Degeneracy of the ϵ_i energy state

$$\rightarrow \quad n_i = g_i e^{-\alpha} e^{-\beta\epsilon_i}$$

$$= A g_i e^{-\beta\epsilon_i}$$

$$A = e^{-\alpha} = \text{constt}$$



$$\sum n_i = N = \sum_i A g_i e^{-\beta\epsilon_i}$$

$$A = \frac{N}{\sum_i g_i e^{-\beta\epsilon_i}}$$

$$n_i = \frac{N g_i e^{-\beta\epsilon_i}}{\sum g_i e^{-\beta\epsilon_i}}$$

Probability:

$$P(\epsilon_i) = \frac{n_i}{N} = \frac{g_i e^{-\beta\epsilon_i}}{\sum_i g_i e^{-\beta\epsilon_i}}$$

$$\underset{\substack{\downarrow \\ \text{Partian fun}^n}}{Z} = \sum g_i e^{-\beta\epsilon_i}$$

→ when state are non-degenerate i.e. $g_i = 1$

$$Z = \sum_i e^{-\beta\epsilon_i} = \text{Sum of all Baltzmann's factor}$$

Prob: calculate the partition fun in case of 1 D H.O.

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \qquad \text{Q. osc}$$

$$E = \frac{p_x^2}{2m} + \frac{1}{2}Kx^2 \qquad \text{cl. osc}$$

$$Z = \sum_n g_n\, e^{-\beta E_n}$$

$$Z = \sum_n e^{-\beta E_n}$$

$$= \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}$$

$$Z = e^{-\beta\hbar\omega/2} + e^{-3\beta\hbar\omega/2} + \cdots$$

$$= \frac{a}{1-r}$$

$$= \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$$

Partition funᶯ for 1D cl. osc:

$$E = \frac{px^2}{2m} + \frac{1}{2}m\omega^2x^2$$

$$Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dp_x}{h}\, e^{-\beta E}$$

rigid rot:

$$E_J = (J^2+J)\frac{\hbar^2}{2I}$$

$$= \int_0^\infty (2J+1)\, e^{-\beta E_J}\, dJ$$

Q. If two ideal dice are rolled once, what is the probability of getting at least one 6.

$\frac{11}{36}$, $\frac{1}{36}$, $\frac{10}{36}$, $\frac{5}{36}$

Partition function:

$$Z = \sum g_i e^{-\beta E_j}$$

$$= \sum e^{-\beta E_j}$$

$$= \frac{1}{h} \int e^{-\beta E_j} \, dx \, dp_x$$

$$= \frac{1}{h^3} \iiiint\!\!\iint e^{-\beta E} \, dq_1 \, dq_2 \, dq_3 \, dp_1 \, dp_2 \, dp_3$$

$$= tr\left(e^{-\beta \hat{H}}\right)$$

1-D Q.H.O :-

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

non-degenerate case = $g_n = 1$

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}$$

1-D classical oscillator:

$$Z = \frac{1}{\beta\hbar\omega} = \frac{K_B T}{\hbar\omega}$$

Partition funn of rigid rotator:

$$Z = \frac{2 I K_B T}{\hbar^2}$$

Partition funn for a molecule of classical ideal gas -

$$E = \frac{p^2}{2m} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$\cancel{Z = \frac{1}{h^3}}$$

$$Z = \int \underbrace{g(p)}_{\text{No. of macrostate or No of non-degenrate state}} dp \, e^{-\beta E}$$

$$Z = \iiiiiint \frac{dx\,dy\,dz\,dp_x\,dp_y\,dp_z}{h^3} e^{-\beta\left(\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right)}$$

$$= \frac{V}{h^3} \int_{-\infty}^{\infty} e^{-\beta p_x^2/2m} dp_x \int_{-\infty}^{\infty} e^{-\beta p_y^2/2m} dp_y \int_{-\infty}^{\infty} \cancel{p_z^2} e^{-\beta p_z^2/2m} dp_z$$

$$Z = \frac{V}{h^3} \int_{-\infty}^{\infty} e^{-\beta p_x^2/2m} dp_x \int_{-\infty}^{\infty} e^{-\beta p_y^2/2m} dp_y \int_{-\infty}^{\infty} e^{-\beta p_z^2/2m} dp_z$$

$$Z = \frac{V}{h^3} \left(\frac{2m\pi}{\beta}\right)^{3/2}$$

$$= \frac{V}{h^3} (2\pi m k_B T)^{3/2}$$

$$= V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

$$Z = \frac{V}{\lambda^3}$$

where $\lambda = \dfrac{h}{\sqrt{2\pi m k_B T}}$

$\downarrow$

De-Broglie wavelength.

Q. Write the partition func for a particle of mass m whose pot. energy is given

$$V(x, y, z) = ax^2 + b(y^2+z^2)^{1/2}$$

Sol^n:

$$E = \frac{p^2}{2m} + V$$

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + ax^2 + b(y^2+z^2)^{1/2}$$

$$Z = \frac{1}{h^3}\iiiiiint e^{-\beta\left[\frac{1}{2m}(p_x^2+p_y^2+p_z^2)+ax^2+b(y^2+z^2)^{1/2}\right]} dx\,dy\,dz\,dp_x\,dp_y\,dp_z$$

$$= \frac{1}{h^3}\int_{-\infty}^{\infty} e^{-\beta ax^2} dx \int_{-\infty}^{\infty} e^{-\beta p_x^2/2m} dp_x$$

ask

$$= \frac{1}{h^3}\sqrt{\frac{\pi}{\beta a}}\left(\frac{2m\pi}{\beta}\right)^{3/2} 2\pi\left(\frac{1}{\beta b}\right)^2$$

Q. In a system of particle fixed in space, each particle has total q. No. J, the mag. q. no. m_j, which has $(2J+1)$ values $-J, (-J+1) \text{----} +J$.

In presence of mag. field B, the energy of a level is given by $E = -m_j B\mu_B g$

$g \to$ Lande g-factor

find partition func^n for such an atom in mag. field

Sol^n:

$$Z = \sum g_i e^{-\beta E}$$

$$= \sum_{m_j=-J}^{+J} 1\cdot e^{+\beta m_j \mu_B B g}$$

$$= e^{-\beta J\mu_B B g} + e^{\beta(-J+1)\mu_B B g} + \cdots + e^{\beta J\mu_B B g}$$

G.P. $\frac{a(1-r^n)}{1-r}$

$$= \frac{\sinh(\beta\mu_B B g(J+1))}{\sinh\left(\frac{\beta\mu_B B g}{2}\right)}$$

$$Z = e^{-\beta J \mu_B B g}\left[1 + e^{\beta\mu_B B g} + e^{2\beta\mu_B B g} + \cdots + e^{2\beta\mu_B B g}\right]$$

$$= \frac{e^{-\beta J \mu_B B g}\left[e^{\beta\mu_B B g(2J+1)} - 1\right]}{e^{\beta\mu_B B g} - 1}$$

$$= \frac{e^{\beta\mu_B B g(J+1)} - e^{-\beta J \mu_B B g}}{e^{\frac{\beta\mu_B B g}{2}}\left[e^{\beta\mu_B B g/2} - e^{-\beta\mu_B B g/2}\right]} = \frac{\sinh(\beta\mu_B g B(J+1))}{\sinh(\beta\mu_B B g/2)}$$

* for a system of N distinguishable particle the partition funⁿ.

$$Z_{system} = (Z_i)^N$$

Prob: Consider a 1-D Ising model of N spins at very low temp. where all spins are aligned $\parallel^{le}$ to each other. There will be few spins flip at temp. T With each spin flip 2J energy is added. In a configuration of N spins, r spins are flip, the energy of the system is

$$E = \underbrace{-NJ}_{\text{all spin are } \parallel^{le}} + \underbrace{2rJ}_{r \text{ flip spin energy}}$$

& No. of configuration is ${}^N C_r$, r varies from 0 to N The partition funⁿ of the system is.

Solⁿ:

$$Z = \sum_r g_r e^{-\beta E_r}$$

$$= \sum_{r=0}^{N} {}^N C_r \, e^{-\beta(-NJ + 2rJ)}$$

$$= (2\cosh\beta J)$$

Boltzmann's definition of probability:

$$P(\epsilon_i) = \frac{n_i}{N} = \frac{g_i e^{-\beta\epsilon_i}}{\sum_i g_i e^{-\beta\epsilon_i}} \to z$$

$N_1 = \frac{N_0 e^{-1}}{z}$

$N_1 = \frac{N_0}{e}$

Prob: In a system, there are three energy states, having energies, $0, K_B T$ & $2K_B T$. The total energy of the system is $1000 K_B T$. find the no. of partic in different state & average En. of each particle.

Soln: N_2 ———— $2K_B T$ find N_0, N_1, N_2

N_1 ———— $K_B T$ $N = ?$

N_0 ———— 0 $\bar{E} = ?$

$N = N_0 + N_1 + N_2$, $N = N_0 + \frac{N_0}{e} + \frac{N_0}{e^2}$ decay formula

$N_0 = \frac{N}{z}$

$N_1 = \frac{N e^{-1}}{z}$

ask $\frac{N_1(\epsilon = K_B T)}{N_0} = \frac{N e^{-\beta K_B T}/z}{N e^{-0}/z}$

$\frac{N_u}{N_l} = \frac{g_u e^{-(E_u - E_l)\beta}}{g_l}$

$$E = 1000 K_B T = N_0 \epsilon_0 + N_1 \epsilon_1 + N_2 \epsilon_2$$

$$= N_0 \times 0 + N_1 K_B T + N_2 \, 2K_B T$$

$$1000 = N_1 + 2N_2$$

$$= \frac{N_0}{e} + \frac{2N_0}{e^2} \Rightarrow N_0 = \frac{1000 e^2}{(e^2+2)}$$

$$\bar{E} = \frac{E}{N} = \frac{K_B T(2+e)}{(1+e+e^2)}$$

Prob: A system of 5 identical but distinguishable particles having energy 3ϵ. The single particle state are available at energies, $0, \epsilon, 2\epsilon, 3\epsilon$.

i) find the ave no. of particles in each energy state

ii) sketch $n(\epsilon)$ with energy.

Sol^n:

En. of the system	Energy of the state 0	ϵ	2ϵ	3ϵ	No. of ways of distribution	Probabilities
3ϵ	4	0	0	1	$\frac{\lfloor 5}{\lfloor 4 \lfloor 0 \lfloor 0 \lfloor 1} = 5$	$\frac{5}{35}$
	3	1	1	0	20	$\frac{20}{35}$
	2	3	0	0	10	$\frac{10}{35}$

$20+10+5 = 35$

(1) $n(0) = n_1 P_1 + n_2 P_2 + \cdots$

$= 4 \times \frac{5}{35} + 3 \times \frac{20}{35} + 2 \times \frac{10}{35}$

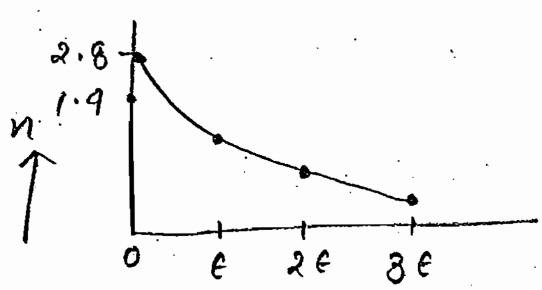
$= 2.8$

$n(\epsilon) = 1.4$

$n(2\epsilon) = 0.571$

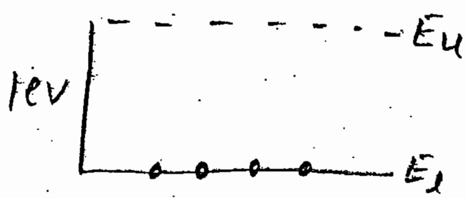
$n(3\epsilon) = .14$

ii)



Prob: The energy required to create a lattice vacancy in a crystal is 1eV. find out the ratio of vacancies at tem 1200K & 300K. a.) e^{-30} b.) e^{30} ✓ c.) e^{-10} d.) e^{10}

Solⁿ:



$$\boxed{n(T) \propto e^{-\Delta E/K_B T}}$$

$$n(1200) \propto e^{-1eV/K_B \times 1200}$$

$$n(300) \propto e^{-1eV/K_B \times 300}$$

ask

$$\frac{n(1200)}{n(300)} = \frac{e^{-1eV/K_B \times 1200}}{e^{-1eV/K_B \times 300}} = \frac{e^{-1200}}{e^{-300}}$$

Q. The states with energy diffⁿ 4.83×10^{-21} J occurs wi relative probability e^2. find temp. (Ans 175K)

ask Q. in terms of Z

A system can take only 3 different energy states of energy $E_1 = 0$, $E_2 = 1.38 \times 10^{-21}$ J, $E_3 = 2.76 \times 10^{-21}$ J

These 3 states occur in 2, 5, 4 different ways respectiv find the probability that at 100K, the system may b

1.) In one of 4- state of energy E_3 $P(E_3) = ?$ $\frac{4e^{-\beta E}}{Z}$

2.) In the G.S. $P(E_1) = \frac{2e^{-\beta 0}}{Z}$

(1.) $\frac{P(E_1)}{P(E_2)} = e^2 = e^{\Delta E/K_B T}$

Q. A system has two normal mode of vibrations with frequenci ~~of vibrations~~ ~~with~~ ω_1 & $\omega_2 (= 2\omega_1)$. Wt is the pot. en. of the system. Also find the probability that at temp. T, The system has energy less than $4\hbar\omega_1$.

Sol^n:

$$V = ax^2 + 4ay^2$$

OR

$$V = \frac{1}{2} m\omega_1^2 (x^2 + 4y^2)$$

$$E_{n_x, n_y} = \left(n_x + \frac{1}{2}\right)\hbar\omega_1 + \left(n_y + \frac{1}{2}\right)\hbar\omega_2 \rightarrow 2\omega_1$$

$$= \left(n_x + 2n_y + \frac{3}{2}\right)\hbar\omega_1$$

$$P(< 4\hbar\omega_1) = \frac{1 \cdot e^{-\beta\frac{3}{2}\hbar\omega_1} + 1 \cdot e^{-\beta\frac{5}{2}\hbar\omega_1} + 2e^{-\beta\frac{7}{2}\hbar\omega_1}}{Z}$$

$$= \frac{e^{-\frac{3}{2}\beta\hbar\omega_1}}{Z}\left(1 + e^{-\beta\hbar\omega_1} + 2e^{-2\beta\hbar\omega_1}\right)$$

$$= \frac{x^{3/2}}{Z}(1 + x + 2x^2)$$

$$x = e^{-\beta\hbar\omega_1}$$

	n_x	n_y
$\frac{3}{2}\hbar\omega_1$	0	0
$\frac{5}{2}\hbar\omega_1$	1	0
	0	1
$\frac{7}{2}\hbar\omega_1$	2	0

degeneracy

Ensembles: Gibbs

→ A collection of large no. of macroscopically identical but essentially independent system, is called Ensemble

OR

→ A system is defined as "coll. of large no. of particles"

Macroscopic identical ÷ System having the same macro-scopic quantities like, T, V, N, E, μ, P etc.

