LCR Parallel Resonance

Expression for Resonant Frequency:

Parallel resonance circuit is shown in fig(1). One branch of the resonant circuit consists resistance R in series with an inductance L. Capacitor C is connected parallel to the RL combination in other branch. An AC source is applied across this parallel combination. The admittance of the circuit is given by

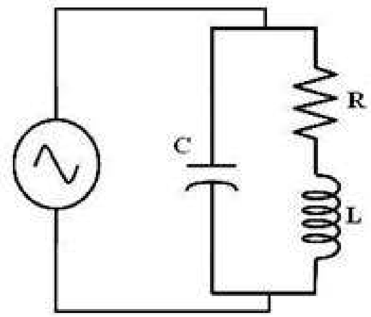

Fig (1)

$$y = \frac{1}{z} = \frac{1}{R + j\omega L} + \frac{1}{1/j\omega C}$$

$$y = \frac{1}{z} = \frac{1}{R + j\omega L} + j\omega C$$

or $$\frac{1}{z} = \frac{1}{(R + j\omega L)} \text{ x } \frac{(R - j\omega L)}{(R - j\omega L)} + j\omega C$$

or $$\frac{1}{z} = \frac{(R - j\omega L)}{(R^2 - j^2\omega^2 L^2)} + j\omega C$$

or $$\frac{1}{z} = \frac{(R - j\omega L)}{(R^2 + \omega^2 L^2)} + j\omega C \qquad (\because j^2 = -1)$$

or $$\frac{1}{z} = \frac{R}{(R^2 + \omega^2 L^2)} - \frac{j\omega L}{(R^2 + \omega^2 L^2)} + j\omega C$$

or $$\frac{1}{z} = \frac{R}{(R^2 + \omega^2 L^2)} - j\omega\left[\frac{L}{(R^2 + \omega^2 L^2)} - C\right] \qquad \text{------------ (1)}$$

At resonance the reactance part of Eqn(1) is equal to zero

$$\therefore \left[\frac{L}{(R^2 + \omega^2 L^2)} - C\right] = 0$$

or $$\frac{L}{(R^2 + \omega^2 L^2)} = C$$

or $$L = C(R^2 + \omega^2 L^2)$$

or $$(R^2 + \omega^2 L^2) = \frac{L}{C} \qquad \text{------------ (2)}$$

or $$\omega^2 L^2 = \frac{L}{C} - R^2$$

or $$\omega_0{}^2 = \frac{L}{CL^2} - \frac{R^2}{L^2}$$

or $$\omega_0{}^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

or $$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

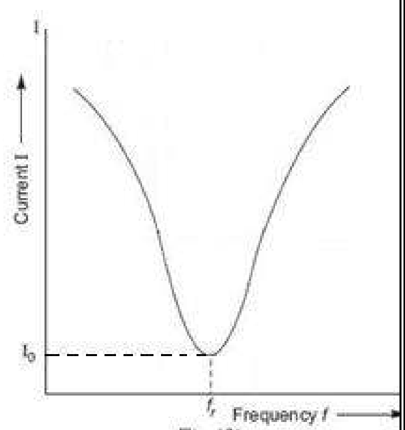

Fig (2)

or $$2\pi f_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \qquad (\because \omega_0 = 2\pi f_0)$$

or $$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \qquad \text{--------------- (3)}$$

where f_0 = resonant frequency of LCR parallel circuit

Dynamic Resistance

The resistance at resonance is known as dynamic resistance. Consider the following equation.

$$\frac{1}{Z} = \frac{R}{(R^2 + \omega^2 L^2)} - j\omega\left[\frac{L}{(R^2 + \omega^2 L^2)} - C\right]$$

at resonance $$\left[\frac{L}{(R^2 + \omega^2 L^2)} - C\right] = 0$$

$$\therefore \frac{1}{Z} = \frac{R}{(R^2 + \omega^2 L^2)} \qquad \text{------------ (1)}$$

we know that $(R^2 + \omega^2 L^2) = \frac{L}{C}$, substituting in Eqn(1), we get

$$\frac{1}{Z} = \frac{R}{L/C}$$

Or $$\frac{1}{Z} = \frac{RC}{L}$$

or $$Z_0 = r_d = \frac{L}{RC} \qquad \text{---------- (2)}$$

where r_d is known as dynamic resistance.

