Conservation of Momentum

Momentum conservation for a system of particles

- So far we talked about point particles. There is a need to:

  (a) generalize it to extended bodies

  (b) to deal with variable mass problem

1. Momentum ($p = mv$) is a more fundamental quantity than m & v separately.

2. Newton's 2nd law should be written as F = $\dot{p}$ instead of 'ma' (for variable m).

3. For a system of particles, **an external Force** causes change of **total momentum** of the system. The internal forces cancel each other.

It will be useful to locate a point for a system of particles where all the mass may be concentrated at. Then the single particle EOM will continue.

Center of Mass

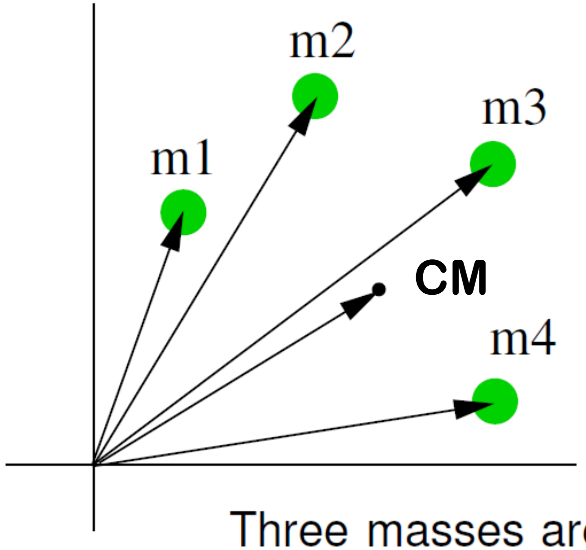
A system has n particles with masses and positions given by

$$m_1,\ m_2, \ldots,\ m_n$$

$$\mathbf{r}_1,\ \mathbf{r}_2, \ldots,\ \mathbf{r}_n$$

Define a **Center of Mass** as

$$\mathbf{R}_{CM} = \frac{1}{M}\left(\sum_i m_i \mathbf{r}_i\right)$$



Example

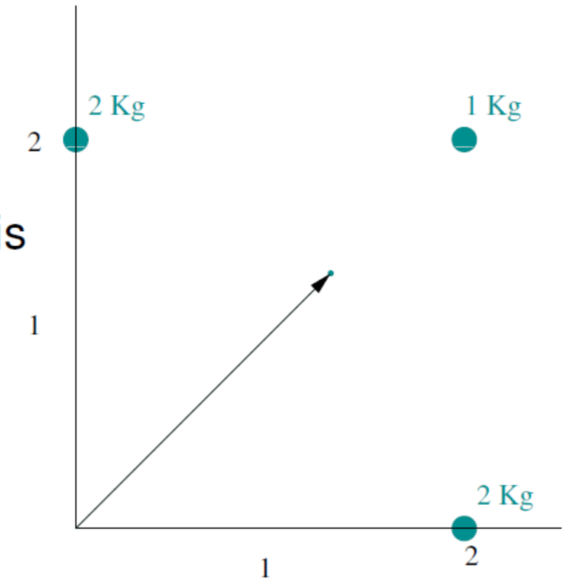
Three masses are kept in a plane as shown in the figure.

$m_1 = 2$ Kg, $m_2 = 2$ Kg and $m_3 = 1$ Kg

$r_1 = 2\mathbf{j}$, $r_2 = 2\mathbf{i}$ and $r_3 = 2\mathbf{i} + 2\mathbf{j}$

Total Mass is 5 Kg. Then Center of Mass is given by

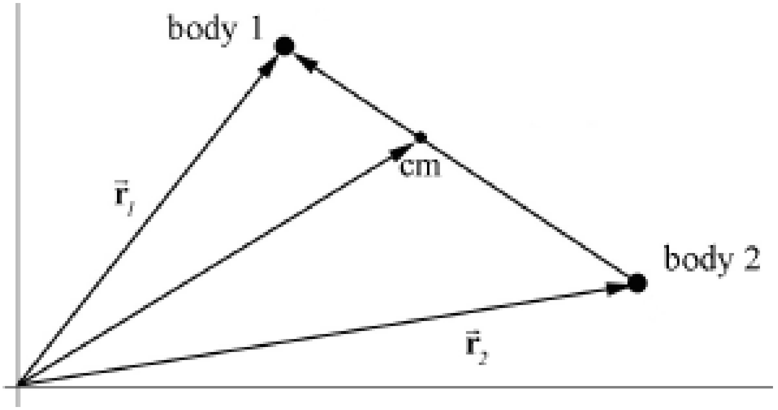
$$\begin{aligned}\mathbf{R}_{cm} &= \frac{1}{5}(2\mathbf{r_1} + 2\mathbf{r_2} + \mathbf{r_3}) \\ &= \frac{6}{5}(\mathbf{i} + \mathbf{j})\end{aligned}$$