→ If we have a coll. of large no. of independent system having macroscopic properties

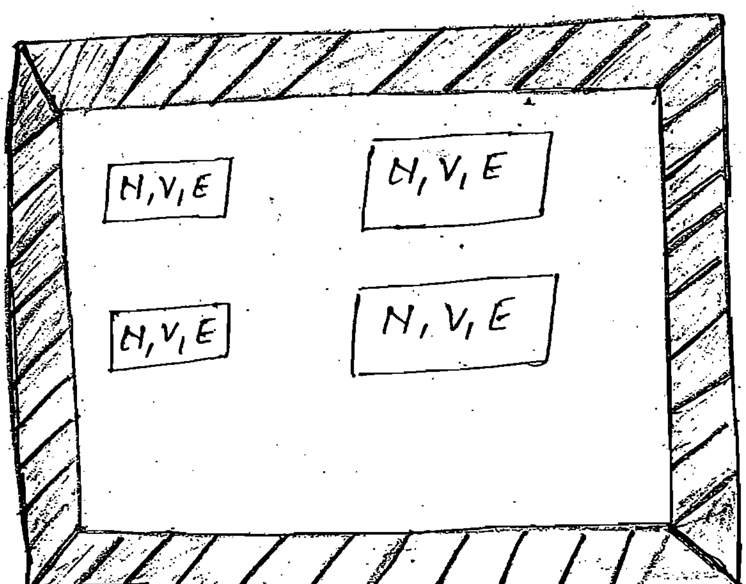
1.) N, V, E same → microcanonical ense. (more restriction)

2.) T, V, N same → canonical En. (less restriction)

3.) T, V, μ same → Grand canonical ensemble (No restriction)

$\boxed{\text{System}}$ → Involving in time → We take the time avereged quantity.

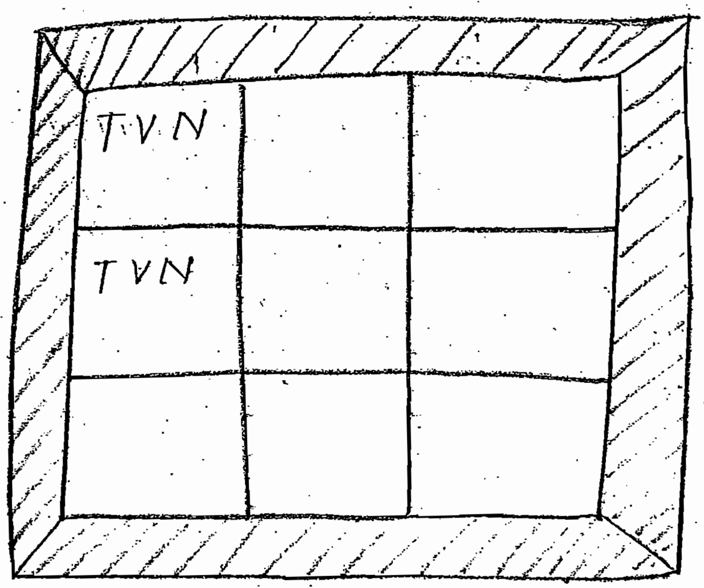
< Time ave. behaviour > = < ensemble avereged behaviour >



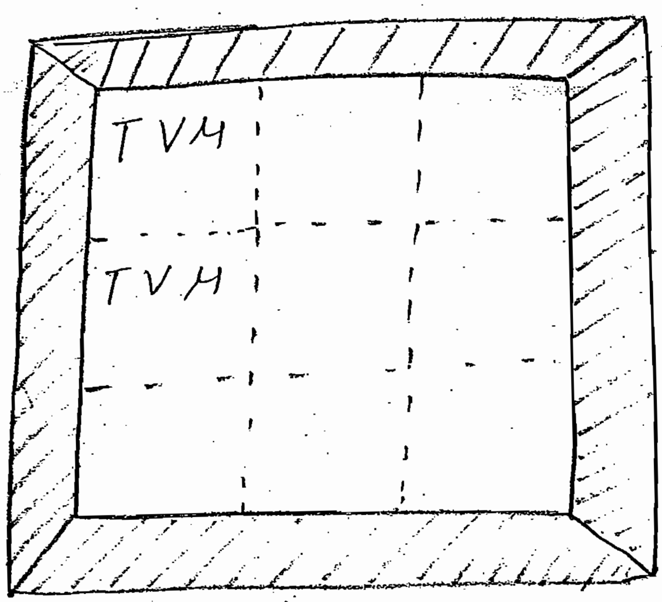
MCE

* Ensemble average of any property would be very close to time average value of that property over a single system.

Properties	MCE
Contact with the environment	No contact ie isolated. ie, No exchange of Energy & matter (particle). ie, walls should be rigid, impenetrable & not conducting (V=0) $dN = 0$



Canonical Ensembles



G. C. E.

Properties	M.C.E.	C.E.	G.C.E.
Contact with the environment	No contact ie isolated.	System in thermal contact with heat reservoir. Walls should be conducting rigid & impentrable	System can exchange both energy & matter.
fluctuations	None	fluctuation in energy only	fluctuation in energy & no. of particle both.
Density function $\rho(E)$	$\rho(E) = \frac{N}{V} = \text{const}$	$\rho(E) = Ae^{-E/K_BT}$	$\rho(E) = e^{(\Omega + nH - E)/K_BT}$
Partition function (Z)	$Z = \Delta T$	$Z = \int e^{-\beta E} d\Gamma$	$Z = \sum_n e^{Hn/T} \cdot Z_n$
wt is the use of Ensembles			

→ Internal energy of an ideal mono atomic gas → $\frac{3}{2} NKT$

Average properties of different quantities →

$\overline{E}$, $\overline{E^2}$, $\overline{P}$, $\overline{\mu}$, ΔE etc

→ $$\overline{X} = \frac{n_1 x_1 + n_2 x_2 + \cdots}{n_1 + n_2 + \cdots}$$

$$= \frac{n_1}{N} x_1 + \frac{n_2}{N} x_2 + \cdots$$

$$= P_1 x_1 + P_2 x_2 + \cdots$$

$$\overline{X} = \sum_i P_i x_i$$

→ $$\overline{E} = \sum_i P_i \epsilon_i$$

$$= \sum_i \frac{e^{-\beta \epsilon_i}}{Z} \epsilon_i$$

$$= \sum_i \frac{e^{-\beta \epsilon_i} \epsilon_i}{\sum e^{-\beta \epsilon_i}}$$

$$\boxed{\overline{E} = -\frac{\partial}{\partial \beta} \ln Z = -\frac{1}{Z} \frac{\partial Z}{\partial \beta}}$$

OR

$$\overline{E} = K_B T^2 \frac{\partial}{\partial T} \ln Z$$, where $\beta = \frac{1}{K_B T}$

→ $$\overline{E^2} = \sum_i P_i \epsilon_i^2$$

$$= \sum_i \frac{e^{-\beta \epsilon_i} \epsilon_i^2}{\sum e^{-\beta \epsilon_i}}$$

$$\boxed{\overline{E^2} = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}}$$

$$\rightarrow \quad (\overline{\Delta E})^2 = (\overline{E^2}) - (\overline{E})^2$$

$$= \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} - \left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right)^2$$

$$\boxed{(\Delta E)^2 = \frac{\partial^2}{\partial \beta^2} \ln Z}$$

OR

$$(\Delta E)^2 = -\frac{\partial}{\partial \beta}\left(-\frac{\partial}{\partial \beta}\ln Z\right)$$

$$= -\frac{\partial}{\partial \beta}(\overline{E})$$

$$= K_B T^2 \frac{\partial \overline{E}}{\partial t}$$

$$= K_B T^2 \left(\frac{3}{2} N K_B\right)$$

for ideal gas

$$(\Delta E)^2 = K_B T^2 \, C_v$$

$$\boxed{\Delta E = K_B T \sqrt{\frac{3N}{2}}}$$

$$\frac{\overline{\Delta E}}{\overline{E}} = \frac{K_B T \left(\frac{3}{2}N\right)^{1/2}}{\frac{3}{2} N K_B T}$$

$$\frac{\Delta \overline{E}}{\overline{E}} = \left(\frac{3}{2}N\right)^{-1/2} = \left(\frac{2}{3N}\right)^{1/2}$$

Relative fluctuation in energy

$$\frac{\Delta E}{E} \propto N^{-1/2} \text{ or } \frac{1}{\sqrt{N}} \qquad 10^{22} \text{ atoms}$$

$$\propto 10^{-11}$$

$\rightarrow 0$, In ideal gas, fluctuations in en. is approx to zero.

Average Pressure:

$$\bar{p} = \sum_i p_i \, P_i$$

$P \rightarrow$ Probability

$p_i \rightarrow$ Pressure

$$\bar{p} = \sum_i \frac{e^{-\beta \epsilon_i}}{\sum_i e^{-\beta \epsilon_i}} \left(-\frac{\partial \epsilon_i}{\partial V}\right)_N \quad \text{—(1)}$$

We KT

$$T\,dS = dE + p\,dV - \mu\,dN$$

At equilibrium $T\,dS \rightarrow 0$ OR Entropy fixed state

$$p = \left(\frac{-\partial \epsilon}{\partial V}\right)_N$$

from eq^n (1)

$$\bar{p} = \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z \Big|_{T,N}$$

β as const

OR

$$= \frac{1}{\beta} \frac{1}{Z} \frac{\partial Z}{\partial V}\Big|_{T,N}$$

$$\bar{p} = \frac{N}{\beta V} = \frac{N k_B T}{V}$$

$$\bar{p} = \frac{2}{3} \frac{\bar{E}}{V}$$

Average chemical potential →

$$\bar{\mu} = \sum_i P_i \mu_i$$

$$= \frac{\sum_i e^{-\beta \epsilon_i} \left(\frac{\partial \epsilon_i}{\partial N}\right)_V}{Z}$$

$$\boxed{\bar{\mu} = -\frac{1}{\beta} \frac{\partial}{\partial N} \ln Z \Big|_{V,T}}$$

Helmholtz free energy (F) and Entropy in terms of Z →

$$F = U - TS$$

$$dF = dU - TdS - SdT$$

We KIT: $TdS = dU + PdV$

UGC POINT ACADEMY

Then

$$dF = -PdV - SdT$$

→ If $dV = 0$

$dT = 0$

$dF = 0$

when $F \Rightarrow$ min^m energy, then system is in equil^m.

$$d[\ln Z(T,V,N)] = \frac{\partial}{\partial \beta} \ln Z \Big|_{V,N} d\beta + \frac{\partial}{\partial V} \ln Z \Big|_{T,N} dV + \frac{\partial}{\partial N} \ln Z \Big|_{T,V} dN$$

$$d[\ln Z(T,V,N)] = -E d\beta + \beta P dV - \beta \mu \, dN \quad \text{—(1)}$$

$$d(\beta E) = \beta dE + E d\beta \quad \text{—(2)}$$

Add (1) + (2)

$$d[\ln Z + \beta \bar{E}] = \beta[dE + PdV - \mu dN]$$
$$= \beta T dS$$
$$= \frac{1}{K_B} dS$$

$$dS = d[K_B \ln Z + K_B \beta \bar{E}]$$

$$\boxed{S = K_B \ln Z + \frac{\bar{E}}{T}}$$

Imp

$$\boxed{F = E - TS = -K_B T \ln Z}$$

$\rightarrow$ $$F = E - TS$$
$$dF = dE - TdS - SdT$$
$$= dE - (dE + PdV - \mu dN) - SdT$$
$$\boxed{dF = -PdV + \mu dN - SdT}$$

4rm this rel^n, we can find that

$$P = -\frac{\partial F}{\partial V}\Big|_{N,T}$$

$$\mu = \frac{\partial F}{\partial N}\Big|_{V,T}$$

Entropy $$S = -\frac{\partial F}{\partial T}\Big|_{N,V}$$

$\rightarrow$ $$E = F + TS$$
$$E = F - T\frac{\partial F}{\partial T}\Big|_{N,V}$$ Helmholtz free energy.

Prob. 1-D.H.O.

$$E = \frac{p^2}{2m} + \frac{1}{2}Kx^2$$

The partition funⁿ $Z = \frac{K_B T}{\hbar\omega}$

find ave thermal En.

Solⁿ:

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\ln Z$$

$$= -\frac{\partial}{\partial\beta}\ln\left(\frac{1}{\beta\hbar\omega}\right)$$

$$= -\beta\hbar\omega \times \frac{1}{\hbar\omega} \times -\frac{1}{\beta^2}$$

$$= \frac{1}{\beta}$$

$$= K_B T$$

$$\langle E \rangle = \frac{1}{2}K_B T + \frac{1}{2}K_B T$$

Q. The hamiltonian of a particle of classical ideal gas is $H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$

wht is the ave. energy of the molecule. Ans $\frac{3}{2}K_B T$

Solⁿ: We KIT

$$Z = \frac{V}{\lambda^3} = \frac{V}{h^3}(2\pi m K_B T)^{3/2}$$

Q. A system can have three energy level $E = 0, \pm\epsilon$. level $E = 0$ is doubly degenerate while other are non-degenerate. Find the ave. energy of the system.

a.) $-\epsilon \tanh \beta\epsilon/2$ b.) $-\epsilon \tanh \beta\epsilon$ c.) $\epsilon \cosh \beta\epsilon$

Sol^n:

$$\text{Partition fun}^n \; Z = 2e^{-0} + e^{-\beta\epsilon} + e^{\beta\epsilon}$$

$$= 2 + e^{\beta\epsilon} + e^{-\beta\epsilon}$$

$$= \left(e^{\beta\epsilon/2} + e^{-\beta\epsilon/2}\right)^2$$

$$= \left(2\cosh \beta\epsilon/2\right)^2$$

$$\bar{E} = \frac{-\partial}{\partial\beta} \ln Z$$

$$= \frac{-\partial}{\partial\beta} \ln\left(2\cosh \beta\epsilon/2\right)^2$$

$$= \frac{-2 \cdot 1}{2\cosh\frac{\beta\epsilon}{2}} \cdot \sinh \beta\epsilon/2 \;\; \epsilon/2$$

$$= -\tanh \frac{\beta\epsilon}{2} \cdot \epsilon$$

$$\bar{E} = -\epsilon \tanh \beta\epsilon/2 \quad \text{Ans}$$

Q. Av. thermal En. 1-D q. osc.

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}$$

$$\bar{E} = \frac{-\partial}{\partial\beta} \ln Z \Big|_{V,N}$$

$$= \cancel{\frac{1}{2}\hbar\omega} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

$$\bar{E} = \frac{h\nu}{e^{h\nu/K_BT} - 1}$$

$$\boxed{* \; P = E/c, \; E = h\nu \quad \omega = 2\pi\nu = 2\pi\frac{c}{\lambda} = Kc}$$

$$\omega = \frac{E}{\hbar}, \quad K = P/\hbar$$

$$\frac{\frac{h}{2\pi} 2\pi\nu}{e^{\beta\frac{h}{2\pi}\cdot\nu 2\pi} - 1}$$

Q. A gas of N particle enclosed in a volume V at temp. T. The logarithm of partition fun^n is given by

$$\ln Z = N \ln[(V-bN)(K_B T)^{3/2}]$$

find (i) Av. Internal En. $\frac{3}{2} N K_B T$

(ii) Eq^n of state of the gas.