Bandwidth

Bandwidth is defined as the band of frequencies between two points on either side of resonant frequency where impedance falls to $\frac{1}{\sqrt{2}}$ times of value at resonance.

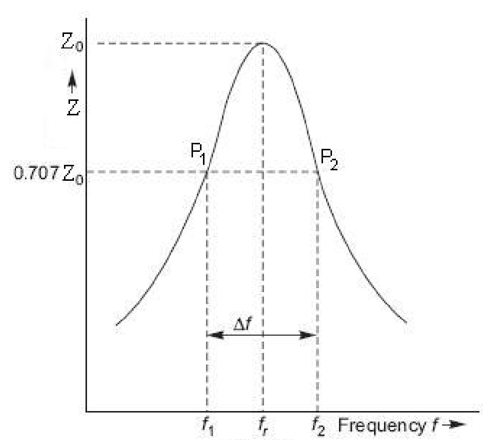

Fig (3)

Impedance of the LCR parallel circuit is given by

$$Z = \frac{(R + j\omega L)\frac{1}{j\omega C}}{\left(R + j\omega L + \frac{1}{j\omega C}\right)}$$

or $$Z = \frac{(R + j\omega L)}{j\omega C\left(R + j\omega L + \frac{1}{j\omega C}\right)}$$

or $$Z = \frac{(R + j\omega L)}{(j\omega RC - \omega^2 LC + 1)}$$

where R << ωL then R can be neglected from the above equation.

$$Z = \frac{j\omega L}{j\omega RC + (1 - \omega^2 LC)}$$

Taking modulus on bothsides of the above equation, we get

$$|Z| = \frac{\omega L}{\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2}} \qquad \text{--------------- (1)}$$

Multiplying numerator and denomination of eqn (1) with $\frac{1}{\omega RC}$, we get

$$|Z| = \frac{\omega L \text{ x } \frac{1}{\omega RC}}{\frac{1}{\omega RC}\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2}}$$

or $$|Z| = \frac{\frac{L}{RC}}{\frac{1}{\omega RC}\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2}}$$

or $$|Z| = \frac{Z_0}{\frac{1}{\omega RC}\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2}} \qquad \left(\because Z_0 = \frac{L}{RC}\right)$$

or $$|Z| = \frac{Z_0 \, \omega RC}{\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2}}$$

but $$|Z| = \frac{Z_0}{\sqrt{2}}$$

$$\therefore \frac{Z_0}{\sqrt{2}} = \frac{Z_0 \, \omega RC}{\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2}}$$

or $$\frac{1}{\sqrt{2}} = \frac{\omega RC}{\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2}}$$

or $$\sqrt{\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2} = \omega RC\sqrt{2} \qquad \text{----------------- (2)}$$

squaring on both sides of equ(2), we get

$$\omega^2 R^2 C^2 + (1 - \omega^2 LC)^2 = 2\omega^2 R^2 C^2$$

or $$(1 - \omega^2 LC)^2 = 2\omega^2 R^2 C^2 - \omega^2 R^2 C^2$$

or $$(1 - \omega^2 LC)^2 = \omega^2 R^2 C^2$$

taking square root on both sides of the above equation, we get

$$1 - \omega^2 LC = \pm \omega RC$$

or $$\omega^2 LC \pm \omega RC - 1 = 0 \qquad \text{--------------- (3)}$$

Eqn (3) is a quadratic equation in ω and will yield two values. The two values ω_1 and ω_2 and they can be obtained as follows.

$$= \frac{\pm RC \pm \sqrt{R^2C^2 + 4LC}}{2LC}$$

or
$$= \frac{\pm RC}{2LC} \pm \frac{\sqrt{R^2C^2 + 4LC}}{2LC}$$

or
$$= \frac{\pm RC}{2LC} \pm \sqrt{\left(\frac{R^2C^2}{4L^2C^2} + \frac{4LC}{4L^2C^2}\right)}$$

or
$$= \frac{\pm R}{2L} \pm \sqrt{\left(\frac{R^2}{4L^2} + \frac{1}{LC}\right)}$$

as R << ωL, therefore $\frac{R^2}{4L^2}$ can be ignored from the above equation, we get

$$= \frac{\pm R}{2L} \pm \sqrt{\frac{1}{LC}} \quad \text{-------------------- (4)}$$