Center of mass

The center of mass vector, $\mathbf{R}_{cm}$, of the two-body system

$$\vec{\mathbf{R}}_{cm} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2}.$$



For a **continuous rigid body**, each point-like particle has mass dm and is located at the position r'. The center of mass is then defined as an integral over the body,

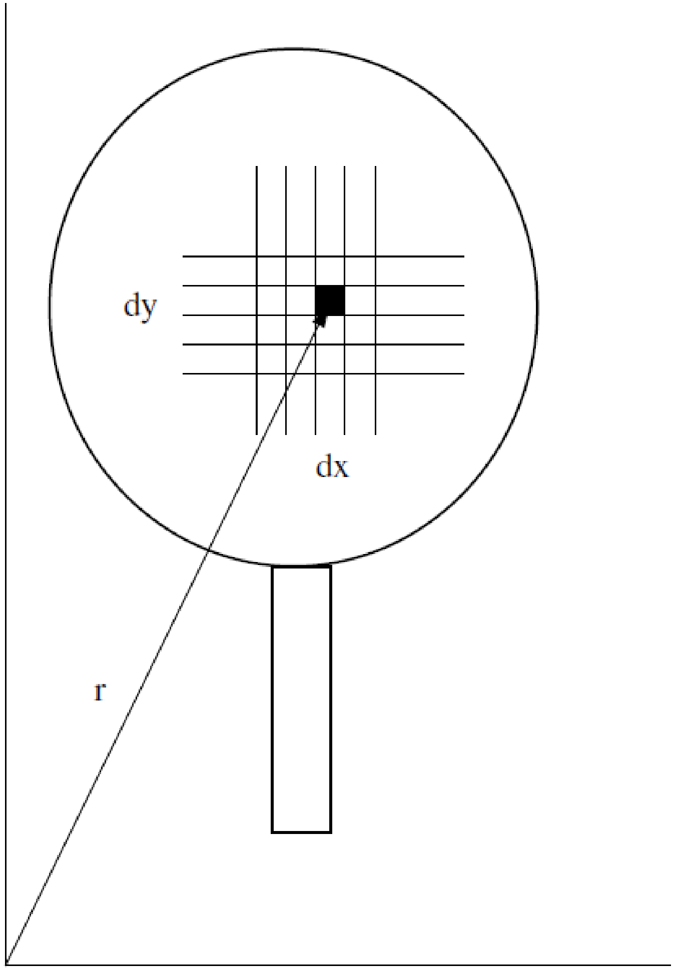
$$\vec{\mathbf{R}}_{cm} = \frac{\int_{\text{body}} dm\, \vec{\mathbf{r}}'}{\int_{\text{body}} dm}.$$

Planar Continuous Bodies

The density is given by $\rho(\mathbf{r})$. An element at $\mathbf{r}$ and of area $dxdy$ has a mass

$dm = \rho(\mathbf{r})dxdy.$

$$\begin{aligned} \mathbf{R}_{cm} &= \frac{1}{M}\sum \mathbf{r}dm \\ &= \frac{1}{M}\int \mathbf{r}\rho(\mathbf{r})dxdy \end{aligned}$$

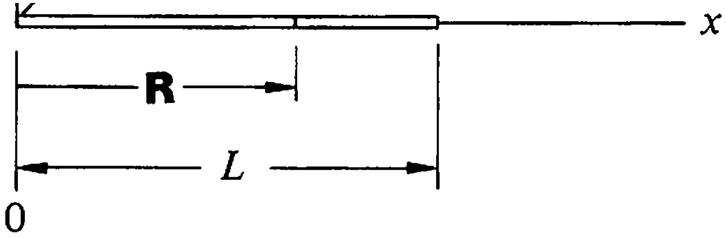


And

(1) $$M = \int \rho(\mathbf{r})dxdy$$

Examples

The mass per unit length λ of a rod of length L varies as $\lambda = \lambda_0(x/L)$, where λ_0 is a constant and x is the distance from the end marked O. Find the center of mass.