Sol^n: (ii) $p = \frac{1}{\beta}\frac{\partial}{\partial V} \ln Z \Big|_{N,T}$

$$p = \frac{N K_B T}{V - bN}$$

$$p(V-bN) = N K_B T \quad \text{(semi Vander-wall eq}^n\text{)}$$

(i) $\bar{E} = -\frac{\partial}{\partial \beta} \ln Z$

$$= -\frac{\partial}{\partial \beta}\left[N \ln (V-bN)(K_B T)^{3/2}\right]$$

Q. The partition fun^n for a system of N particles is

$$V^N\left(\frac{2\pi m K_B T}{h^2}\right)^{3N/2}$$

find

(i) $\langle E \rangle$ $\frac{3}{2}NK_BT$

(ii) Eq^n of state $PV = NK_BT$

Q. A gas of N non-interacting particle is in thermal equilibrium at temp. T. Each particle can be any of the possible non-degenerate states of energy $0, 2\epsilon$ & 4ϵ. The average energy per particle of the gas, when $\beta\epsilon$ is 2ϵ, 3ϵ, $2\epsilon/3$, ϵ.

Q. The partition fun^n of 1-D H.O. is

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} \qquad \langle n \rangle = \frac{1}{e^{\hbar\omega}-1}$$

find the average no. of oscillations or quanta.

Q.2

$$Z = 1+e^{-2\beta\epsilon}+e^{-4\beta\epsilon}$$

$$\langle E \rangle = \frac{-\partial}{\partial\beta}\ln Z = \frac{0+2\epsilon\, e^{-2\beta\epsilon}+4\beta e^{-4\beta\epsilon}}{1+e^{-2\beta\epsilon}+e^{-4\beta\epsilon}}$$

$\beta\epsilon \ll 1$

$$\frac{6\epsilon}{3} = 2\epsilon$$

Ave En. of System.

(3) Av. Osc $\langle n \rangle = \sum_{n=0}^{\infty} \frac{n e^{-\beta E_n}}{Z}$ $\qquad E_n = (n+\frac{1}{2})\hbar\omega$

$n = 0, 1, 2$

$$= \frac{0\, e^{-\beta\hbar\omega/2} + 1.e^{-\frac{3}{2}\hbar\omega} + 2e^{-5/2\,\hbar\omega} + \dots}{Z}$$

$$\langle n \rangle = \frac{e^{-\frac{3}{2}\hbar\omega}\left(1+2e^{-\beta\hbar\omega}+3e^{-2\beta\hbar\omega}+\dots\right)}{Z}$$

$$1+2x+3x^2+\cdots = (1-x)^{-2}$$

$$1+x+x^2+\cdots = (1-x)^{-1}$$

$$\langle n \rangle = \frac{e^{-\frac{3}{2}\hbar\omega}}{Z} \frac{1}{(1-e^{-\beta\hbar\omega})^2}$$

$$= \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$\langle n \rangle = \frac{1}{e^{\beta\hbar\omega}-1}$$

$$\langle n \rangle = \frac{1}{e^{\hbar\omega/k_BT}-1}$$

Ave. No. of photon or phonons or quanta.

→ Oscillator emits & absorbed energy in the form of quanta.

2 ———— $\frac{5}{2}\hbar\omega$

1 ———— $\frac{3}{2}\hbar\omega$

$n=0$ ———— $\frac{\hbar\omega}{2}$

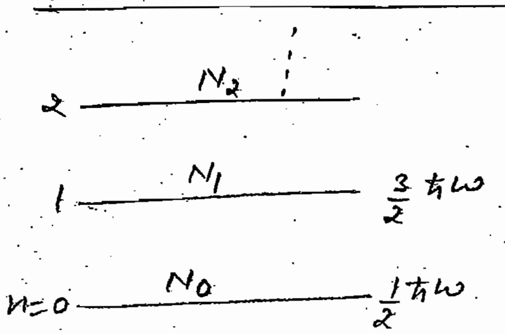
$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$E_n = nh\nu$$

$$E_n = 0, h\nu, 2h\nu \cdots$$

Av. No. of qu. $= 0, 1, 2, 3 \cdots$

Total No. of Oscillator :



$$N = N_0 + N_1 + N_2 + \cdots$$

$$= N_0 + N_0 e^{-\hbar\omega\beta} + N_0 e^{-2\beta\hbar\omega} + \cdots$$

$$N = \frac{N_0}{1-e^{-\beta\hbar\omega}}$$

Total No. of Osc.

$$E = N_0 E_0 + N_1 E_1 + \cdots$$

$$= N_0\left(\frac{1}{2}\hbar\omega\right) + N_0 e^{-\beta\hbar\omega}\left(\frac{3}{2}\hbar\omega\right) + \cdots$$

$$= \frac{1}{2} N_0 \hbar\omega \left[1 + 3e^{-\beta\hbar\omega} + 5e^{-2\beta\hbar\omega} + \cdots\right]$$

$$= \frac{1}{2} \frac{N_0 \hbar\omega}{(1-e^{-\beta\hbar\omega})^3}$$

wrong

$$Av.\,En = \frac{N\hbar\omega}{e^{\beta\hbar\omega}-1} = N\langle E \rangle =$$

→ If G is given

$$G = NK_BT(1-\ln Z)$$

$$G = E + PV - TS$$

Q. $Z = \dfrac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$

Calculate Av. no. of oscillators.

Solⁿ:

Av. En. of osc $= \dfrac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}$

Av. no of Osc $= \dfrac{1}{e^{\hbar\beta\omega}-1}$

Q. for a non-relativistic fermi gas

$$E \propto \frac{N^{5/3}}{V^{2/3}}$$

The pressure of gas is.

$\frac{1}{3}\frac{E}{V}$, $\frac{2}{3}\frac{E}{V}$, $\frac{5}{2}\frac{E}{V}$

Solⁿ:

$$\bar{p} = \left(-\frac{\partial E}{\partial V}\right)_N$$

$$\bar{p} = \frac{2}{3}\frac{E}{V}$$

Q. for a certain system the energy of each state is given by

$$E_s = f_s(T) - Nk_BT\ln\left(\frac{V}{V_0}\right)$$

$V_0 =$ constt

(i) Write the partition funⁿ of the system.

(ii) Ave Pressure of the system

(iii) $\bar{\mu}$

$$Z = \sum_s e^{-\beta E_s} \rightarrow \text{value}$$

$$\bar{P} = \frac{Nk_BT}{V}, \quad \bar{\mu} = -\frac{1}{\beta}\frac{\partial}{\partial N}\ln Z\Big|_{T,V}$$

Sol^n:

$$Z = \sum_S e^{-\beta E_S} \qquad = \left(\frac{V}{V_0}\right)^{N} \sum_S e^{-\beta E_S(T)}$$

$$= \sum_S e^{-}$$

$$\mu = -\frac{1}{\beta}\frac{\partial}{\partial N}\left[N \ln\left(\frac{V}{V_0}\right)\right]$$

$$= -\frac{1}{\beta} \ln\left(\frac{V}{V_0}\right)$$

$$= -K_B T \ln\left(\frac{V}{V_0}\right)$$

$$\bar{P} = \frac{N K_B T}{V}$$

Q. for a certain system

$$Z(T, V, N) = \left(\frac{\beta}{\beta_0}\right)^{-\frac{3N}{2}} \left(\frac{V}{V_0}\right)^{N}$$

find $\bar{E}$, F, $\bar{P}$, $\bar{\mu}$, S, eq^n of state, & $\Delta E/\bar{E}$

Sol^n:

Q. The partition funⁿ for a certain system is given as

$$Z = \left(\frac{\beta}{\beta_0}\right)^{-N} e^{-N\left(\frac{V}{V_0}\right) - 10}, \quad \beta_0, V_0 \text{ are constt}$$

find $\bar{E}$, F, $\bar{P}$, $\bar{\mu}$, S, eqⁿ of states. $\Delta E/\bar{E}$.

Average thermal energy of 1-D classical osc →

HO

Some important formula:

$$\cancel{H} = E + PV$$

$$= NK_BT^2\left[\frac{\partial}{\partial T}(\ln Z)\right]_V + NK_BT$$

$$G = H - TS$$

$$= NK_BT^2\left[\frac{\partial}{\partial T}(\ln Z)\right]_V + NK_BT - NK_B \ln Z - E$$

$$G = K_BT(1 - \ln Z)$$

Specific heat

$$C_V = T \frac{\partial S}{\partial T}\Big|_V$$

$$= -T\left(\frac{\partial^2 f}{\partial T^2}\right)_V$$

Isothermal compressibility

$$K_T = \frac{1}{B} = -V\left(\frac{\partial P}{\partial V}\right)_T$$

$$= V\left[\frac{\partial^2 f}{\partial V^2}\right]_T$$

$$= \left[V \frac{\partial^2}{\partial V^2}(-NK_BT \ln Z)_T\right]$$

Q. The partition fun for an interacting gas (real) is assumed to:

$$Z = \left(\frac{V-Nb}{N}\right)^{N} \left(\frac{m k_B T}{2\pi \hbar^2}\right)^{3N/2} e^{N^2 a^2 / V k_B T}$$

a, b are constt

Show that the pressure of the gas

$$P = \frac{N k_B T}{(V-Nb)} - \frac{N^2 a^2}{V^2} \Rightarrow \left(P + \frac{N^2 a^2}{V^2}\right)(V - Nb) = N k_B T$$

Sol^n:

Q. A system containing N non-interacting localised particles of spin 1/2 & mag. moment μ, each kept in constt. external m.f. H is normal equilibrium at temp. T. The magnetisation of the system is $N\mu \coth\left(\frac{\mu H}{K_B T}\right)$, $N\mu \tanh\left(\frac{\mu H}{K_B T}\right)$ ✓

Sol^n:

$$M = -\frac{\partial F}{\partial H}\bigg|_{T,V,N}$$

$$F = -NK_BT \ln Z$$

$$= -NK_BT \ln\left[2\cosh\left(\frac{\mu H}{K_BT}\right)\right]$$

$$M = N\mu \tanh\left(\frac{\mu H}{K_BT}\right)$$

$$Z = \Sigma e^{-\beta \epsilon_i}$$

$$= e^{-\beta(\mu H)} + e^{\beta \mu H}$$

$$Z = 2\cosh \beta \mu H$$

Q. An ensemble of N, three level system with energies $-\epsilon_0$, 0, $+\epsilon_0$ in thermal equilibrium at temp. T. If $\beta\epsilon_0 = 2$. wt is the prob. of finding the system in the level $\epsilon = 0$

Probability $P = \frac{g_i e^{-\beta \epsilon_i}}{\Sigma_i g_i e^{-\beta\epsilon_i}}$

$\frac{\cosh 2}{2}$, $(\cosh 2)^{-1}$, $(2\cosh 2)^{-1}$, $(1+2\cosh 2)^{-1}$

Q. $E = \frac{bS^3}{VN}$, $T = ?$ $\quad \frac{bS^2}{VN}$, $\frac{3bS^2}{VN}$ etc

Sol^n: $Z = \sum_i e^{-\beta\epsilon_i} = 1 + e^{-\beta\epsilon_0} + e^{\beta\epsilon_0}$

$$P = \frac{g_i e^{-\beta\epsilon_i}}{\sum_i g_i e^{-\beta\epsilon_i}} \Rightarrow \frac{e^{-\beta\epsilon_i}}{\sum_i e^{-\beta\epsilon_i}} = \frac{1}{1+e^{-2}+e^{2}}$$

$$= \frac{1}{1+\left(\frac{e^{-2}+e^{2}}{2}\right)\cdot 2} = \frac{1}{1+2\cosh 2}$$

$$= (1+2\cosh 2)^{-1}$$

Imp $\boxed{T = \left.\frac{\partial E}{\partial S}\right|_{V, N}}$

$\cancel{dE = TdS}$

$TdS = dE + PdV - \mu dN$

$dF = -SdT - PdV + \mu dN$

Average thermal energy of 1-D classical osc. $\Rightarrow$

1-D Cl. H.O.

$$E = \frac{P_x^2}{2m} + \frac{1}{2}Kx^2$$

$$\langle E\rangle = \frac{1}{2m}\langle P_x^2\rangle + \frac{1}{2}K\langle x^2\rangle$$

$$= \frac{1}{2\cancel{m}}\cancel{m}K_BT + \frac{1}{2}K_BT$$

$$= \frac{1}{2}K_BT + \frac{1}{2}K_BT$$

$$\boxed{\langle E\rangle = K_BT}$$

$\rightarrow$ Average thermal energy corresponding to each quadratic term of momentum & position is $\frac{1}{2}K_BT$. (Equipartition theorem)

$$\langle E\rangle = \frac{\int H(q,p)e^{-\beta H(q,p)}\,dq\,dp}{\int e^{-\beta H(q,p)}\,dq\,dp}$$

If $\qquad \bar{X} = \Sigma x_i P_i$

$$= \frac{\Sigma x e^{-\beta H(q,p)}}{\Sigma e^{-\beta H(q,p)}}$$

$$\langle E\rangle = \frac{\iint\left(\frac{P_x^2}{2m} + \frac{1}{2}Kx^2\right)e^{-\beta\left(\frac{P_x^2}{2m}+\frac{1}{2}Kx^2\right)}dx\,dp_x}{\iint e^{-\beta\left(\frac{P_x^2}{2m}+\frac{1}{2}Kx^2\right)}dx\,dp_x}$$

Unit

$dq\,dp \Rightarrow$ Joule $\cdot$ sec

Average thermal energy of 3-D q. osc. →

$$\boxed{\langle E \rangle = 3 K_B T}$$

Q. A system has N particle having p.E. $ax^2 + by^2$. The Av. thermal En. of the system is

$= 2 N K_B T$

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + ax^2 + by^2$$

$$\langle E \rangle = 2 K_B T$$

* Non-interacting cl. ideal gas having N particles, Av. thermal En.

