Chapter 7 – Poisson's and Laplace Equations

A useful approach to the calculation of electric potentials
Relates potential to the charge density.
The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

E = electric field
ρ = charge density
ε_0 = permittivity

The electric field is related to the electric potential by a gradient relationship

$$E = -\nabla V$$

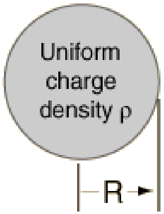
Therefore the potential is related to the charge density by Poisson's equation

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}$$

In a charge-free region of space, this becomes Laplace's equation

$$\nabla^2 V = 0$$

Potential of a Uniform Sphere of Charge



Total charge

$Q = \frac{4}{3}\pi R^3 \rho$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} = \frac{-\rho}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = \frac{-\rho}{\varepsilon_0}$$

outside $\quad \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = 0$, solution of form $\frac{a}{r} + b \qquad a = \frac{Q}{4\pi\varepsilon_0} = kQ \qquad V = \frac{Q}{4\pi\varepsilon_0 r}$

inside $\quad V = cr^2 + d \qquad 2c + 4c = \frac{-\rho}{\varepsilon_0}$ giving $c = \frac{-\rho}{6\varepsilon_0}$

$$\frac{-\rho R^2}{6\varepsilon_0} + d = \frac{Q}{4\pi\varepsilon_0 R}$$ giving $$d = \frac{Q}{4\pi\varepsilon_0 R} + \frac{\rho R^2}{6\varepsilon_0}$$

$$V = \frac{\rho}{6\varepsilon_0}\left[R^2 - r^2\right] + \frac{Q}{4\pi\varepsilon_0 R} = \frac{\rho}{6\varepsilon_0}\left[R^2 - r^2\right] + \frac{\rho R^2}{3\varepsilon_0}$$

Poisson's and Laplace Equations

From the point form of Gaus's Law

Del_dot_ $D = \rho_V$

Definition D

$D = \varepsilon E$

and the gradient relationship

$E = -$DelV

Del_D = Del_(εE) = $-$Del_dot_$(\varepsilon$DelV$) = \rho_V$

Del_DelV $= \dfrac{-\rho_V}{\varepsilon}$

Poisson's Equation

Laplace's Equation

if $\rho_V = 0$

Del_dot_D $= \rho_V$

Del_Del = Laplacian

The divergence of the gradient of a scalar function is called the Laplacian.

Poisson's and Laplace Equations

$$\mathrm{LapR} := \left[\frac{d}{dx}\left(\frac{d}{dx} V(x,y,z) \right) + \frac{d}{dy}\left(\frac{d}{dy} V(x,y,z) \right) + \frac{d}{dz}\left(\frac{d}{dz} V(x,y,z) \right) \right]$$

$$\mathrm{LapC} := \frac{1}{\rho} \cdot \frac{d}{d\rho}\left(\rho \cdot \frac{d}{d\rho} V(\rho,\phi,z) \right) + \frac{1}{\rho^2} \cdot \left[\frac{d}{d\phi}\left(\frac{d}{d\phi} V(\rho,\phi,z) \right) \right] + \frac{d}{dz}\left(\frac{d}{dz} V(\rho,\phi,z) \right)$$

$$\mathrm{LapS} := \left[\frac{1}{r^2} \cdot \frac{d}{dr}\left(r^2 \cdot \frac{d}{dr} V(r,\theta,\phi) \right) \right] + \frac{1}{r^2 \cdot \sin(\theta)} \cdot \frac{d}{d\theta}\left(\sin(\theta) \cdot \frac{d}{d\theta} V(r,\theta,\phi) \right) + \frac{1}{r^2 \cdot \sin(\theta)^2} \cdot \frac{d}{d\phi}\frac{d}{d\phi} V(r,\theta,\phi)$$

Examples of the Solution of Laplace's Equation

D7.1

Given

$$V(x,y,z) := \frac{4 \cdot y \cdot z}{x^2 + 1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\varepsilon o := 8.854 \cdot 10^{-12}$$

$$V(x,y,z) = 12$$

Find: V @ and ρv at P

$$LapR := \left[\frac{d}{dx}\left(\frac{d}{dx}V(x,y,z)\right) + \frac{d}{dy}\left(\frac{d}{dy}V(x,y,z)\right) + \frac{d}{dz}\left(\frac{d}{dz}V(x,y,z)\right)\right]$$

$$LapR = 12$$

$$\rho v := LapR \cdot \varepsilon o$$

$$\rho v = 1.062 \times 10^{-10}$$

Uniqueness Theorem

Given is a volume V with a closed surface S. The function V(x,y,z) is completely determined on the surface S. There is only one function V(x,y,z) with given values on S (the boundary values) that satisfies the Laplace equation.

