# LCR Series Resonance

## Resonant Frequency

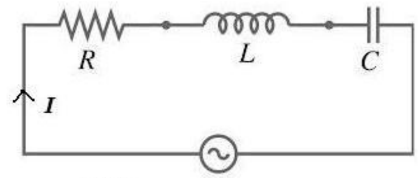

Fig (1)

The LCR series resonance circuit is shown in fig(1), in which resistance R, inductor L, capacitor C are connected in series with and ac voltage source. The impedance of the circuit is given by

$$Z = R + X_L + X_C$$

where $X_L$ is inductive reactance = $j\omega L$

$X_c$ is capacitive reactance = $\frac{1}{j\omega C}$

$$Z = R + j\omega L + \frac{1}{j\omega C}$$

or $$Z = R + j\omega L - \frac{j}{\omega C}$$

or $$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) \quad \text{----------------} \ (1)$$

At resonance reactive component of eqn(1) is equal to zero, therefore

$$\left(\omega L - \frac{1}{\omega C}\right) = 0$$

or $$\omega L = \frac{1}{\omega C}$$

or $$\omega^2 L = \frac{1}{C}$$

or $$\omega_0^{\ 2} = \frac{1}{CL} \quad \text{-------------} \ (2)$$

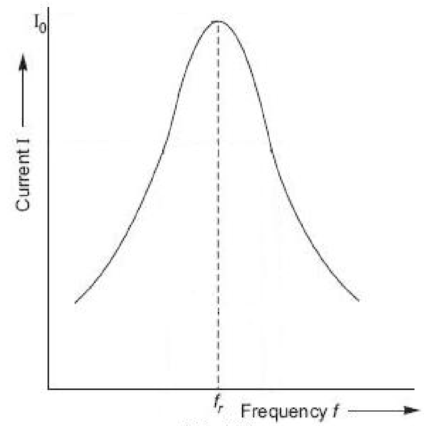

Fig (2)

or $$\omega_0 = \frac{1}{\sqrt{CL}} \quad \text{-------------} \ (3)$$

$\omega_0$ is resonant angular frequency

As $\omega_0 = 2\pi f_0$ sustituting this value in eqn (3), we get

$$2\pi f_0 = \frac{1}{\sqrt{CL}}$$

or $$f_0 = \frac{1}{2\pi\sqrt{CL}} \quad \text{--------------} \ (4)$$

where $f_0$ is resosnant frequency.

At resosnance the resistance of the series circuit is zero and the impedance is equal to resistance R. At any other frequency the magnitude of the impedance is

$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ which is always grater than R. In series resonance circuit, at resonance the impedance is minimum and the current is maximum.

At frequencies lower than resonance the capacitive reactance of the circuit is large as compared to the inductive reactance $\left(\frac{1}{\omega C} > \omega L\right)$ and the total reactacne is capacitive.

At frequencies higher than resonance the inductive reactance is large as compared to the capactive reactance $\left(\omega L > \frac{1}{\omega C}\right)$ and the total reactance is inductive.

At resonant frequency inductive and capacitive reactances are equal $\left(\omega L = \frac{1}{\omega C}\right)$ and the circuit is purely resistive.

**Band Width**

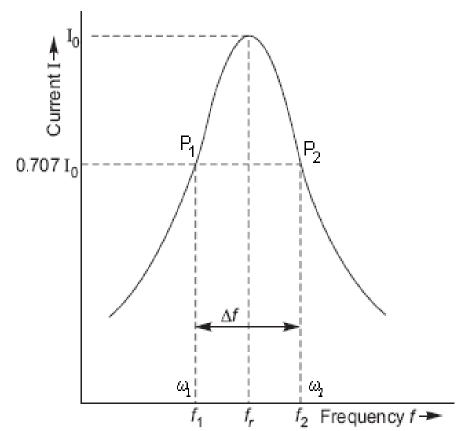

Fig (3)

The band width is defined as range of frequencies specified between two points on either side of resonant frequency where current falls to $\frac{1}{\sqrt{2}}$ times or power falls to half of its value at resinance. $P_1$ and $P_2$ are known as half power points shown in fig(3) on either side of the resonant frequency. Then the distance between $P_1$ $P_2$ represents the band width.