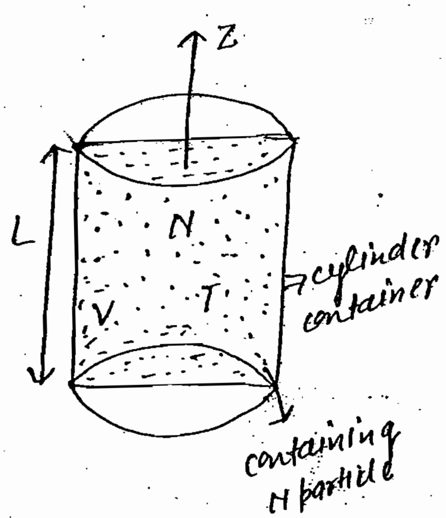
$$\boxed{\langle E \rangle = \frac{3}{2} N K_B T}$$

Q. A system of N cl. non-interacting particles each of mass m at temp. T and is confined by the external pot. $V = \frac{1}{2} A r^2$, where A = constt. in 3-D

find the Av. Int. En of the system.

$\frac{3}{2} N K_B T$, $\frac{5}{2} N K_B T$, $3 N K_B T$ ✓, $\frac{1}{2} N K_B$

Q.



$L << R$

$mgL >> K_B T$

$\langle E \rangle$ of the system

$\frac{3}{2} N K_B T$, $\frac{5}{2} N K_B T$ ✓

Solⁿ: Hamiltonian of the particle

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

$\frac{3}{2} K_B T$

$$\langle E \rangle = \frac{\iiiint\!\!\iint H e^{-\beta H}\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z}{\iiiint\!\!\iint e^{-\beta H}\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z}$$

Q. A particle confined in a region $x \geq 0$ by a pot. which linear increases with x as $V(x) = u_0 x$, u_0 constt. Find mean position of the particle at temp. T

$$\frac{K_B T}{u_0}, \quad \frac{K_B T}{2u_0} \text{ etc} \qquad \int_0^\infty t^n e^{-at}\,dt = \frac{\Gamma{n+1}}{a^{n+1}}$$

Sol^n:

$$\langle x \rangle = \frac{\iint x\, e^{-\beta H(q,p)}\, dq\, dp}{\iint e^{-\beta H(q,p)}\, dq\, dp}$$

$$= \frac{\int_0^\infty x\, e^{-\beta u_0 x}\, dx}{\int_0^\infty e^{-\beta u_0 x}\, dx}$$

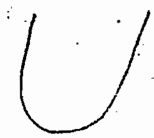
Thermal Expansion of Solids:

Av. value of thermal expansion Cl. H.O. treatment

$$\langle x\rangle = \frac{\iint x\, e^{-\beta H}\, dx\, dP_x}{\iint e^{-\beta H}\, dx\, dP_x}$$

$$H = \frac{p_x^2}{2m} + ax^2 \quad \underline{H.O}$$

Parabolic pot.

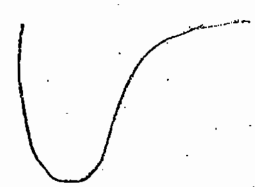


$$\langle x\rangle = \frac{\int_{-\infty}^{\infty} x\, e^{-\beta a x^2}\, dx}{\int_{-\infty}^{\infty} e^{-\beta a x^2}\, dx} = 0$$

Cl. An H.O. treatment:

$$V(x) = ax^2 \underbrace{- bx^3 - cx^4}$$

Anharmonic terms
b, c are smaller than a



$(1+bx^3+cx^4)$

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty} x\, e^{-\beta(ax^2 - bx^3 - cx^4)}\, dx}{\int_{-\infty}^{\infty} e}$$

$$\boxed{\langle x\rangle = \frac{3b\,(K_BT)}{4a^2}}$$

$\langle E\rangle$

Coeff. of thermal expansion:
OR
Linear expansion coeff.

$$\alpha = \frac{d\langle x\rangle}{dt} = \frac{3}{4a^2} b K_B = \text{const}$$

but expansion
$\alpha \to 0$ at $T \to 0$

Quantum anharmonic osc. treatment:

$$\langle x\rangle = \frac{3}{4a^2} b \langle E\rangle$$

$$\langle x\rangle = \frac{3}{4a^2} b \left(\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right)$$

$$\alpha = \frac{d\langle x\rangle}{dT}$$

$$= \frac{3}{4a^2} b \left[\frac{(\hbar\omega)^2}{K_B T^2} \frac{e^{\hbar\omega/K_B T}}{(e^{\hbar\omega/K_B T}-1)^2}\right]$$

AT $T \to 0$
$\alpha \to 0$

Q. Consider a system of two Ising spins S_1 & S_2 taking values ± 1 with internal energy $\epsilon = -J S_1 S_2$ When it is in thermal equilibrium at temp. T. The ave. en. varies as $C/K_B T$ for large T.
Then C varies as $\cancel{-J^2}$, J^2, $-J$, $+J$

Soln:

$$\langle E\rangle = \frac{-\partial}{\partial\beta} \ln Z$$

$$Z = \sum g e^{-\beta\epsilon} = 2e^{-\beta J} + 2e^{\beta J}$$

	S_1	S_2
$\epsilon = +J$	+1	-1
	-1	+1
$-J$	+1	+1

$$\langle E\rangle = \frac{-[-2Je^{-\beta J} + 2Je^{\beta J}]}{2e^{-\beta J} + 2e^{\beta J}}$$

$\tanh x \to x$
$x \to 0$

$x \to 0$
$\cos \to 1$
$\sin \to x$

$$= \frac{J[e^{-J/K_BT} + e^{J/K_BT}]}{e^{-J/K_BT} + e^{J/K_BT}}$$

$$\langle E\rangle = \frac{J\left[1 - \frac{J}{K_BT} - 1 - \frac{J}{K_BT}\right]}{2}$$

$$\langle E\rangle = -\frac{J^2}{K_BT}$$

Q. The partition fu^n of a system is $Z = e^{\alpha T^3 V}$, find $\mu, \bar{P}, S, U$

α is constt. $\bar{\mu} = 0$

$\bar{P} = \alpha K_B T^4$, $S = 4\alpha K_B T^3 V$, $U = 3\alpha K_B T^4 V$

Q. The free en. of a gas having N particle, volume V & temp. T is given as

$$F = -NK_BT \ln\left[\frac{a_0 V (K_BT)^{5/2}}{N}\right]$$

find Internal energy of the gas.

$$\ln z = \ln\left[\frac{a_0 V}{N\beta^{5/2}}\right]$$

$$E = -\frac{\partial}{\partial\beta}\ln z$$

$$= \frac{5}{2}\frac{N}{\beta}$$

Q. The entropy

$$S = NK\ln\left[\frac{V}{h^3}\left(\frac{4\pi mE}{3N}\right)^{3/2}\right] + \frac{3}{2}NK_B \text{ in MCE}$$

Sacur-Tetrode eq^n

obtain
(i) I.E. $\to \frac{3}{2}NK_BT$
(ii) C_V
(iii) $P \Rightarrow \frac{P}{T} = \left.\frac{\partial S}{\partial V}\right|_{N,T}$
(iv) $\sigma = \frac{S}{K_B}$
statistical error

for system of N-particle $F = -k_B T \ln(Z^N)$

$$\ln Z = \ln Z_1^N$$

$$= \ln\left[\frac{g_0 V}{N \beta^{5/2}}\right]^N$$

Q. The helmholtz energy F in case of canonical ensembl is given as

$$F = -N\tau \ln\left(\frac{eV}{N\lambda^2}\right)$$

$$\lambda = \frac{h}{\sqrt{2\pi m\tau}}, \quad \tau = k_B T$$

find σ & U (I.E.)

$$\sigma = -\frac{\partial F}{\partial \tau} = N \ln\left(\frac{eV}{n\lambda^2}\right)$$

$$u = -\tau^2 \frac{\partial}{\partial \tau}\left(\frac{F}{\tau}\right)_{V,N} = \frac{3}{2} N\tau$$

The entropy of mixing of two ideal gases →
(Gibbs paradox)

for a molecule of ideal gas

$$E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$Z_1 = \frac{V}{\lambda^3} = V\left(\frac{2\pi m K_B T}{h^2}\right)^{3/2}$$

for N molecules

$$Z_{system} = (Z_1)^N$$

$$= V^N\left(\frac{2\pi m K_B T}{h^2}\right)^{3N/2}$$

$$= V^N\left(\frac{2\pi m}{\beta h^2}\right)^{3N/2}$$

$$\bar{E} = -\frac{\partial}{\partial \beta}\ln Z$$

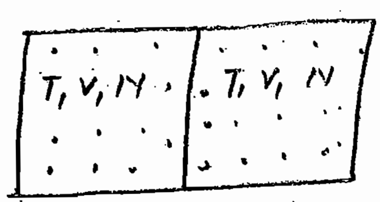
$$\bar{E} = \frac{3}{2} N K_B T$$

$$\bar{P} = \frac{2}{3}\frac{E}{V} = \frac{N K_B T}{V}$$

$$S = K_B \ln Z + \frac{\langle E \rangle}{T}$$

$$S = K \ln V^N + \frac{3}{2} N K \ln\left(\frac{2\pi m K_B T}{h^2}\right) + \frac{3}{2} N K_B$$

→ Now we consider the mixing of two ideal gas



ideal gas molecule is non-interacting so entropy remain same after & before.

Before mixing the entropy of the system

$$S = S_1 + S_2 = 2S_1$$

$$S = 2NK_B \ln V + 3NK_B \ln\left(\frac{2\pi m K_B T}{h^2}\right) + 3NK_B$$

After mixing the entropy of the system

$N \to 2N, \quad V \to 2V, \quad T \to K$

$$S' = 2NK_B \ln 2V + 3NK_B \ln\left(\frac{2\pi m K_B T}{h^2}\right) + 3NK_B$$

Change in entropy

$Z = (Z_1)$

$$\Delta S = S' - S$$

$$\boxed{\Delta S = 2NK_B \ln 2}$$

→ Mixing of two same ideal gases is reversible proces

$\Rightarrow \Delta S = 0$

Two results are contradictor particle is distinguishable.

This contrary result is Klas Gibbs paradox.

* How to treat the problem arise in the result
→ modifying the partition function

$$Z = \frac{(Z_1)^N}{\underline{|N}} \quad \text{(particle is indistinguishable)}$$

$$Z = \frac{V^N}{\underline{|N}}\left(\frac{2\pi m K_B T}{h^2}\right)^{3N/2}$$

Entropy before mixing

$$S = 2NK_B \ln V - 2K_B \ln \underline{|n} + 3NK_B \ln\left(\frac{2\pi m K_B T}{h^2}\right) + 3N$$

$\parallel$
$2S_1$

After mixing

$$S' = S$$

$$S' = NK_B \ln V - K_B \ln |\underline{N} + \frac{3}{2} NK_B \ln\left(\frac{2\pi m K_B T}{h^2}\right) + \frac{3}{2}NK_B$$

$V \to 2V, \quad N \to 2N, \quad T \to K$

$$S' = 2NK_B \ln 2V - K_B \ln |\underline{2N} + 3NK_B \ln\left(\frac{2\pi m K_B T}{h^2}\right) + 3NK_B$$

$$\Delta S = S' - S$$

$$= 2NK_B \ln 2V - 2NK_B \ln V - K_B \ln |\underline{2N} + 2K_B \ln |\underline{N}$$

$$= 2NK_B \ln 2 - [2NK_B \ln \overset{\ln 2 + \ln N}{2N} - 2\cancel{N}K_B] + 2N\cancel{K_B} \ln N - 2\cancel{N}K_B$$

$$\Delta S = 0$$

→ Thus the Gibbs paradox is removed by taking into account the indistinguishable of particles. The modifying & correct expression of $Z(N, V, T)_{\text{idealgas}}$

$$Z(N, V, T) = \frac{V^N}{|\underline{N}} \left(\frac{2\pi m K_B T}{h^2}\right)^{\frac{3N}{2}}$$

$$E = -\frac{\partial}{\partial \beta} \ln Z = \frac{3}{2} NK_B T$$

$$P = \frac{2}{3}\left(\frac{E}{V}\right)$$

$$S = NK_B \ln\left[\left(\frac{V}{N}\right)\left(\frac{4\pi m E}{3N}\right)^{3/2}\right] + \frac{5}{2} NK_B$$

Which is called Sackur - Tetrode eqn.

Maxwell's distribution law of molecular speeds (c) →

Used Boltzmann's distribution law.