Application: The theorem of uniqueness allows to make statements about the potential in a region that is free of charges if the potential on the surface of this region is known. The Laplace equation applies to a region of space that is free of charges. Thus, if a region of space is enclosed by a surface of known potential values, then there is only one possible potential function that satisfies both the Laplace equation and the boundary conditions.

Example: A piece of metal has a fixed potential, for example, V = 0 V. Consider an empty hole in this piece of metal. On the boundary S of this hole, the value of V(x,y,z) is the potential value of the metal, i.e., V(S) = 0 V. V(x,y,z) = 0 satisfies the Laplace equation (check it!). Because of the theorem of uniqueness, V(x,y,z) = 0 describes also the potential inside the hole

Examples of the Solution of Laplace's Equation

Example 7.1

Assume V is a function only of x – solve Laplace's equation

$$V = \frac{V_o \cdot x}{d}$$

Examples of the Solution of Laplace's Equation

Finding the capacitance of a parallel-plate capacitor

Steps

1 – Given V, use E = - DelV to find E
2 – Use D = εE to find D
3 - Evaluate D at either capacitor plate, D = Ds = Dn *an*
4 – Recognize that ρs = Dn
5 – Find Q by a surface integration over the capacitor plate

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon \cdot S}{d}$$

Examples of the Solution of Laplace's Equation

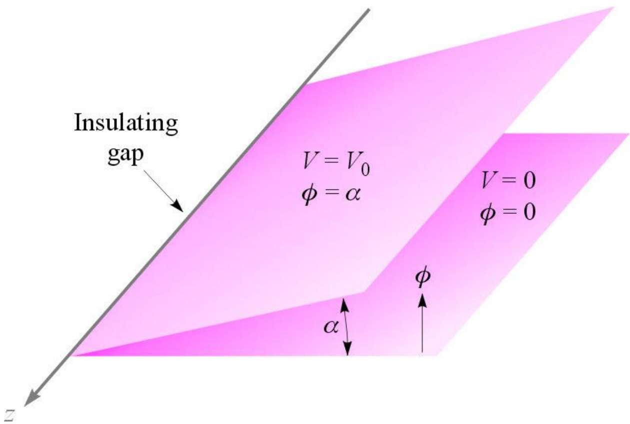
Example 7.2 - Cylindrical

$$V = V_o \cdot \frac{\ln\left(\frac{b}{\rho}\right)}{\ln\left(\frac{b}{a}\right)}$$

$$C = \frac{2 \cdot \pi \cdot \varepsilon \cdot L}{\ln\left(\frac{b}{a}\right)}$$

Examples of the Solution of Laplace's Equation

Example 7.3



Examples of the Solution of Laplace's Equation

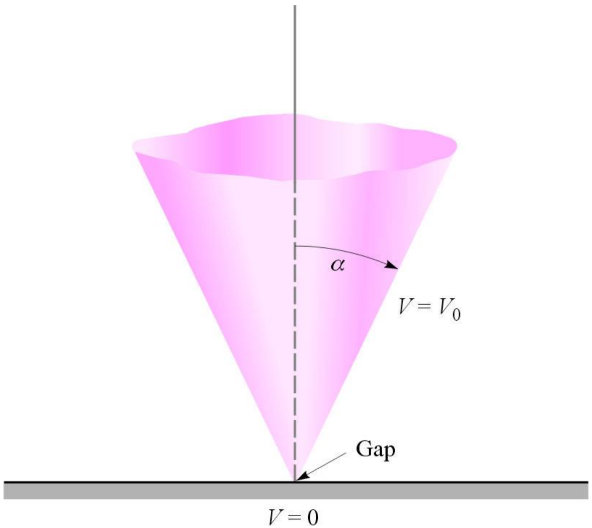
Example 7.4 (spherical coordinates)

$$V = V_o \cdot \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

$$C = \frac{4 \cdot \pi \cdot \varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