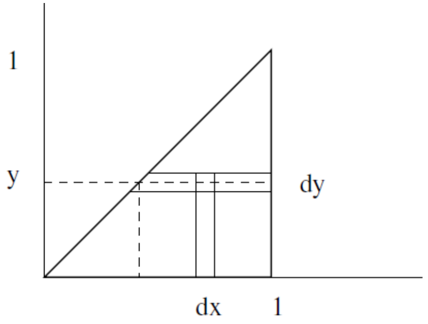


$$M = \int dm = \int_0^L \frac{\lambda_0 x\, dx}{L} = \tfrac{1}{2}\lambda_0 L.$$

$$\mathbf{R}_{cm} = \frac{2}{\lambda_0 L}\int_0^L (x\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})\,\frac{\lambda_0 x\, dx}{L}$$

$$= \frac{2}{L^2}\,\frac{\hat{\mathbf{i}}}{3}\,x^3\Big|_0^L = \tfrac{2}{3}L\hat{\mathbf{i}}.$$

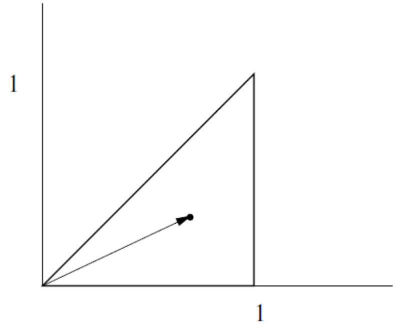
A triangular sheet with uniform density ρ_0 is placed as shown in the figure. Cleary $M = \rho_0/2$



$$\int_0^1 dy \int_y^1 x\, dx = \int_0^1 dy \left(\frac{1}{2} - \frac{y^2}{2}\right) = \frac{2}{3}$$

This gives

$$\mathbf{R}_{cm} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$$



$$\mathbf{R}_{cm} = \frac{1}{M}\int \rho_0(x\mathbf{i} + y\mathbf{j})\,dx\,dy$$

$$= 2\left(\int x\,dx\,dy\right)\mathbf{i} + 2\left(\int y\,dx\,dy\right)\mathbf{j}$$

The center of mass of a system of bodies can be found by treating each body as a particle concentrated at its center of mass.

Equations of Motion

What are CM coordinates good for?

Now, by definition,

$$
\begin{aligned}
M\mathbf{R}_{cm} &= \sum m_i \mathbf{r}_i \\
M\ddot{\mathbf{R}}_{cm} &= \sum m_i \ddot{\mathbf{r}}_i
\end{aligned}
$$

But for each particle, labled by i,

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i = \mathbf{F}_i^{ext} + \mathbf{F}_i^{int}$$

Hence,

$$M\ddot{\mathbf{R}}_{cm} = \sum \mathbf{F}_i^{ext} + \sum \mathbf{F}_i^{int}$$

But by Newton's third law, all internal forces appear in pairs and are equal and opposite. Thus in the following summation all internal forces cancel each other out.

$$\sum_i \mathbf{F}_i^{int} = 0$$

Thus Equation of Motion for Center of Mass of any system

$$M\ddot{\mathbf{R}}_{CM} = \mathbf{F}^{ext} = \sum_i \mathbf{F}_i^{ext}$$

One point $\mathbf{R}_{cm}$ traces the same motion as that of a single particle of mass M under the influence of a force $\mathbf{F}^{ext}$

Translational Motion of the Center of Mass

The velocity of the center of mass is given by $\vec{\mathbf{V}}_{\text{cm}} = \frac{1}{m^{\text{total}}} \sum_{i=1}^{i=N} m_i \vec{\mathbf{v}}_i = \frac{\vec{\mathbf{p}}^{\text{total}}}{m^{\text{total}}}$.