$dN(c)\,dc \Rightarrow$ No. of molecu in dc range

$$\overset{dc}{\underset{c \qquad\qquad c+dc}{\longleftrightarrow}}$$

$dN(c)\,dc$

$p \qquad p+dp$

$\epsilon \qquad \epsilon+d\epsilon$

$k \qquad k+dk$

If gas contain N molecules → momentum

$$\frac{dN}{N} = \underset{\text{Probab.}}{dP} = \frac{\underset{\text{No. of microstate or degeneracy}}{g(p)}\,dp\,e^{-\beta\epsilon}}{Z}$$

$$= A\,g(p)\,dp\,e^{-\beta\epsilon}$$

$$= A\,\frac{V\,4\pi p^2}{h^3}\,dp\,e^{-\beta\epsilon}$$

$$dP = \frac{AVp^2 e^{-\beta(p^2/2m)}}{2\pi^2\hbar^3}\,dp$$

$$\int dP = \frac{AV}{2\pi^2\hbar^3}\int_0^\infty e^{-\beta(p^2/2m)}\,p^2\,dp$$

$$1 = \frac{AV}{2\pi^2\hbar^3}\sqrt{\frac{2\pi}{(\beta/4m)^3}}$$

$$\boxed{A = \frac{(2\pi)^{3/2}\,\hbar^3}{V}\left(\frac{\beta}{4m}\right)^{3/2}}$$

$$dP = \frac{4}{\sqrt{\pi}} \left(\frac{2}{2mK_BT}\right)^{3/2} p^2 \, e^{-\beta p^2/2m} \, dp$$

$p = mc$

$$\boxed{dP = \frac{dN}{N} = 4\pi \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-mc^2/2K_BT} \, c^2 dc}$$

$c \to 0 \text{ to } \infty$

$$\boxed{dN = 4\pi N \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-mc^2/2K_BT} \, c^2 dc}$$

<u>Average speed :</u> $(\bar{c})$

$$\bar{c} = \frac{\int c \, dP}{\int dP}$$

$$\int dP = 1$$

$$\bar{c} = \int c \, dP$$

$$= \int_0^\infty c \frac{dN}{N} = \frac{\int c \, dN}{\int dN}$$

$$\boxed{\underset{\text{Average speed}}{\bar{c}} = \sqrt{\frac{8K_BT}{\pi m}}}$$

$$\overline{c^2} = \frac{\int c^2 dN}{\int dN}$$

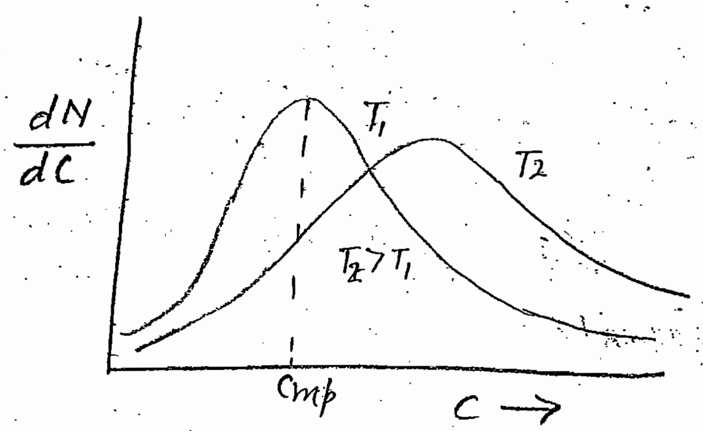
$$= \frac{3K_BT}{m}$$

$$\boxed{\left(\overline{C^2}\right)^{1/2} = \sqrt{\frac{3K_BT}{m}} = C_{rms}\ \text{root mean square}}$$

$C_{rms} > \bar{C}_{Ave} > C_{mp}$ (most probable)

root mean square

$\sqrt{3} : \sqrt{\frac{8}{\pi}} : \sqrt{2}$



$A_1 = A_2$ (area)

$\int \frac{dN}{dc} \cdot dc = N$

Breadth × height

Most probable speed (C_{mp}):

followed by max. no. of molecules

at $C = C_{mp}$

$$\frac{d}{dc}\left(\frac{dN}{dc}\right) = 0$$

$$\rightarrow \frac{dN}{dc} = 4\pi N \left(\frac{m}{2\pi K_B T}\right)^{3/2} c^2 e^{-mc^2/2K_BT}$$

peak depend upon temp.

Maxwell's distribution law of velocity distribution $\Rightarrow$

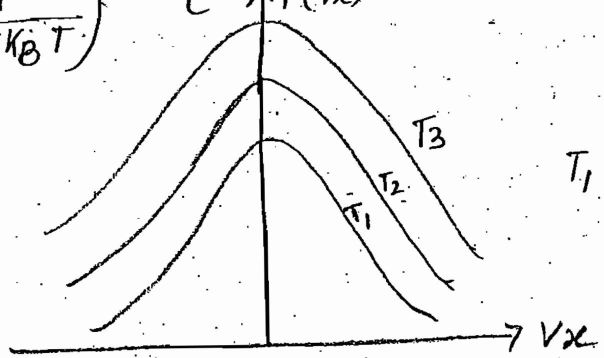
$$\underset{V_x}{+}\overset{dN}{\text{———}}\underset{V_x+dV_x}{+}$$

$$dP = \frac{dN(V_x)}{N} = \left(\frac{m}{2\pi K_B T}\right)^{1/2} e^{-mV_x^2/2K_BT}\, dV_x$$

$$dP(v) = \frac{dN(v)}{N} = \frac{dN(V_x)\, dN(V_y)\, dN(V_z)}{N}$$

$$= \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-mv^2/2K_BT}\, \frac{dV_x\, dV_y\, dV_z}{dv}$$

$$\frac{dP}{dV_x} = \left(\frac{m}{2\pi K_B T}\right)^{1/2} e^{-mV_x^2/2K_BT}$$



$T_1 > T_2 > T_3$

i) $\dfrac{dP(V_x)}{dV_x}$ = max. at $V_x = 0$

ii) Probability distribution curve is symmetric
$P(-V_x) = P(V_x)$

Most Probable velocity

$\Rightarrow \dfrac{d}{dV_x}\left(\dfrac{dP}{dV_x}\right) = 0$ wrong

$V_x = 0$

OR $\dfrac{d}{dV_x}[P(V_x)] = 0$ right

Max. value of

$$\cancel{dP(V_x)\Big|_{max}}\ \text{wrong} = \cancel{dP(V_x)\Big|_{V_x=0}} = \cancel{\left(\frac{m}{2\pi K_B T}\right)^{1/2}}$$

oR

$$P(V_x)\Big|_{max} = P(V_x)\Big|_{V_x=0} = \left(\frac{m}{2\pi K_B T}\right)^{1/2}$$

iv) <u>Average velocity:</u>

$$\langle V_x \rangle = \int_{-\infty}^{\infty} V_x\, P(V_x)\, dV_x$$

$$= 0$$

v) $\langle V_x^2 \rangle$

$$= \int_{-\infty}^{\infty} V_x^2\, P(V_x)\, dV_x$$

$$= \frac{K_B T}{m}$$

vi)

$$\langle v^2 \rangle = \langle V_x^2 \rangle + \langle V_y^2 \rangle + \langle V_z^2 \rangle$$

$$= \frac{K_B T}{m} + \frac{K_B T}{m} + \frac{K_B T}{m}$$

$$V_{rms} = \sqrt{\frac{3K_B T}{m}}$$

vii) Velocity at which $P(V_x)$ falls $1/e$ time of max. value =

$$V_x = \sqrt{\frac{2K_B T}{m}}$$

viii) $\overline{v_x v_y} = \iint_{-\infty}^{\infty} v_x v_y \, P(v_x v_y) \, dv_x \, dv_y$

$= 0$

ix) $\overline{v_x^2 v_y v_z} = 0$

x.) $\overline{(\alpha + \beta v_x)^2} = \alpha^2 + \beta^2 \frac{K_B T}{m}$

xi) $\overline{(\alpha v_x - \beta v_y)^2} = (\alpha^2 + \beta^2) \frac{K_B T}{m}$

xii) $\overline{(\alpha v_x + \beta v_y - v_z)^2} = (\alpha^2 + \beta^2 + 1) \frac{K_B T}{m}$

Quantum Statistical Mechanics :

→ Applicable when the particles are identical & indistingu

Fermi-Dirac Statistics

Particle have spin

$S = \frac{1}{2}, \frac{3}{2}$ ----- (half odd integral spin)

Ex: e^-, p, n etc. Ω^-, Hole

These particles are K/as fermions.

→ fermions follow the pauli exclusion principle.

→ Condensation property nt exist.

Bose-Einstein st.

Particle have spin

$S = 0, 1, 2$ ..... (integral)

Ex: ${}_2He^4$, ${}_1H^1$ atom, ${}_3Li^7$ etc [redacted] α, positron, ${}_1H^2$

These particles are K/as Bosons

→ Bose-Eins condensation is possible.

→ They donot follow P.E.P.,

→ No. of distribution is greater than fermions.

→ In case of Boson ${}_{11}Na^{23}$

In case of atom $= n + p + e$

$= A - Z + Z + Z$

$= A + Z \rightarrow$ even → Boson

Nucleus

: $A - Z + Z = A \rightarrow$ even

Ex: ${}_{11}Na^{23}$, ${}_{11}Na^{24}$

Bosons   fermions

→

a   b

q. states

$\psi_{ab}^{MB}(12) = \phi_a(1)\,\phi_b(2)$

$$\psi_{ab}^{F}(1,2) = \frac{1}{\sqrt{2}}\left[\phi_a(1)\,\phi_b(2) - \phi_a(2)\,\phi_b(1)\right]$$

F-D sts wave funⁿ is antisymmetric.

$$\boxed{\psi_a^{F}(1,2) = 0 \quad , \text{ when } a \equiv b}$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} \phi_a(1) & \phi_a(2) \\ \phi_b(1) & \phi_b(2) \end{vmatrix}$$

Slater Determinent.

for Bosons:

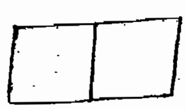
$$\psi_{ab}^{B}(1,2) = \frac{1}{\sqrt{2}}\left[\phi_a(1)\,\phi_b(2) + \phi_a(2)\,\phi_b(1)\right]$$

→ wave funⁿ is symmetric.

$a \equiv b$

$$\underset{a\equiv b}{\psi_{ab}^{B}(1,2)} = \sqrt{2}\,\phi_a(1)\,\phi_a(2)$$

Distribution of two particles in two degenerate cells →



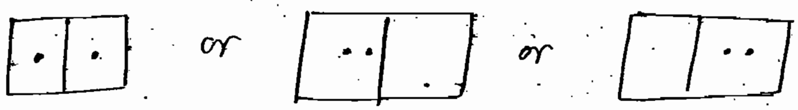
case-1) M-B/classical :

$$\boxed{\Omega = g_i^{\,n_i}}$$

A	B	B	A	AB			BA

2.) <u>Bosons</u> :

No. of ways of distribution $\boxed{\Omega = {}^{n_i+g_i-1}C_{n_i}}$



$$\Omega_{B-E} = {}^{2+2-1}C_2 = {}^{3}C_2 = 3$$

3.) <u>Fermions</u> :

$$\boxed{\Omega_{F-D} = {}^{g_i}C_{n_i}}$$

$\Longrightarrow$

No. of ways of distribution of N particles in different cells 1, 2 ---- i ---- having degeneracies g_1, g_2 ---- etc.

(i) <u>When particle are classical</u> :

$$\boxed{\Omega_{M-B} = \prod_i \frac{\lfloor N}{\lfloor n_i} (g_i)^{n_i}}$$

UGC POINT ACADEMY

2.) <u>Bosons</u> :

$$\boxed{\Omega_B = \prod_i {}^{n_i+g_i-1}C_{n_i}}$$

3.) <u>Fermions</u> :

$$\boxed{\Omega_{F-D} = \prod_i \left({}^{g_i}C_{n_i}\right)}$$

Imp → ratio of probilities of finding the two particles in same state to that probability of finding particles in different states.

$$\boxed{r_{MB} : r_{BE} : r_{FD} = 1 : 2 : 0}$$

Q. Partition funⁿ for a system of two particles each of which can occupy any one of energy level 0 & ϵ.

Solⁿ:

(i) If particle are classical:

Distri.		E_i
0	ϵ	
AB		0
	AB	ϵ
A	B	ϵ
B	A	ϵ

UGC POINT ACADEMY

$$Z = \sum_i g_i e^{-\beta \epsilon_i}$$

$$= g_1 e^{-\beta\epsilon_1} + g_2 e^{-\beta\epsilon_2} + g_3 e^{-\beta\epsilon_3}$$

$$= 1 \cdot e^{-0} + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon}$$

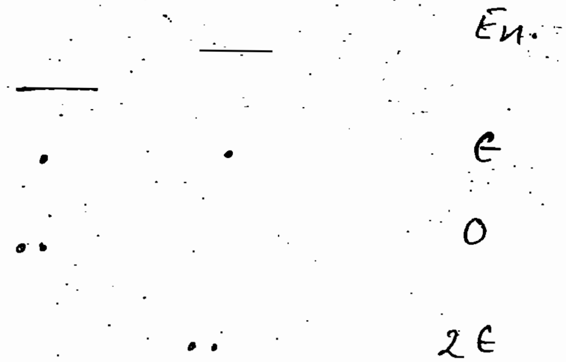
$$\boxed{Z = (1+e^{-\beta\epsilon})^2}$$

(ii) fermions:

0	ϵ	E_i
•	•	ϵ

$$Z = e^{-\beta\epsilon}$$

iii) <u>Bosons :</u>



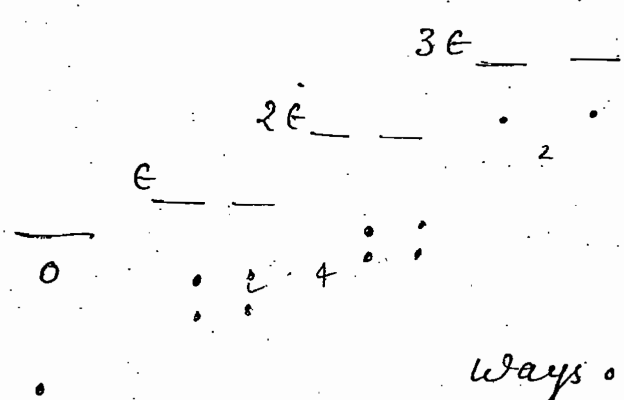
$$Z = e^{-\beta\epsilon} + e^{-0} + e^{-2\beta\epsilon}$$

$$\boxed{Z = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

Q. Two particles are to be distributed in different cells having energies $0, \epsilon, 2\epsilon$ ---.
The gr state is non-degenerate, while the excit states are doubly degenerate.
Find the ways of distribution such that the En of the system is 3ϵ if particles are :

<u>Sol^n ;</u>

(i) <u>Fermions :</u>



ways of distribution $= 6$

$$g_iC_{n_i} = {}^4C_2 = \frac{\lfloor 4}{\lfloor 2 \lfloor 2} = \frac{\overset{2}{\cancel{4}} \times 3 \times \cancel{2}}{\cancel{\lfloor 2}\,\cancel{\lfloor 2}}$$

$$= 6$$

ii) Bosons:

ask

No. of ways = 6

$^{n_i+g_i-1}C_{n_i}$ $\quad$ $^{2+4-1}C_2$

$^5C_2$

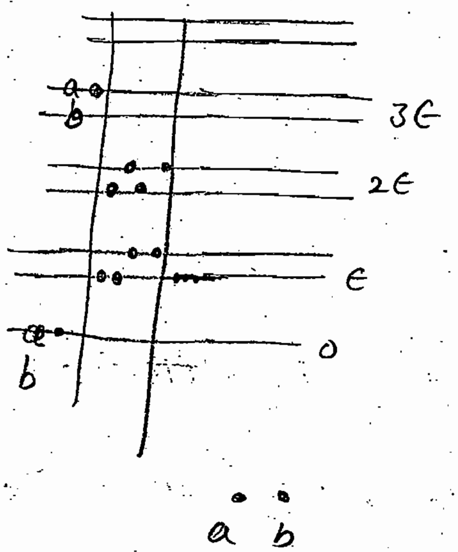
$\frac{\lfloor 5}{\lfloor 2 \lfloor 3} = \frac{5 \times 4^2 \times \lfloor 3}{\lfloor 2 \lfloor 3} = 10$