Examples of the Solution of Laplace's Equation

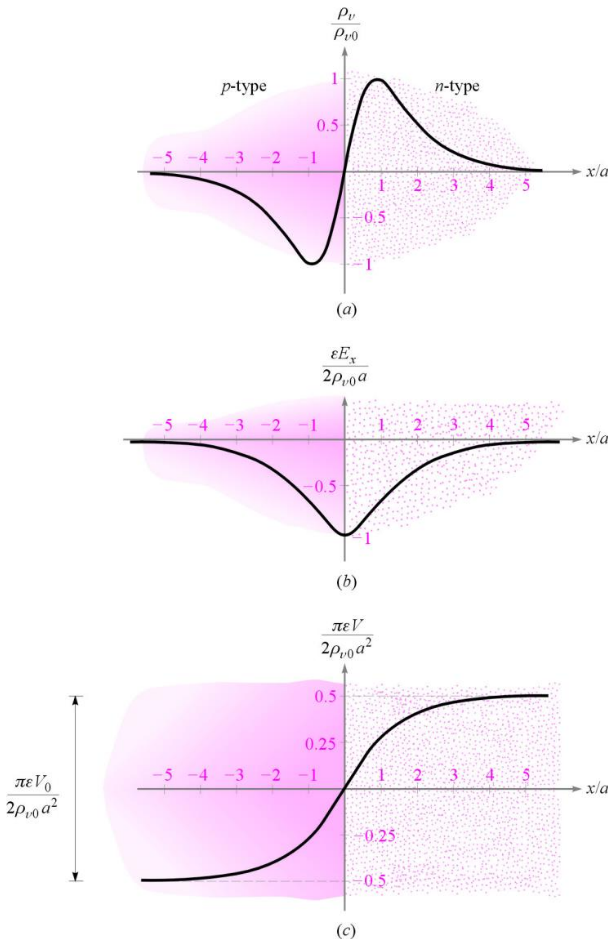
Example 7.5

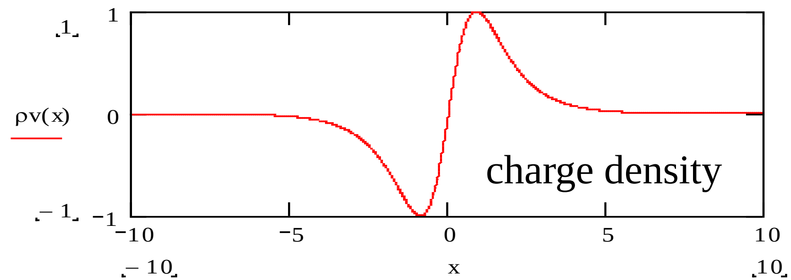


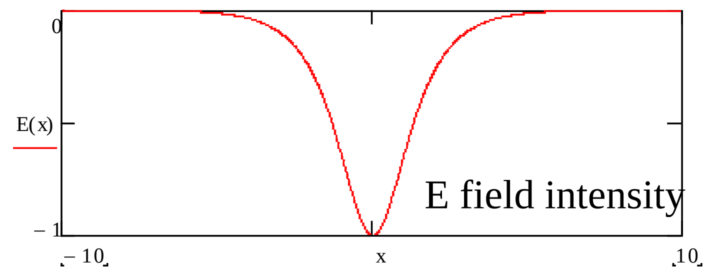
$$V = V_o \cdot \frac{\ln\left(\tan\left(\frac{\theta}{2}\right)\right)}{\ln\left(\tan\left(\frac{\alpha}{2}\right)\right)}$$

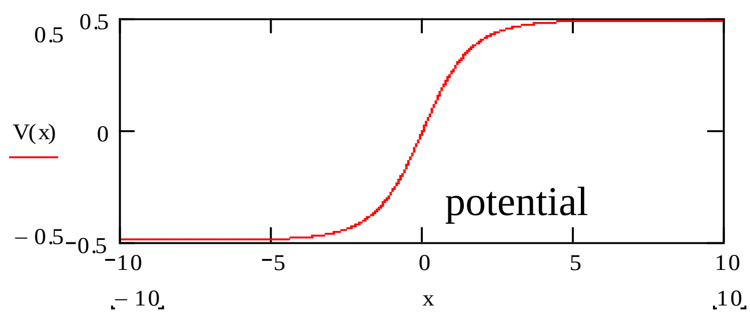
$$C = \frac{2 \cdot \pi \cdot \varepsilon \cdot r_1}{\ln\left(\cot\left(\frac{\alpha}{2}\right)\right)}$$