Band width (BW) = $\Delta\omega_0 = (\omega_2 - \omega_1)$

or $\Delta f_0 = (f_2 - f_1)$.

An expression for band width of series resonant circuit can be derieved as follows.

The current in the LCR circuit is given as

$$I = \frac{E_0}{Z} \text{ -------------- (1)}$$

Substituting Z value in eqn (1), we get

$$I = \frac{E_0}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

Taking modulus on both sides, we get

$$|I| = \frac{\sqrt{E_0{}^2}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

or $$I = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \text{ ------------- (2)}$$

current at where the power falls to half of its resonant value is given by

$$I = \frac{I_0}{\sqrt{2}} \qquad \text{but } I_0 = \frac{E_0}{R}$$

Therfore $I = \frac{E_0}{R\sqrt{2}}$ substituting I value in equ (2), we get

$$\frac{E_0}{R\sqrt{2}} = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

or
$$\frac{1}{R\sqrt{2}} = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = R\sqrt{2}$$

Squaring on both sides, we get

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$$

or
$$\left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2 - R^2$$

or
$$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

taking square root on both sides, we get

$$\omega L - \frac{1}{\omega C} = \pm R \qquad \text{----------------} \quad (3)$$

Eqn(3) is a quadratic equation in $\omega$ and will yield two values. The two value of $\omega$ are $\omega_1$ and $\omega_2$. From fig(3) $\omega_2 > \omega_1$.

Therefore
$$\omega_2 L - \frac{1}{\omega_2 C} = R \qquad \text{-------------} \ (4)$$

$$\omega_1 L - \frac{1}{\omega_1 C} = -R \qquad \text{--------------} \ (5)$$

Adding eqn (4) and eqn(5), we get

$$\omega_2 L - \frac{1}{\omega_2 C} + \omega_1 L - \frac{1}{\omega_1 C} = R + (-R)$$

$$\omega_2 L + \omega_1 L - \frac{1}{\omega_2 C} - \frac{1}{\omega_1 C} = R - R$$

$$L(\omega_2 + \omega_1) - \frac{1}{C}\left(\frac{1}{\omega_2} + \frac{1}{\omega_1}\right) = 0$$

Dividing the above equation with L on both sides, we get

$$\frac{L(\omega_2 + \omega_1)}{L} - \frac{1}{LC}\left(\frac{1}{\omega_2} + \frac{1}{\omega_1}\right) = 0$$

or $$(\omega_2+\omega_1)-\frac{1}{LC}\left(\frac{1}{\omega_2}+\frac{1}{\omega_1}\right)=0$$

or $$(\omega_2+\omega_1)-\frac{1}{LC}\left(\frac{\omega_2+\omega_1}{\omega_1\omega_2}\right)=0$$

or $$(\omega_2+\omega_1)-\omega_0^{\;2}\left(\frac{\omega_2+\omega_1}{\omega_1\omega_2}\right)=0 \qquad \left(\because \omega_0^{\;2}=\frac{1}{LC}\right)$$

or $$(\omega_2+\omega_1)=\omega_0^{\;2}\left(\frac{\omega_2+\omega_1}{\omega_1\omega_2}\right)$$

or $$1=\frac{\omega_0^{\;2}}{\omega_1\omega_2}$$

or $$\omega_0^{\;2}=\omega_1\omega_2 \qquad \text{---------} \ (6)$$

or $$f_0^{\;2}=f_1f_2 \qquad \text{----------} \ (7)$$

The resonant frequency is the geometric mean of the two half power frequencies.

Subtracting equ (4) from eqn (5), we get

$$\omega_2L-\frac{1}{\omega_2C}-\left(\omega_1L-\frac{1}{\omega_1C}\right)=R-(-R)$$

or $$\omega_2L-\frac{1}{\omega_2C}-\omega_1L+\frac{1}{\omega_1C}=R+R$$

or $$\omega_2L-\omega_1L-\frac{1}{\omega_2C}+\frac{1}{\omega_1C}=R+R$$

or $$L(\omega_2-\omega_1)+\frac{1}{C}\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right)=2R$$

or $$L(\omega_2-\omega_1)+\frac{1}{C}\left(\frac{\omega_2-\omega_1}{\omega_2\omega_1}\right)=2R$$