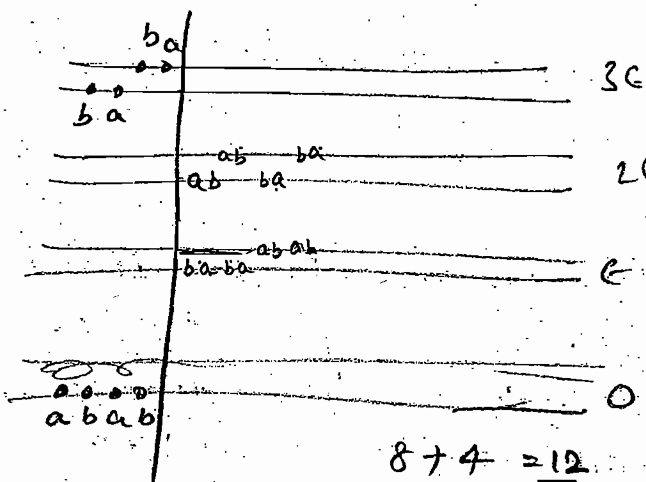
iii) Classical:

$g_i^{n_i}$

No. of ways = 12

$3^2 = 9$





8 + 4 = 12

Fermi- Dirac statistics:

Distribution funⁿ for fermions $f(\epsilon) \rightarrow$

We KIT, for M-B st.

$$f(\epsilon_i) = \frac{n_i}{g_i} = \frac{1}{e^{\alpha+\beta\epsilon_i}} = \frac{1}{e^{(\epsilon_i-\mu)/K_BT}}$$

In F-D st.

$$\Omega_{F-D} = {}^{g_i}C_{n_i}$$

$$= \frac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i}$$

At equil^m

$$\delta(\log \Omega_{F-D}) = 0$$

$$\left.\begin{aligned} N - \sum_i n_i &= 0 \\ E - \sum_i n_i\epsilon_i &= 0 \end{aligned}\right\} \text{constraints eq}^n\text{s}$$

$$f\langle\epsilon_i\rangle = \frac{n_i}{g_i} = \langle n_i\rangle$$

$$= \frac{1}{e^{\beta(\epsilon_i-\mu)}+1}$$

2nd method:

Partition funⁿ

$$Z = \sum_i e^{-\beta(\epsilon_i-\mu)}$$

$$Z = \sum_{n_1, n_2 \cdots} e^{-\beta[n_1\epsilon_1 + n_2\epsilon_2 + \cdots] - \mu(n_1 + n_2 + \cdots)}$$

or

$$Z = \sum_{n_1=0}^{1} e^{-n_1\beta(\epsilon_1 - \mu)} \sum_{n_2} e^{-n_2\beta(\epsilon_2 - \mu)} \cdots$$

or

$$Z = \left(1 + e^{-\beta(\epsilon_1 - \mu)}\right) \times \left(1 + e^{-\beta(\epsilon_2 - \mu)}\right)$$

$$Z = \prod_i \left(1 + e^{-\beta(\epsilon_i - \mu)}\right)$$

$$\log_e Z = \sum_i \log\left(1 + e^{-\beta(\epsilon_i - \mu)}\right)$$

$$\sum_i \langle n_i \rangle = N = +\frac{1}{\beta}\frac{\partial}{\partial \mu}(\log_e Z) = \frac{1}{\beta}\sum_i \frac{\beta e^{-\beta(\epsilon_i - \mu)}}{(1 + e^{-\beta(\epsilon_i - \mu)})}$$

$$TdS = dE + PdV - \mu dN + Nd\mu$$
$$N = -\frac{\partial E}{\partial \mu}\Big| = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \mu}\Big|_{T,V}$$

$$\mu = -\frac{1}{\beta}\frac{\partial}{\partial N}\ln Z\Big|_{T,V}$$

$$\boxed{\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}}$$

Ave. no. of particle in ith cells = Occupation no.

<u>Occupation probability:</u>

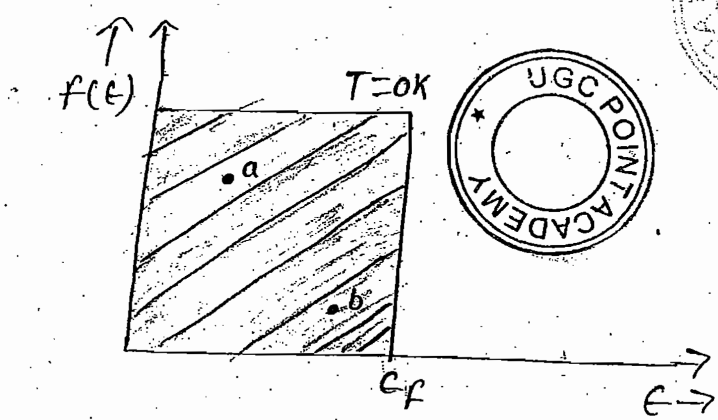
$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}$$

$$\boxed{f(\epsilon) = \frac{1}{e^{(\epsilon-\epsilon_f)/K_BT}+1}}$$

$\beta = \frac{1}{K_BT}$ $\qquad \mu \sim \epsilon_f$

<u>at $T = 0K$</u>

$$f(\epsilon) = 1 \qquad \epsilon < \epsilon_f$$
$$= 0 \qquad \epsilon > \epsilon_f$$



<u>Expression for</u>
<u>$\epsilon_f(T=0K)$ for fermions or fermi gas at 0K →</u>

No. of states in En. range
$\epsilon \rightarrow \epsilon + d\epsilon$
$p \rightarrow p + dp$

$$g(p)\,dp = \frac{V\;4\pi p^2\,dp}{h^3}$$

$$g(\epsilon)\,d\epsilon = \frac{V\;4\sqrt{2}\,\pi\,m^{3/2}\,E^{1/2}\,dE}{h^3}$$

<u>No. of e⁻ (fermions)</u>

$$dn(\epsilon) = \underbrace{g(\epsilon)\,d\epsilon \times \overbrace{f(\epsilon)}^{\rightarrow 1 \text{ at } 0^\circ K}}_{\text{No. of filled states}} \times g_s \rightarrow (2s+1)\ \text{fermions}$$

Total no. of fermions

$$N = \int dN(\epsilon) = \int dn(\epsilon)$$

$$= (2S+1)V \frac{4\sqrt{2}\,\pi m^{3/2}}{h^3} \int_0^{E_f(0)} \underset{\substack{\downarrow \\ 1 \text{ If } T=0K}}{f(\epsilon)} E^{1/2}\, dE$$

$$N = \frac{(2S+1)\, 4\sqrt{2}\,\pi m^{3/2}}{h^3} \frac{E_{f(0)}^{3/2}}{3/2}$$

$$\boxed{E_f(0) = \frac{h^2}{2m}\left[\frac{3N}{4\pi V g_s}\right]^{2/3}}$$

$$E_f(0) \propto \left(\frac{N}{V}\right)^{2/3}$$

$$\boxed{E_f(0) \propto n^{2/3}}$$

$n \rightarrow$ fermion density

Average energy of fermion in Fermi-Dirac gas. →

$$\bar{\epsilon} = \frac{\int_0^{E_f(0)} \epsilon\, dn}{\int_0^{E_f(0)} dn}$$

$$= \frac{\int_0^{E_f(0)} \epsilon\ \epsilon^{1/2}\, dn}{\int_0^{E_f(0)} \epsilon^{1/2}\, dn}$$

$$\boxed{\bar{\epsilon} = \frac{3}{5} E_f(0)}$$

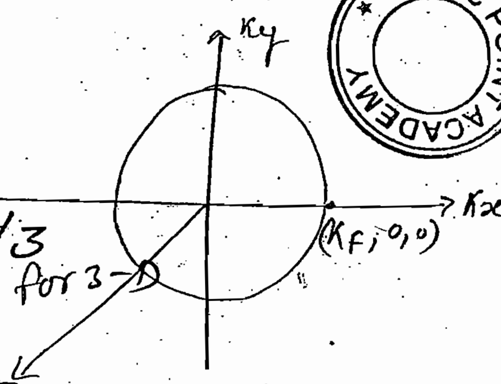
Ave. En. of gas:

$$\overline{E}_{gas} = N\overline{E} = \frac{3}{5} N E_f(0)$$

At 0 K →

$$E_f(0) = \frac{p_f^2}{2m} = \frac{\hbar^2 K_f^2}{2m}$$ $K_f \to$ fermi wave vect

$$= \frac{\hbar^2}{2m}(K_x^2 + K_y^2 + K_z^2)$$



$K_F \propto n^{1/3}$

$P_F \propto n^{1/3}$

$v_f \propto n^{1/3}$ (velocity)

$\lambda_f \propto n^{-1/3}$

$T_F \propto n^{2/3}$

fermi temp → energy

$$K_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} \text{ for 3-D}$$

$$K_F = \left(\frac{2\pi N}{A}\right)^{1/2} \to 2D$$

fermi surface at 0K → spherical

$$K_F = \left(\frac{N\pi}{2a}\right) \to 1\text{-D}$$

Pressure of the gas:

$$= \frac{1}{3} g_s \left(\frac{\overline{E}_{gas}}{V}\right)$$

$$= \frac{1}{3}(2S+1) \frac{\frac{3}{5} N E_f(0)}{V}$$

$$= \frac{1}{5}(2S+1)\left(\frac{N}{V}\right) E_f(0) \quad \text{—①}$$

$$PV = \frac{1}{3} g_s \overline{E}$$

$$PV = \frac{2}{3} \overline{E}$$

$$\bar{E} = a\left(\frac{N}{V}\right)^{5/3} \cdot \frac{1}{V^{2/3}} = a\,\frac{N^{5/3}}{V^{2/3}}$$

$$P = -\left.\frac{\partial E}{\partial V}\right|_N = \frac{2}{3}a\,(N)^{5/3}\,V^{-5/3}$$

$$PV = \frac{2}{3}\bar{E} = \frac{1}{3}g_s\bar{E}$$

from (1)

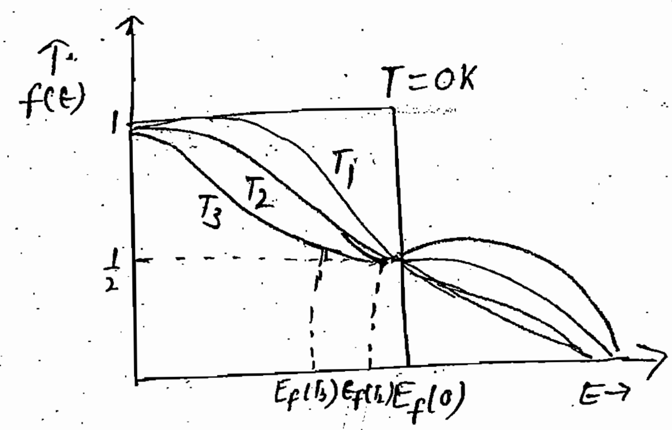
$$\boxed{P = \frac{1}{3}\left(\frac{N}{V}\right)^{5/3}\left(\frac{6\pi^2}{2s+1}\right)^{2/3}\frac{\hbar^2}{m}}$$

$$\boxed{P \propto N^{5/3}}$$

At 0K

$$P_{MB} = 0$$

$$\boxed{P_{FD} > P_{MB}}$$



$$f(E) = \frac{1}{e^{(E-E_f)/K_BT}+1}$$

$$T \neq 0K$$

$$E = E_f$$

$$\boxed{f(E) = \frac{1}{2}}$$

$$\boxed{E_F(T) = E_F(0)\left[1 - \frac{5\pi^2}{12}\left(\frac{K_B T}{E_F(0)}\right)^2\right]}$$

→ Pressure increases when are E· is less.

Q. Show that at $T = 0K$ in e^- gas, where $\epsilon << E_F(0)$

$$\frac{N(E_F - \epsilon \to E_F)}{N_{total}} = \frac{3}{2}\frac{\epsilon}{E_F}$$

Sol^n:

$$\frac{\int_{E_F-\epsilon}^{E_F} dn}{\int_0^{E_F} dn} = \frac{\int_{E_F-\epsilon}^{E_F} E^{1/2}\, dE}{\int_0^{E_F} E^{1/2}\, dE}$$

$$= \left[1 - \left(\frac{E_F - \epsilon}{E_F}\right)^{3/2}\right]$$

$$= \left[1 - \left(1 - \frac{\epsilon}{E_F}\right)^{3/2}\right]$$

$$= \frac{3}{2}\frac{\epsilon}{E_F}$$

Q. for a 2-D e^- gas no. of e^- per unit area is given b

$$n = \frac{4\pi m K_B T}{h^2} \log\left(e^{E_F/K_B T} + 1\right)$$

Sol^n:

$$\frac{N}{A} = n = \frac{4\pi m K_B T}{h^2} \log\left(^{E_F/K_B T} + 1\right)$$

$$\frac{dN}{A} = dn = g_s \times g(K)\, dK \times f$$

$$= 2 \times \frac{K}{2\pi}\, dK \times \frac{1}{e^{(E_F - E_F)/K_B T} + 1}$$

$$= \frac{2m}{2\pi h^2}\int_0^{\infty} \frac{1}{e^{(E - E_F)/K_B T} + 1}\, d\epsilon$$

Q. Show that de-Broglie wavelength of e^- at Fermi surface at 0K is

$$2\left(\frac{\pi}{3n}\right)^{1/2}.$$

Hint: $\lambda_{dB} = \dfrac{h}{P_f} = \dfrac{h}{\sqrt{2mE_f}}$

Q. Show that the partition funⁿ of a relativistic gas of N mono atomic malecule having the energy momentum relation $E = pc$ is

$$\frac{1}{\underline{|N}}\left[\frac{8\pi V}{h^3}\left(\frac{K_B T}{c}\right)^3\right]^N$$

$\rightarrow g(p)dp \equiv g(\epsilon)d\epsilon$

Also find $\langle E \rangle$, $\langle p \rangle$.