Let $\frac{R}{2L} = \alpha$, and we know that $\sqrt{\frac{1}{LC}} = \omega_0$, substituting these values in the equ (4) we get

$$= \pm\alpha \pm \omega_0$$

Two roots can be written as follows

$\omega_2 = \omega_0 + \alpha$ and $\omega_1 = \omega_0 - \alpha$ (since angular frequency can not be negative)

Therefore Bandwidth $\therefore \Delta\omega = \omega_2 - \omega_1$

substituting ω_1 and ω_2 values in the above equation, we get

$$\Delta\omega = \omega_0 + \alpha - (\omega_0 - \alpha)$$

or $\Delta\omega = \omega_0 + \alpha - \omega_0 + \alpha$

or $\Delta\omega = 2\alpha$ ---------------- (5)

but $\alpha = \frac{R}{2L}$

therefore $\Delta\omega = \frac{2R}{2L}$

or $\Delta\omega = \frac{R}{L}$

or $\Delta\omega = \omega_2 - \omega_1 = (2\pi f_2 - 2\pi f_1) = \frac{R}{L}$

or $2\pi(f_2 - f_1) = \frac{R}{L}$

$$(f_2 - f_1) = \frac{R}{2\pi L}$$

Therefore bandwidth $\Delta f = (f_2 - f_1) = \frac{R}{2\pi L}$ ------------- (6)

Quality Factor (Q – Factor)

When voltage V is applied to the parallel LCR circuit, the current flowing through the circuit at resonance is given by

$$I = \frac{V}{r_d} \qquad \text{------------ (1)}$$

Where r_d is dynamic resistance = $\frac{L}{RC}$

Substituting r_d value in eqn (1), we get

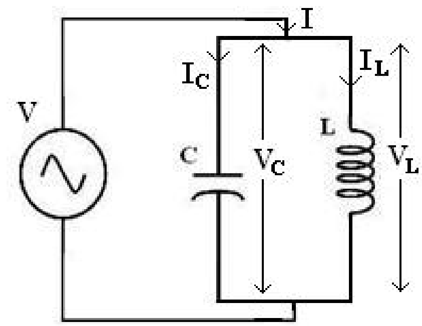

Fig (4)

$$I = \frac{V}{L/RC}$$

or $$I = \frac{VRC}{L}$$

from above equation $$V = \frac{IL}{RC} \qquad \text{----------- (2)}$$

Voltage across the capacitor = V_C

Voltage across the inductor = V_L

From fig (4) voltage across capacitor (V_C) = voltage across inductor (V_L) = applied voltage (V) (Since the capacitor and the inductor are parallel to the applied voltage)

Voltage across capacitor $$V = I_C x \frac{1}{\omega_O C}$$

$$I_C = V\omega_O C \qquad \text{---------- (3)}$$

Substituting V value from eqn(2) in eqn(3) we get

$$I_C = \frac{IL}{RC}\omega_O C$$

or $$I_C = \frac{IL\omega_O}{R}$$

Voltage across the inductor is $$V = I_L x \omega_O L$$

or $$I_L = \frac{V}{\omega_O L} \qquad \text{--------- (4)}$$

substituting V value from eqn(2) in eqn (4), we get

$$I_L = \frac{IL}{RC} x \frac{1}{\omega_O L}$$

or $$I_L = \frac{I}{RC\omega_O} \qquad \text{--------- (5)}$$

The Q-factor is defined as the magnification of the current at resonance

$$Q = \frac{I_C}{I} = \frac{I_L}{I}$$

From eqn(4) we have $$\frac{I_C}{I} = \frac{\omega_O L}{R}$$

$$\therefore \quad \frac{I_C}{I} = Q = \frac{\omega_O L}{R} \qquad \text{------------- (6)}$$

From eqn (5) we have $\frac{I_L}{I} = \frac{1}{\omega_O RC}$

$$\therefore \quad \frac{I_L}{I} = Q = \frac{1}{\omega_O RC} \qquad \text{---------- (7)}$$