Examples of the Solution of Poisson’s Equation



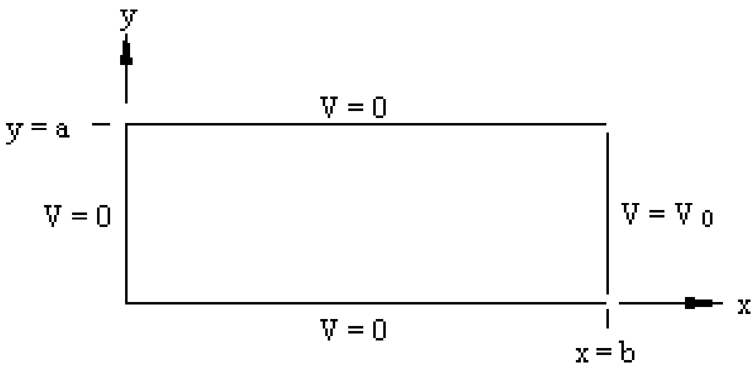






Product Solution Of Laplace's Equation

Referring to the figure below and the specific boundary conditions for the potential on the sides of the structure ...



... choose the dimensions a and b of the box and the potential boundary condition V_0.

$b := 0.5$ Length of box in the x direction (m).

$a := 0.5$ Length of box in the y direction (m).

$V_0 := 2$ Impressed potential on the wall at x = b (V).

Product Solution Of Laplace's Equation

Define potential function

The solution for the potential everywhere inside the rectangular box structure is given as an infinite series. It is not possible to numerically add all of the infinite number of terms in this series. Instead, we will choose the maximum number of terms n_{max} to sum:

$n_{max} := 41$ Maximum n for summation.

$n := 1, 3 .. n_{max}$ Only the odd n terms are summed since all even n terms are zero

The potential V everywhere inside the structure was determined in Example 3.24 to be:

$$V(x,y) := \frac{4 \cdot V_0}{\pi} \cdot \sum_n \frac{1}{n \cdot \sinh\left(\frac{n \cdot \pi \cdot b}{a}\right)} \cdot \sinh\left(\frac{n \cdot \pi \cdot x}{a}\right) \cdot \sin\left(\frac{n \cdot \pi \cdot y}{a}\right)$$

Product Solution Of Laplace's Equation

Plot V versus x at y = a/2

We will generate three different plots of this potential. The first is V as a function of x thr
the center of the box structure. The other two plots will show the potential within the int
of the box in the xy plane. Choose the number of points to plot V in the x and y direction

$npts := 50$ Number of points to plot V in x and y.

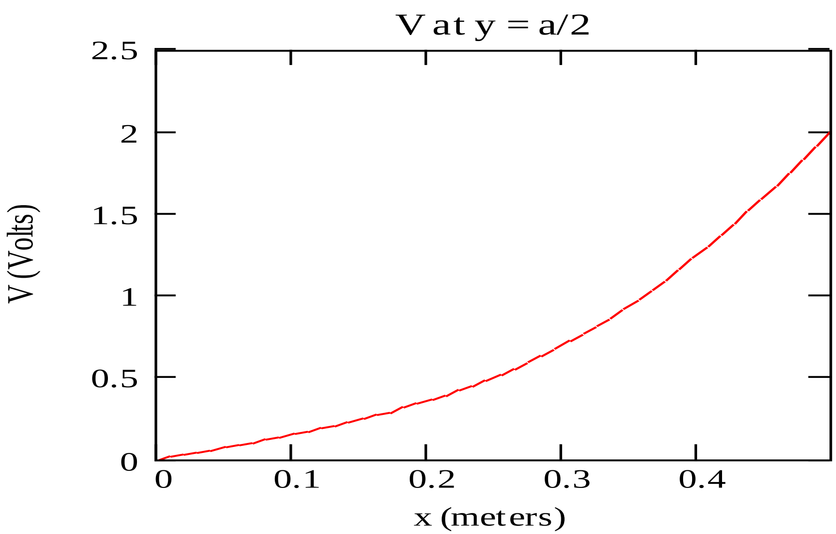
$x_{end} := b$ $y_{end} := a$ x and y ending points (m).