Solⁿ:

$$Z = \Sigma g_i e^{-\beta\epsilon_i}$$

$$= \int g(\epsilon)d\epsilon\, e^{-\beta\epsilon_i}$$

$$= \int g(p)\,dp\, e^{-\beta pc}$$

$$= \int_0^\infty \frac{4\pi p^2 dp\, V}{h^3}\, e^{-\beta pc}$$

$$= \frac{8\pi V}{h^3}\left(\frac{K_B T}{c}\right)^3$$

$$Z_{gas} = \frac{1}{N!} (Z_1)^N$$

$$= \frac{1}{\underline{|N}} \left[\frac{8\pi V}{h^3} \left(\frac{K_B T}{c} \right)^3 \right]^N$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z \Big|_{V, N}$$

$$=$$

$$\langle P \rangle = \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z \Big|_{T, N}$$

$$\langle P \rangle = \frac{1}{3} \frac{\langle E \rangle}{V}$$

→ If case is non-relativistic then $E = p^2/2m$ treated this prob

Q. Show that the partition funn of a non-relativistic ga of N monoatomic malecules having the en. mom reln $E = p^2/2m$ is

$$\langle P \rangle = \frac{2}{3} \frac{\langle E \rangle}{V}$$

$$Z_1 = \Sigma g_i \, e^{-\beta E_i}$$

$$Z_1 = \frac{V}{\lambda^3}$$

Q. Calculate the fermi energy & average en of e^- in eV for sodium atom assuming it has one free e^- per atom.

Given $\rho = 0.977\ gm/cm^3$ $A = 23$

$\frac{3}{5}\bar{E}_0$

Soln:

$$E_f = \frac{h^2}{2m_{e^-}}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

$$\rho = \frac{M}{V} = \frac{N \cdot m}{V}$$

Q. 3 spin 1/2 fermions are distributed in 2 non-degenerate cells no. of ways of distribution

ϵ	ϵ'
↑	↑↓
↓	↑↓
↑↓	↑
↑↓	↓

4 ways

4 spin 1/2 → 1-ways

5 spin 1/2 → 0-ways

Bose-Einstein statistics:

Distribution fun^n:

$$Z = \sum_i e^{-\beta(\epsilon_i - \mu)}$$

$$= \sum_{n_1, n_2} e^{-\beta[(n_1\epsilon_1 + n_2\epsilon_2 + \cdots) - \mu(n_1 + n_2 + \cdots)]}$$

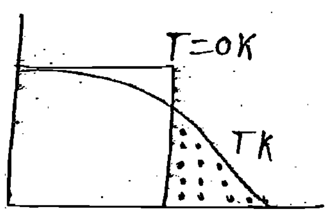
$$= \sum_{n_1=0}^{\infty} e^{-\beta n_1(\epsilon_1 - \mu)} \times \sum_{n_2} e^{-\beta n_2(\epsilon_2 - \mu)}$$

$$\sum x^n = \frac{1}{1-x}$$

$$Z = \frac{1}{1-e^{-\beta(\epsilon_1 - \mu)}} \times \frac{1}{1-e^{-\beta(\epsilon_2 - \mu)}} \times \cdots$$

$$Z = \prod_i \frac{1}{[1 - e^{-\beta(\epsilon_i - \mu)}]}$$

Electronic specific heat(C_e) →



No. of excited e's in range $d\epsilon$

$$N_{excited} = \frac{8\sqrt{2}\, V \pi m^{3/2}}{h^3} \, d\epsilon$$

$$d\epsilon = K_B T$$

$$N_{excited} = \frac{3}{2} \frac{N K_B T}{E_f(0)}$$

Excitation Energy

$$E = N_{exe} \times K_B T$$

$$= \frac{3}{2} \frac{N K_B^2 T^2}{E_f(0)}, \quad E_f(0) = K_B T_F$$

$$C_e = \frac{dE}{dT} = \frac{3 N K_B^2 T}{K_B T_F}$$

$$\boxed{C_e = 3 N K_B \left(\frac{T}{T_F}\right)}$$

$$C_e \propto T$$

$$\boxed{C_e = AT}$$

Bose-Einstein distribution function

1st page

$$Z = \prod_i \frac{1}{[1-e^{-\beta(\epsilon_i-\mu)}]}$$

$$\ln Z = -\sum_i \ln[1-e^{-\beta(\epsilon_i-\mu)}]$$

$$\sum \langle n_i \rangle = N = \frac{1}{\beta}\frac{\partial}{\partial \mu} \ln Z \Big|_{T,V}$$

$$\sum_i \langle n_i \rangle = \sum_i \frac{1}{e^{\beta(\epsilon_i-\mu)}-1}$$

$$\boxed{\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i-\mu)}-1}}$$

Occupation no. for Bosons:

$$\langle n \rangle = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

$$f_{BE}(\epsilon) = \langle n \rangle = \frac{1}{e^{\beta\epsilon}\cdot e^{-\beta\mu}-1}$$

$$e^{-\beta\mu} \gg 1$$

$$f_{BE} = f_{M.B}$$

Q. st. change into cl. st.

Classical ideal gas:

$$\mu = -\frac{1}{\beta}\frac{\partial}{\partial N}\ln Z\Big|_{T,V}$$

$$Z = \frac{V^N}{h^{3N} N!}(2\pi m k_B T)^{3N/2}$$

$$= N\ln\left((2\pi m k_B T)^{3/2} + N\ln N - N\ln N + N\right)$$

$$\mu = -\frac{1}{\beta}\left[\ln\left\{\frac{V}{N}\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\right\}\right]$$

for validity of CSM

$$e^{-\beta\mu} >> 1$$

$$= \frac{V}{N}\left(\frac{2\pi m K_B T}{h^2}\right)^{3/2} >> 1$$

$$\Rightarrow \frac{1}{n\lambda^3} >> 1$$

$$\boxed{n\lambda^3 << 1}$$

validity cond^n for CSM.

When T high

$$n = \frac{N}{V} = \text{low}$$

particle behave like cl. gas

When T is Low

$$n = \frac{N}{V} = \text{High}, \quad n\lambda^3 >> 1$$

particle behave like quantum gas particle.

<u>Ideal Bose gas →</u>
<u>(Photon gas)</u>

for photons gas

$$Z_{pho} = \prod_i \frac{1}{[1-e^{-\beta(\epsilon_i-\mu)}]}$$

With $\mu=0$, (min en. of a particle or photon $E=mc^2=0$)

$$Z_{photon} = \prod_i \frac{1}{(1-e^{-\beta\epsilon_i})}$$

$$\ln Z = \sum_i \ln \frac{1}{(1-e^{-\beta \epsilon_i})}$$

$$\ln Z|_{\text{Photon gas}} = 2 \times \int g(p)\,dp \;\ln Z|_{\text{photon}}$$

ω freq find 2 mods.
Transvers mode for photon
ω ⇝ 2 dir^n

$$= 2\times \int g(\omega)\,d\omega \,\ln\left(\frac{1}{1-e^{-\beta \epsilon_i}}\right)$$

$$g(p)\,dp = \frac{4\pi p^2\,dp\,V}{h^3} = \frac{4\pi E^2\,dE\,V}{h^3 c^3}$$

$$E = \hbar\omega$$

$$g(\omega)\,d\omega = \frac{V \times 4\pi \hbar^3 \omega^2\,d\omega}{h^3 c^3}$$

$$\ln Z|_{ph.\,gas} = -\int_0^\infty \frac{\omega^2 V}{\pi^2 c^3} \ln(1-e^{-\beta\hbar\omega})\,d\omega$$

$$\boxed{\ln Z|_{ph.\,gas} = \frac{V\pi^2}{45 c^3 (\beta\hbar)^3}}$$

$$F = -K_B T \ln Z = -K_B T \frac{V\pi^2}{45 c^3 (\beta\hbar)^3}$$

Av. En. of photon gas:

$$\langle E \rangle = \frac{-\partial}{\partial\beta} \ln Z \Big|_{N,V}$$

$$= \frac{3V\pi^2}{45 c^3 \hbar^3}\left(\frac{1}{\beta^4}\right)$$

$$\boxed{\langle E \rangle \propto T^4}$$

$$\langle P \rangle = \frac{1}{\beta}\frac{\partial}{\partial V}\ln Z\Big|_{T,N}$$

$$= \frac{\pi^2}{45 c^3 \hbar^3 \beta^4}$$

$$\boxed{\langle P \rangle \propto T^4}$$

N.R

$$PV = \frac{S}{3}\langle E \rangle$$

$$\langle P \rangle = \frac{2}{3}\frac{\langle E \rangle}{V}$$

$$\rightarrow \frac{\langle P \rangle}{\langle E \rangle} = \frac{1}{3V}$$

$\Rightarrow \langle P \rangle = \frac{1}{3}\frac{\langle E \rangle}{V}$ → Relativistic case

$$\boxed{\langle P \rangle = \frac{\langle U \rangle}{3}}$$

$\langle U \rangle \Rightarrow$ En. density

Specific heat at constt volume

$$C_V = \frac{\partial}{\partial t}\langle E \rangle$$

$$= \frac{32\pi^5 V K_B^4 T^3}{15(hc)^3}$$

$$\boxed{C_V \propto T^3}$$

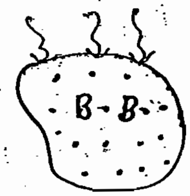
* Entropy $S = \frac{-\partial F}{\partial T}\Big|_{N,V}$

$$= \frac{32\pi^5 K_B^4 V T^3}{45(hc)^3}$$

$$\boxed{S \propto T^3}$$

Planck's radiation formula of BBR →

B.B. has atomic oscillators and radiation has discrete En. packets (photons)



Energy of osc.

$$E_n = nh\nu$$

$$n = 0, 1, 2, \ldots$$

Total no. of osc.

$$N = N_0 + \underbrace{N_1}_{N_0 e^{-h\nu/K_BT}} + N_2 + \cdots = \frac{N_0}{1 - e^{-h\nu/K_BT}}$$

Total En. of osc.

$$E = E_0 N_0 + E_1 N_1 + \cdots = \frac{N_0 h\nu\, e^{-h\nu/K_BT}}{(1 - e^{-h\nu/K_BT})^2}$$

Av. En. of an osc.

$$= \frac{E}{N}$$

$$= \frac{\hbar\omega}{e^{\hbar\omega/K_BT} - 1} = \frac{h\nu}{e^{h\nu/K_BT} - 1}$$

No. of oscillators in freq. range $\nu \to \nu + d\nu$

$$= 2 \times \frac{4\pi p^2 dp\, V}{h^3}$$

Energy density of black body having freq. $\nu \to \nu + d\nu$

$$U_\nu d\nu = \frac{\text{no. of osc. in range } \nu \to \nu + d\nu}{\text{volume}} \times E_{osc}$$

$$\boxed{U_\nu d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/K_BT} - 1} d\nu}$$

Planck's radiation formula

$$\boxed{p = E/c = h\nu/c}$$

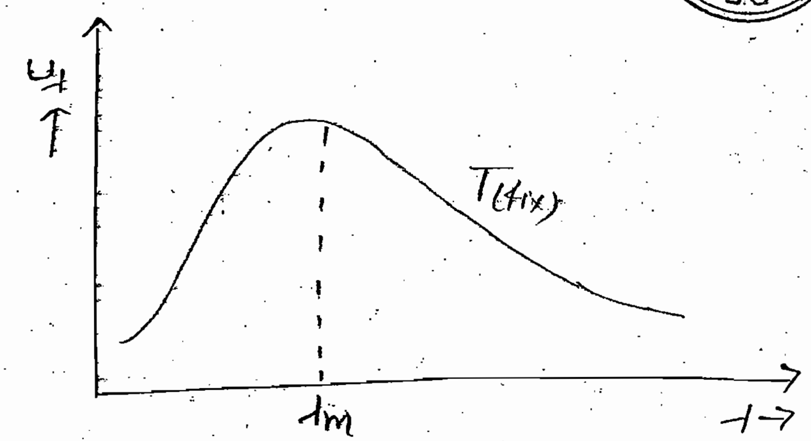
<u>In terms of wavelength</u>

$$\nu = c/\lambda$$

$$d\nu = -\frac{c}{\lambda^2}\, d\lambda$$

$$\boxed{u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\, \frac{1}{e^{hc/\lambda k_B T} - 1}\, d\lambda}$$

$$\boxed{u_\lambda = \frac{8\pi hc}{\lambda^5}\, \frac{1}{e^{hc/\lambda k_B T} - 1}\, d\lambda}$$



$\longrightarrow$ <u>BB spectra is continuous.</u> (ie it absorb completely, completely emitted)

(i) In low wavelength limit

$$\frac{hc}{\lambda k_B T} >> 1$$

$$u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5} \times \cancel{1}\left(e^{-hc/\lambda kT}\right)$$

$$\boxed{u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\, e^{-hc/\lambda kT}\, d\lambda}$$

This is Wein's Law.

(ii) long wavelength limit

$$\frac{hc}{\lambda K_B T} << 1 \Rightarrow e^{hc/\lambda K_B T} = 1 + \frac{hc}{\lambda K_B T}$$

$$\boxed{u_\lambda d\lambda = \frac{8\pi K_B T}{\lambda^4} d\lambda}$$

Rayleigh's - Jean's law.