The total momentum is then expressed in terms of the velocity of the center of mass by

$$\vec{\mathbf{p}}^{\text{total}} = m^{\text{total}} \vec{\mathbf{V}}_{\text{cm}}.$$

The total external force is equal to the change of the total momentum of the system,

$$\vec{\mathbf{F}}^{\text{total}}_{\text{ext}} = \frac{d\vec{\mathbf{p}}^{\text{total}}}{dt} = m^{\text{total}} \frac{d\vec{\mathbf{V}}_{\text{cm}}}{dt} = m^{\text{total}} \vec{\mathbf{A}}_{\text{cm}}, = m^{total} \ddot{\vec{R}}_{cm}$$

where $\vec{\mathbf{A}}_{\text{cm}}$, is the acceleration of the center of mass.

The system behaves as if all the mass is concentrated at the center of mass and all the external forces act at that point. This is an over simplification. The shape of the body and the point of application of force matters.

The same force on the same mass with different shape may lead to different types of motion.

$$\vec{F}^{total}_{ext} = m^{total} \ddot{\vec{R}}_{cm}$$

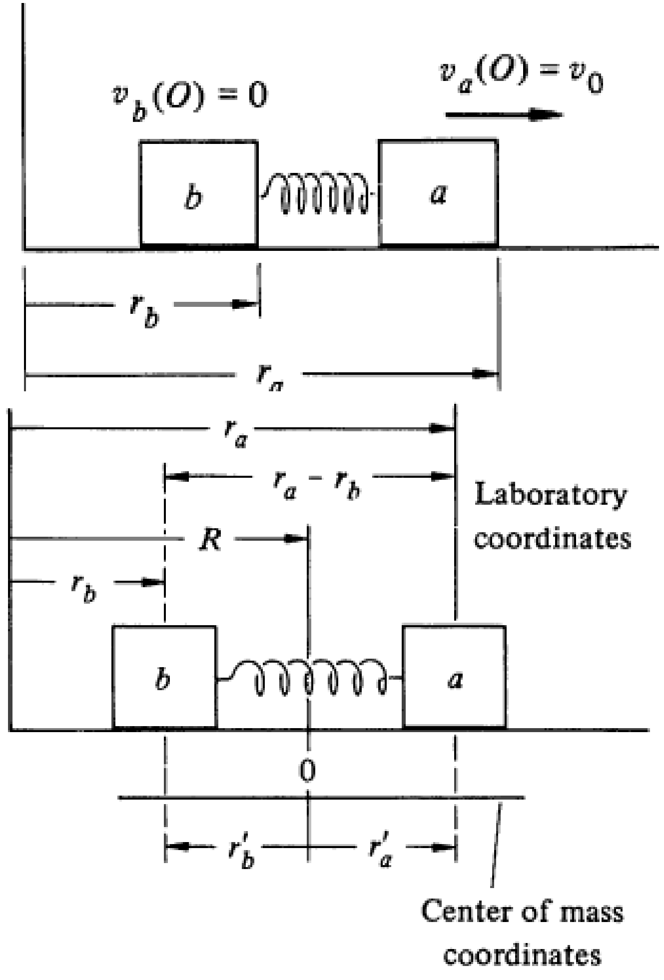
Note EOM describes translational motion.

Center of mass Theorem

The center of mass of a system of particles (rigid or non-rigid) moves as if the entire mass were concentrated at that point and all external forces act there.

An example

Two identical blocks *a* and *b* both of mass *m* slide without friction on a straight track. They are attached by a spring of length *l* and spring constant *k*. Initially they are at rest. At *t* = 0, block *a* is hit sharply, giving it an instantaneous velocity v_0 to the right. Find the velocities for subsequent times.



$$R = \frac{mr_a + mr_b}{m + m} = \frac{1}{2}(r_a + r_b).$$

$$r'_a = r_a - R = \tfrac{1}{2}(r_a - r_b)$$

$$r'_b = r_b - R = -\tfrac{1}{2}(r_a - r_b) = -r'_a.$$

$$r_a - r_b - l = r'_a - r'_b - l,$$

$$m\ddot{r}'_a = -k(r'_a - r'_b - l)$$

$$m\ddot{r}'_b = +k(r'_a - r'_b - l),$$

EOM In CM frame

$$m(\ddot{r}'_a - \ddot{r}'_b) = -2k(r'_a - r'_b - l).$$

Letting $u = r'_a - r'_b - l, \quad m\ddot{u} + 2ku = 0.$

$u = A \sin \omega t + B \cos \omega t,$ where $\omega = \sqrt{2k/m}.$

Difference in CM coordinates executes SHM

$m_1\ddot{r}_a = -k(r_a - r_b + l), \quad m_2\ddot{r}_b = -k(r_b - r_a - l).$

Applying initial conditions:

$t = 0,\ u(0) = 0 \qquad B = 0.$

Since $u = r'_a - r'_b - l = r_a - r_b - l,$

At t=0, $\dot{u}(0) = v_a(0) - v_b(0) = A\omega \cos(0) = v_0, \qquad A = v_0/\omega$

Therefore, $u = (v_0/\omega) \sin \omega t.$

Since $v'_a - v'_b = \dot{u}$, and $v'_a = -v'_b$, we have $v'_a = -v'_b = \frac{1}{2} v_0 \cos \omega t.$

The laboratory velocities are:

$$v_a = \dot{R} + v'_a \qquad \dot{R} = \tfrac{1}{2}[v_a(0) + v_b(0)]$$

$$v_b = \dot{R} + v'_b. \qquad = \tfrac{1}{2} v_0.$$

$$v_a = \frac{v_0}{2}(1 + \cos \omega t) \quad v_b = \frac{v_0}{2}(1 - \cos \omega t).$$

The masses move to the right on the average, but they alternately come to rest in a push me-pull-you fashion.

Conservation of momentum

The total external force *F* acting on a system is related to the total momentum *P* of the system by

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}. \qquad \Delta\vec{\mathbf{p}}_{\text{system}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} dt \equiv \vec{\mathbf{I}}.$$

Impulse

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings.

$$\mathbf{F} = 0, \qquad d\mathbf{P}/dt = 0.$$

The total momentum is constant.

- No matter how strong the interactions among an isolated system of particles.
- No matter how complicated the motion is.

The change in the total momentum of the system is zero, $\Delta\vec{\mathbf{p}}_{\text{system}} = \vec{\mathbf{0}};$

Initial momentum: $\vec{\mathbf{p}}_0^{\text{total}} = m_1\vec{\mathbf{v}}_{1,0} + m_2\vec{\mathbf{v}}_{2,0} + \cdots.$ Final momentum: $\vec{\mathbf{p}}_f^{\text{total}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f} + \cdots.$

$$\vec{\mathbf{p}}_0^{\text{total}} = \vec{\mathbf{p}}_f^{\text{total}}. \qquad p_{0x}\hat{i} + p_{0y}\hat{j} + p_{0z}\hat{k} = p_{fx}\hat{i} + p_{fy}\hat{j} + p_{fz}\hat{k}$$

Component wise conserved

A few points about the conservation law

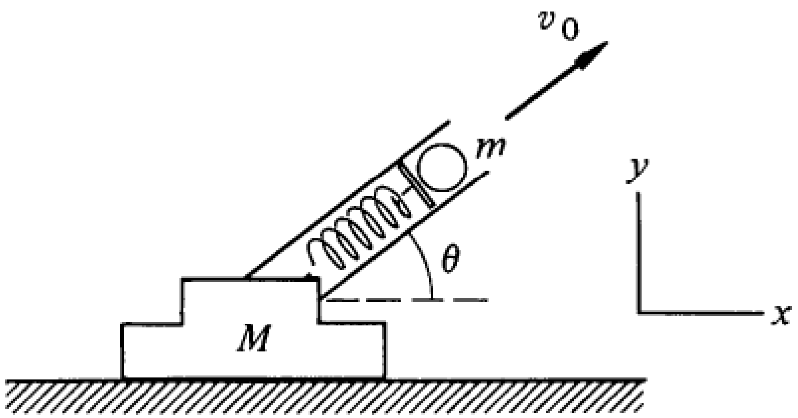
- Conservation of momentum holds true even in areas where Newtonian mechanics proves inadequate, including the realms of quantum mechanics and relativity. So it is more fundamental.