Generate a list of x_i and y_j points at which to plot the potential:

$i := 0 .. npts - 1$ $j := 0 .. npts - 1$

$x_i := i \cdot \frac{x_{end}}{npts - 1}$ $y_j := j \cdot \frac{y_{end}}{npts - 1}$

Now plot the potential as a function of x through the center of the box at y = a/2:



For a rectangular box with

$b = 0.5$ (m)

$a = 0.5$ (m)

and

$V_0 = 2$ (V)

Product Solution Of Laplace's Equation

Computed

$V\left(b, \frac{a}{2}\right) = 2.0303$ (V)

Exact

$V_0 = 2.0000$ (V)

For $n_{max} = 41$, the percent error in the potential at this point is:

$$\text{Error} := \frac{V\left(b, \frac{a}{2}\right) - V_0}{V_0} \cdot 100$$

$\text{Error} = 1.515$ (%)

Product Solution Of Laplace's Equation

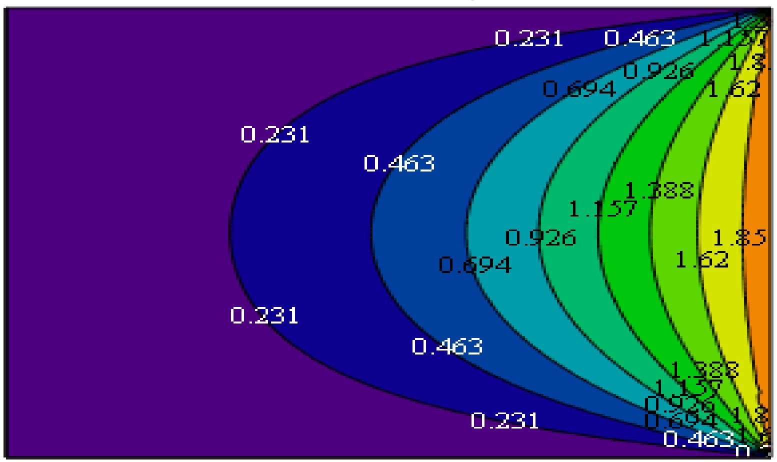
Plot V throughout the inside of the box

Now we will plot the potential throughout the interior of the rectangular box structure. compute V at the matrix of points x_i and y_j:

$$Potential_{i,j} := V(x_i, y_j)$$

Now generate a contour plot of $Potential_{i,j}$

Plot of V(x,y)



$y_{end} = 0.500$ (m)

For a box with

$b = 0.5$ (m)

$a = 0.5$ (m)

and

$V_0 = 2$ (V)

$y = 0$ (m)

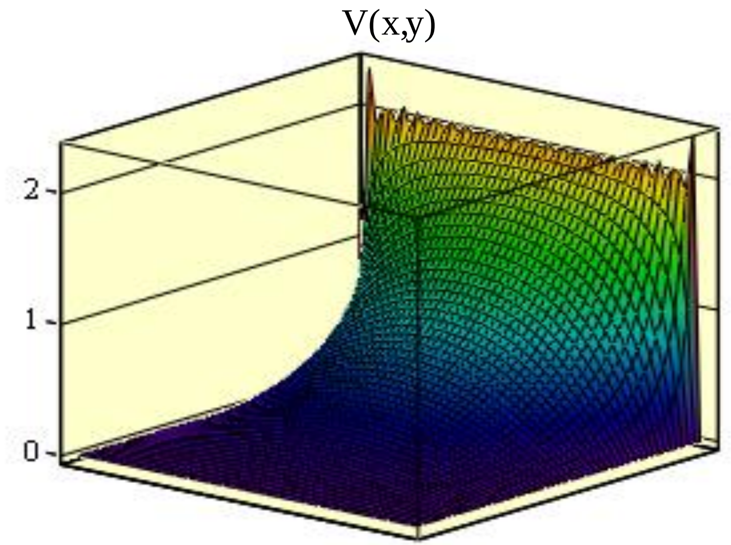
Potential

$x = 0$ (m)

$x_{end} = 0.500$ (m)

We can observe in this plot that the potential is a complicated function of x and y. (Th plot below may help in visualizing the variation of V throughout the interior of this box potential is symmetric about the plane y = a/2 which we would expect since the box boundary conditions are both symmetric about this same plane.

Product Solution Of Laplace's Equation



Potential

For a box with

$b = 0.5 \quad (m)$

$a = 0.5 \quad (m)$

and

$V_0 = 2 \quad (V)$

The jagged edge on the potential at the far wall is due to numerical error and is a nonphysical result. The potential along that wall should exactly equal

$V_0 = 2 \quad (V)$

since that is the applied potential along that wall. This jaggedness in the numerical solution can be reduced by increasing the number of terms in the infinite summation (n_{max}) for V and/or increasing the number of points to plot in the contour and surface plots (npts).