(iii) u_λ is max. at

$\lambda = \lambda m$

$$\Rightarrow \frac{du_\lambda}{d\lambda} = 0$$

$$\Rightarrow \frac{3(e^{hc/\lambda m K_B T} - 1)}{e^{hc/\lambda m K_B T}} = \frac{hc}{\lambda m K_B T}$$

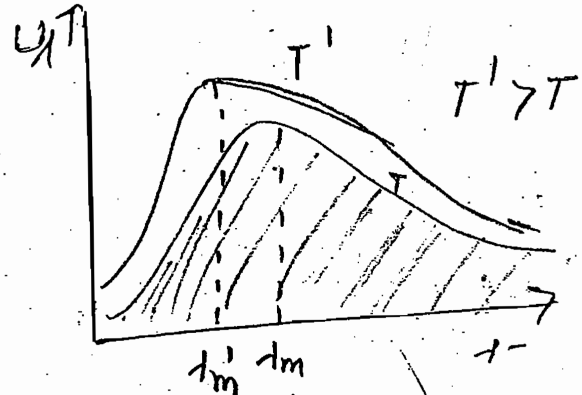
$$\Rightarrow 3 = \frac{hc}{\lambda m K_B T}$$

$$\Rightarrow \lambda m \cdot T = \frac{hc}{3 K_B} = \text{constt.}$$

$$\boxed{\lambda m \cdot T = \frac{hc}{2.965 K_B} = b}$$

Weins'. displacement law

$\lambda_m' T' = \lambda m T$



(iv) Area enclosed by the curve

$$\propto T^4$$

(Steafan's Law)

$$\text{Area} = \int_0^{\infty} u_\lambda \, d\lambda$$

$$= \int_0^{\infty} \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda KT} - 1} \, d\lambda$$

$$\frac{hc}{\lambda KT} = x$$

$$= \propto T^4 \int_0^{\infty} x^3 \frac{dx}{e^x - 1}$$

Steafan's Law:

$$E = \sigma A e (T^4 - T_0^4) t$$

$$e = 1 \text{ for B.B.}$$

$$\sigma = 5.67 \times 10^{-8} \text{ watt}/m^2$$

$$\sigma = \frac{2\pi^5 K_B^4}{15 h^3 c^2}$$

Q. $T_0 = 27^\circ C$, $T_A^{00} = 373^\circ C$, $T_B^{00} = 573^\circ C$, Body A & B are spherical of radius $\frac{r_A}{r_B} = \frac{1}{2}$. find

$$\frac{\text{Rate of cooling of body A}}{\text{Rate of cooling of body B}} = \frac{1}{4}\left(\frac{T_A^4 - T_0^4}{T_B^4 - T_0^4}\right)$$

$\Rightarrow$ Rate of cooling $\propto (T - T_0)$

if $(T - T_0)$ is small

This is Newton's Law of cooling.

Adiabatic expansion of black body $\rightarrow$

$$dQ = dU + PdV$$

In adi.ex. $\leftarrow 0 = dU + PdV$

$$= d(uV) + PdV$$

$$= udV + Vdu + PdV$$

$$= udV + Vdu + \frac{u}{3}dV$$

$$\Rightarrow 0 = \frac{4u}{3}dV + Vdu$$

$$= \frac{4}{3}\frac{dV}{V} + \frac{du}{u}$$

$$\boxed{uV^{4/3} = \text{constt}}$$

$$u \propto T^4$$

$$\boxed{T^4 V^{4/3} = \text{constt}} \rightarrow P^{1/4} V^{1/3} = \text{constt}$$

$$\boxed{PV^{4/3} = \text{constt}}$$

$$\boxed{TV^{1/3} = \text{constt}} \quad \text{or} \quad \boxed{VT^3 = \text{constt}}$$

$$\boxed{TV^{\gamma-1} = \text{constt}}$$

$\gamma = \frac{C_P}{C_V} = \frac{4}{3} \rightarrow$ Triatomic gas

If adiabatically in case of BB.

$T \rightarrow 2T$

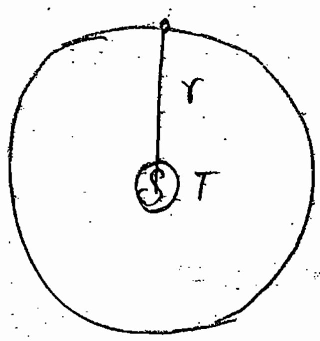
$V \rightarrow \frac{V}{8}$

Measurement of Sun's temp →

By pyrometer

Total radiation pyrometer (Based on stefan's law)

optical pyrometer (Based on Be Wein's displacement)



En. radiated by the sun/sec

$= \sigma \cdot 4\pi R_s^2 T_s^4$

$= S \times 4\pi r^2$

↓

En. falling/Area.sec on earth's surface. (solar constt)

(Poynting vector) unit w/m²

$S = 2\,cal/min\text{-}cm^2$

Bose – Einstein condensation :

particle-den Gr.S. → sharp increase ⇒ condensation.

Occupation No.

$$N = \sum_s \langle n_s \rangle = \sum_s \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}$$

$$\langle n \rangle = \frac{1}{e^{\beta(\epsilon - \mu)} - 1}$$

for gr. state $\epsilon = 0$

$$\langle \bar{n} \rangle = \frac{1}{e^{-\beta\mu} - 1}$$

for Bosons μ must be –ve.
as $\langle n \rangle$ can never be –ve.

$$\Rightarrow \langle n \rangle = \frac{1}{e^{\beta|\mu|} - 1}$$

T → Increases
$|\mu|$ → Increases

T → Decreases
$|\mu|$ → decreases.

As no. of particle is conserved.
The temp. at which
$|\mu| \to 0$, when T is very low

is called $T \to T_B$ (Bose temp. or condensation temp.)

No. of Bosons in the Bose gas

$$N = \int g(p)\, dp \times f_{BE}(\epsilon)$$

$$g(p)\, dp = \frac{4\pi p^2 dp . V}{h^3} = \frac{V p^2 dp}{2\pi^2 \hbar^3}$$

$$N = \frac{V}{2\pi^2\hbar^3}\int_0^\infty \frac{p\,dp}{e^{\beta(p^2/2m-\mu)}-1}$$

at $T = T_B$, $\mu = 0$

$$N = \frac{V}{2\pi^2\hbar^3}\int_0^\infty \frac{p^2\,dp}{(e^{p^2/2mK_BT}-1)}$$

let $y^2 = \frac{p^2}{2mK_BT}$

$$2y\,dy = \frac{2p\,dp}{2mK_BT_B}$$

$$dp = \sqrt{2mK_BT_B}\,dy$$

$$N = \frac{V(2mK_BT_B)^{3/2}}{2\pi^2\hbar^3}\int_0^\infty \frac{y^2}{e^{y^2}-1}\,dy$$

$$N = \frac{V(2mK_BT_B)^{3/2}}{2\pi^2\hbar^3}\left(\frac{2.612\sqrt{\pi}}{4}\right)$$

$$\boxed{T_B = \left[\frac{N}{2.612\,V}\right]^{2/3}\frac{2\pi\hbar^2}{mK}}$$

$$T_B \propto \left(\frac{N}{V}\right)^{2/3}$$

$$T_B \propto n^{2/3}$$

$$T_B \propto \frac{1}{m}$$

→ what happens when T decreases beyond T_B
$(T < T_B)$

There is a problem:

Problem is that zero weightage is given to the gr st. $\epsilon = 0$ as $g(\epsilon) \propto \epsilon^{1/2}$

If $\epsilon = 0$, $g(\epsilon = 0) = 0$

But $\langle n \rangle = \dfrac{1}{e^{\beta(\epsilon - \mu)} - 1}$

if $\epsilon = 0$ & $\mu = 0$

$\langle n \rangle = \infty$

This problem can be solved by isolating the state & let gr st. has N_0 particles.

Then

$$N = N_0 + \frac{V}{2\pi^2\hbar^3}\int_0^\infty \frac{p^2\,dp}{(e^{p^2/2mk_BT} - 1)} \rightarrow \frac{\frac{V(2mk_BT_B)^{3/2}}{2\pi^2\hbar^3}\left(\frac{2.612\sqrt{\pi}}{4}\right)}{(T_B)^{3/2}}$$

$$\boxed{N = N_0 + N\left(\frac{T}{T_B}\right)^{2/3}}$$

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$$N_0 = N\left[1 - \left(\frac{T}{T_B}\right)^{2/3}\right]$$

$T < T_B$

$T \to 0$ (population increases in gr. st.)

$N_0 \to N$ (i.e, all particles goes over to the gr state)

Energy of Bosonic system below T_B →

$\langle n \rangle = \dfrac{1}{e^{\beta|\mu|} - 1}$

$T < T_B$

En(E) of Bosonic system below T_B

$$E = \int g(E)\,d\epsilon \times f(\epsilon) \times \epsilon$$

$$= \frac{V}{2\pi^2\hbar^3}\int \frac{p^2}{2m}\,\frac{p^2\,dp}{e^{\beta(p^2/2m)} - 1}$$

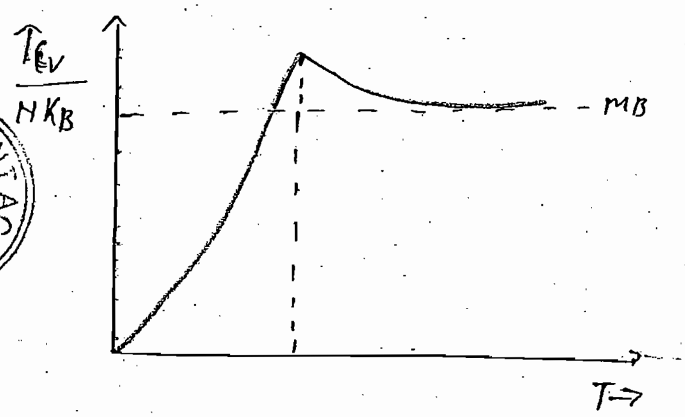
$$= \frac{V}{4\pi^4\hbar^3 m}(2mK_BT)^{5/2}\int_0^\infty \frac{y^4}{e^{y^2}-1}dy$$

$$E = \frac{V}{4\pi^4\hbar^3 m}(2mK_BT)^{5/2}\,\frac{3}{8}\times 1.341\sqrt{\pi}$$

$$E = 0.77\, NK\,\frac{T^{5/2}}{T_B^{3/2}}$$

$$C_V = \frac{dE}{dT}$$

$$C_V \propto T^{3/2}$$



Q. Determine the Bose temp. of Bosons each of mass of 6.65×10^{-27} Kg and spin 0, their concentration being $10^{26}/m^3$ Then also determine the fraction N_0/N of the Bosons in gr. st. at a temp. $0.2\,T_B$

$$T_B = 1.32\,K, \quad \frac{N_0}{N} = 0.91$$

Q. The excitation of 3-D solid are bosonic in nature with their freq. ω & wave vector K. which are related as $\omega \propto K^2$. In large wavelength limit the chemical pot. is zero the behaviour of sp. heat depends on T as

$$T,\ T^3,\ \checkmark T^{3/2},\ T^{5/2}.$$

Ans: $E = \int \epsilon\, g(\epsilon)\, d\epsilon\ \ f(\epsilon)$

$$= \int \epsilon \frac{4\pi p^2 dp}{h^3} \frac{1}{e^{\beta\epsilon}-1}$$

$$\propto \int \epsilon \, \epsilon^{1/2} \frac{1}{e^{\beta\epsilon}-1}$$

$$\int_0^\infty \frac{\epsilon^{3/2}}{e^{\beta\epsilon}-1} d\epsilon$$

$$\propto T^{5/2} \int_0^\infty \frac{x^{3/2}\, dx}{e^x - 1}$$

$C_V \propto T^{3/2}$

→ Condensation of photons does not possible.

$$\langle n \rangle = \frac{1}{e^{\beta \epsilon} - 1}$$

No. is not conserved in case of photon.

$\epsilon \propto p^{s}$

$s = 1$ condensation not possible.

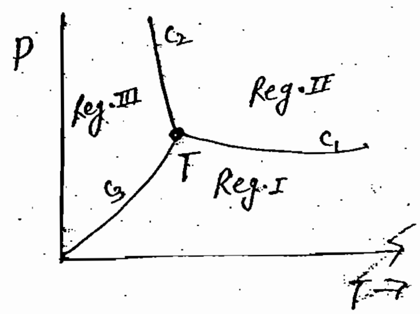
Q. condensation is possible for

✓ $0 < s < 1$ $\qquad$ $s < 2$

$0 < s < 2$ $\qquad$ $0 < s < 3$

Phase transition :

Ex: Water – vapour



T → Triple pt.
Point where all the three phase, solid, liquid & gas coe. in equilibrium with each other.

In case of water the co-ordinate of triple pt.
$(0.075°C,\ 76cm)$

Reg. I	Reg II	Reg III
Gas	liquid	solid
Curve 1	Curve 2	Curve 3
Curve of vaporisation	Curve of fusion	Curve of sublimation.

Critical point :

The pt. at which liquid vapour equilibrium lying ends. Above critical point, there is no further phase transition.

Prob: The vapour pressure P (in mm of Hg) of a solid at temp. T is expressed by $\ln P = 23 - \frac{3863}{T}$ & that of liquid phase by $\ln P = 19 - \frac{3063}{T}$

Q. At transition pt. which is correct one-state to other $\rightrightarrows$ P, T

$G = 0$

$dG =$ constt

$dG = 0$

None

free En G i.e $G(T, P)$ is continuous at T.P.

$\underline{dG = 0}$

* $G = H - TS \quad = U + PV - TS$

$dG = \quad dU + PdV + VdP - TdS - SdT$

$dG = \quad VdP - SdT$

or

$dG = VdP - SdT - MdH$

Types of transitions

Ist order

1^{st} derivative of G is discontinuous

Ex: V, S, M, Density

$V \to$ volume, $S \to$ Entropy, $M \to$ magnetization

IInd order

2nd order derivative is discontinuous & 1st order derivative is continuous

C, χ_m, compressibility

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)$$

Ising model :-

A model used to explain ferromagnetism easily.

$$E = -J\sum_{ij} S_i S_j - \vec{\mu} H \sum S_i$$

$\vec{\mu}$ → dipole moment of atom; H → mag. field

$-J\sum_{ij} S_i S_j$ → Interaction En. with nearest neighbouring atom

S_i, S_j can take value ± 1 only.

$J \to$ exchange En.

If $H = 0$

$$E = -J\sum_{i,j} S_i S_j$$

let there are 2 atom

$$E = -J\sum S_1 S_2 = -J\sum_{S_1} S_1 \sum_{S_2} S_2$$

$$E = e^{\beta J}\sum_{\cancel{S_1 S_2 = \pm 1}} S_1 S_2 \quad \text{or} \quad \sum_{S_1}\sum_{S_2} e^{\beta J\, S_1 S_2}_{\,S_1 S_2 = \pm 1}$$

$$E = e^{\beta J}\sum_{\cancel{S_1}} S_1(+1) + e^{\beta J}\sum S_1(-1) = \sum_{S_1} e^{\beta J S_1(+1)} + \sum_{S_1} e^{\beta J \Sigma S_1}$$

$$\boxed{E_2 = 2^2(\cosh\beta J)}$$

If there are 3 atom

$$E_3 = 2^3(\cosh\beta J)^2$$

In case of N-particle

$$\boxed{Z_n = 2^N (\cosh \beta J)^{N-1}}$$

$$F = -K_B T \ln Z_N$$

If $H \neq 0$

$$\boxed{Z_N = 2^N (\cosh \beta J)^{N-1} (2 \cosh \beta \mu H)^N}$$

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