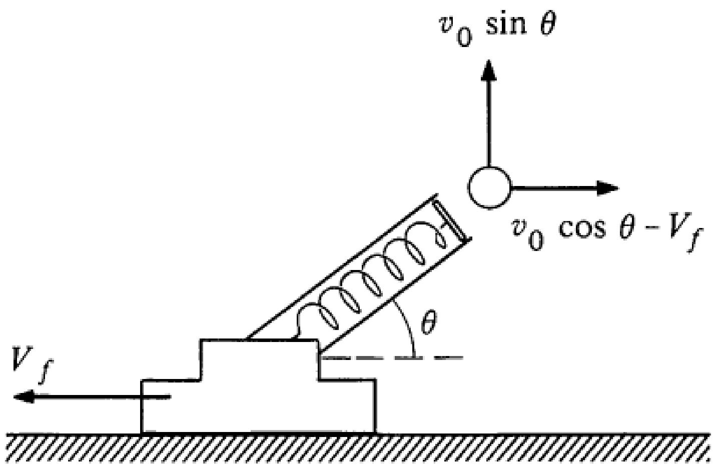
- Conservation of momentum can be generalized to apply to systems like the electromagnetic field, which possess momentum but not mass.

The momentum of a system is conserved if the net external force on the system is zero

Spring Gun Recoil

A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble is v_0. What is the final motion of the gun?





By conservation of momentum: Since there are no horizontal external forces,

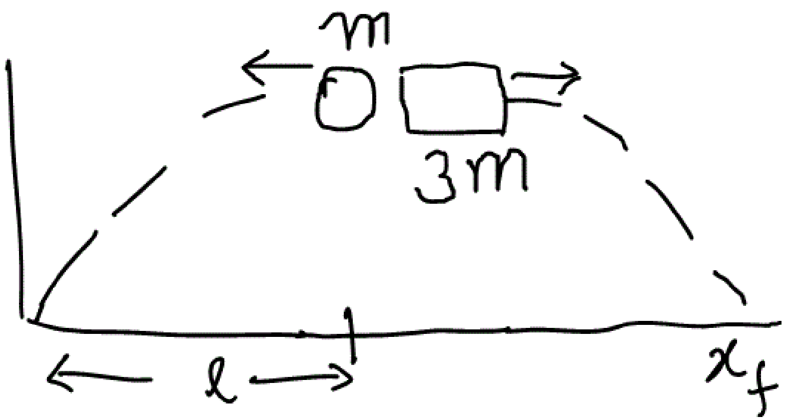
$P_{x,\text{initial}} = 0,$ the system is initially at rest. $0 = \dfrac{dP_x}{dt}.$ $P_{x,\text{initial}} = P_{x,\text{final}}.$

After the marble has left the muzzle, the gun recoils with some speed V_f and its final horizontal momentum is MV_f, to the left. The final horizontal speed of the marble relative to the table is $v_0 \cos\theta - V_f$.

Therefore, $$0 = m(v_0 \cos\theta - V_f) - MV_f \longrightarrow V_f = \frac{mv_0 \cos\theta}{M+m}.$$

Exploding Projectile

An instrument-carrying projectile of mass M accidentally explodes at the top of its trajectory. The horizontal distance between launch point and the explosion is l. The projectile breaks into two pieces that fly apart horizontally. The larger piece, m_2, has three times the mass of the smaller piece, m_1. The smaller piece returns to earth at the launching station. Neglect air resistance and effects due to the earth's curvature. How far away from the original launching point does the larger piece land?



$$M = m_1 + m_2,\ m_2 = 3m_1, m_1 = \frac{1}{4}M, m_2 = \frac{3}{4}M$$

Momentum conservation:

$$MV = -m_1 v_1 + m_2 v_2,\quad v_1 = V,\quad Vt = l$$

$$v_2 = \frac{5}{3}V,\ x_2 = v_2 t = \frac{5}{3}l,\ x_f = l + \frac{5}{3}l = \frac{8}{3}l$$

Center of mass method:

$$R_{cm} = \frac{m_1 x_{1f} + m_2 x_{2f}}{M} = 2l,\ x_{1f} = 0,\ x_{2f} = 2l\frac{M}{m_2} = \frac{8}{3}l.$$

Is it always possible to reduce a many body problem to center of mass coordinate?