Dividing the above equation with L on both sides, we get

$$\frac{L(\omega_2-\omega_1)}{L}+\frac{1}{LC}\left(\frac{\omega_2-\omega_1}{\omega_2\omega_1}\right)=\frac{2R}{L}$$

or $$(\omega_2-\omega_1)+\frac{1}{LC}\left(\frac{\omega_2-\omega_1}{\omega_2\omega_1}\right)=\frac{2R}{L}$$

or $$(\omega_2-\omega_1)+\omega_2\omega_1\left(\frac{\omega_2-\omega_1}{\omega_2\omega_1}\right)=\frac{2R}{L} \qquad \left(\because \omega_0^{\;2}=\frac{1}{LC}=\omega_1\omega_2\right)$$

or $$(\omega_2-\omega_1)+(\omega_2-\omega_1)=\frac{2R}{L}$$

or $$2(\omega_2-\omega_1)=\frac{2R}{L}$$

or $$\left(\omega_2 - \omega_1\right) = \frac{R}{L} \text{ --------------- (8)}$$

or $$\Delta\omega = \frac{R}{L}$$

or $$\left(2\pi f_2 - 2\pi f_1\right) = \frac{R}{L} \qquad \left(\because \ \omega_2 = 2\pi f_2 \text{ and } \omega_1 = 2\pi f_1\right)$$

or $$2\pi\left(f_2 - f_1\right) = \frac{R}{L}$$

or $$\left(f_2 - f_1\right) = \frac{R}{2\pi L}$$

$$\text{Band width} = \left(f_2 - f_1\right) = \frac{R}{2\pi L} \text{ -------------- (9)}$$

**Quality Factor (Q – factor)**

The degree of selectivity or sharpness of the series resonant circuit is defined in terms of quality factor also known as Q-factor. The quality factor is defined as

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} \text{ -------------- (10)}$$

From eqn (8) we have $\left(\omega_2 - \omega_1\right) = \frac{R}{L}$, substituting this in eqn (10), we get

$$Q = \frac{\omega_0}{R/L}$$

or $$Q = \frac{\omega_0 L}{R}$$

The quality factor is defined as the ratio of inductive reactance to total resistance of the circuit.

**Selectivity**

The varation of peak value of current with the frequency of applied emf is shown in fig (4). In figure two curves are plotted. One when the circuit resistance is low i.e., R and the other for the circuit resistance is high i.e., 2R. If the circuit resistance is low the curve of current has a sharp peak and the circuit is said to be sharply resonant or highly selective. If the circuit resistance is high the peak is brodened and the selectivity is poor. Selectivity of a circuit is given as

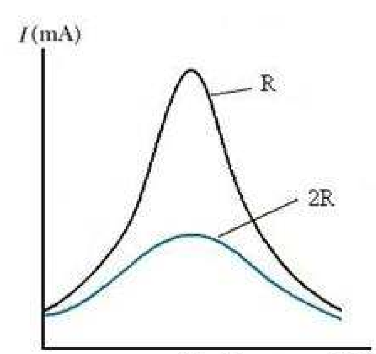

Fig (4)

$$\text{Selectivity} = \frac{\Delta\omega}{\omega_0}$$

The quality factor of the series LCR circuit is given as

$$Q = \frac{\omega_0 L}{R}$$

or $\quad QR = \omega_0 L$

or $\quad \frac{R}{L} = \frac{\omega_0}{Q}$

or $\quad \omega_2 - \omega_1 = \frac{\omega_0}{Q} \qquad \left(\because \ \omega_2 - \omega_1 = \frac{R}{L}\right)$

or $\quad \frac{\omega_2 - \omega_1}{\omega_0} = \frac{1}{Q}$

$$\text{Selectivity} = \frac{\omega_2 - \omega_1}{\omega_0} = \frac{1}{Q}$$

Or $\quad \text{Selectivity} = \frac{\Delta\omega_0}{\omega_0} = \frac{1}{Q}$

Or $\quad \text{Selectivity} = \frac{BW}{\omega_0} = \frac{1}{Q}$

Thus the band width is inversely proportional to quality factor or q-factor. As the value of Q increases, band width decreases and sharpness of resonance or selectivity of the circuit increases.